Final Exam, Math100, December 16, 2014

(R. Boltje)

Show all your work. No calculator, cell phones, electronic devices, textbooks, or notes allowed.

Maximum score: 50 points

Time: 3 hours

1. (7 points) Suppose that I and J are sets. Suppose further that, for each element $i \in I$, one has a set A_i and, for each $j \in J$, one has a set B_j. Show that

$$\left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcup_{j\in J} B_j\right) = \bigcup_{(i,j)\in I\times J} (A_i \cap B_j)\,.$$

2. (5 points) Give a precise definition of the following notions:

(a) $a \equiv b \pmod{n}$ for $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$.

(b) A *function* $f\colon A \to B$.

(c) *Injectivity* of a function $f\colon A \to B$.

(d) An *equivalence relation* on a set A?

(e) A set A is *countably infinite*.

3. (7 points) Show that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for all natural numbers n.

4. (8 points) Show that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then $ac \equiv bd \pmod{n}$.

5. (8 points) Prove that there exist infinitely many prime numbers.

6. (8 points) Let $f\colon A \to B$ be a function.

(a) Show that $X \subseteq f^{-1}(f(X))$ for all subsets X of A.

(b) Show that $f(f^{-1}(Y)) \subseteq Y$ for all subsets Y of B.

7. (7 points) Prove or disprove: If a relation is transitive and symmetric then it is reflexive.