Final Exam, Math 118, Winter 2000

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1. For which prime numbers p ($p \neq 2, 11$) is $11 + p\mathbb{Z}$ a square in $\mathbb{Z}/p\mathbb{Z}$?

2. Let p be the prime number 821.
(a) Is the congruence $X^2 \equiv -2006 \mod p$ solvable in $\mathbb{Z}$?
(b) Is the congruence $X^2 \equiv 3X + 5 \mod p$ solvable in $\mathbb{Z}$?

3. Compute the continued fraction expansion of
(a) $\sqrt{n^2+1}$ for all $n \in \mathbb{N}$,
(b) $\sqrt{n^2-1}$ for all $n \in \mathbb{N}$, $n \geq 2$.

4. (a) Compute the continued fraction expansion of $\sqrt{58}$.
(b) Has $X^2 - 58Y^2 = -1$ a solution in $\mathbb{N}^2$? If yes, compute its fundamental solution.
(c) Compute the fundamental solution of $X^2 - 58Y^2 = 1$.

5. Let $\xi \in \mathbb{R} \smallsetminus \mathbb{Q}$ with $\xi > 0$ and let p_n/q_n be the n-th convergent of ξ (with $\gcd(p_n, q_n) = 1$), $n \in \mathbb{N}_0$. Moreover, define

$$f_0 := 0, \;\; f_1 := 1, \;\; f_n := f_{n-1} + f_{n-2} \text{ for } n \geq 2.$$

The numbers f_n, $n \in \mathbb{N}_0$, are called the *Fibonacci numbers.*
(a) Show that

$$f_n = \frac{1}{\sqrt{5}}\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \frac{1}{\sqrt{5}}\Big(\frac{1-\sqrt{5}}{2}\Big)^n$$

for $n \in \mathbb{N}_0$.
(b) Show that $p_n \geq f_n$ for all $n \in \mathbb{N}_0$.
(c) Show that $q_n \geq f_{n+1}$ for all $n \in \mathbb{N}_0$.
(d) Is it possible that $p_n = f_n$ and $q_n = f_{n+1}$ for all $n \in \mathbb{N}_0$? If yes, for which ξ?