Show all your work. No calculator, textbooks, or notes allowed.

Maximum score: 30 points

Time: 1 hour 30 minutes

1. Suppose that $G$ and $H$ are groups and that $f: G \to H$ is a homomorphism.
   (a) Show that if $U$ is a subgroup of $G$ then $f(U)$ is a subgroup of $H$. (3 points)
   (b) Show that if $V$ is a subgroup of $H$ then $f^{-1}(V)$ is a subgroup of $G$. (4 points)

Solution: (a) We need to show that
   (i) $1_H \in f(U)$,
   (ii) if $x, y \in f(U)$ then $xy \in f(U)$, and
   (iii) if $x \in f(U)$ then $x^{-1} \in f(U)$.

Proof of (i): Since $U \leq G$, we have $1_G \in U$ and therefore $1_H = f(1_G) \in f(U)$.

Proof of (ii): Let $x, y \in f(U)$. Then there exist $a, b \in U$ such that $x = f(a)$ and $y = f(b)$. Since $U \leq G$, we have $ab \in U$. Since $f$ is a homomorphism, we obtain $xy = f(a)f(b) = f(ab) \in f(U)$.

Proof of (iii): Let $x \in f(U)$. Then there exists $a \in U$ such that $x = f(a)$. Since $U \leq G$, also $a^{-1} \in U$. Since $f$ is a homomorphism, this yields $x^{-1} = f(a)^{-1} = f(a^{-1}) \in f(U)$.

(b) We need to show that
   (i) $1_G \in f^{-1}(V)$,
   (ii) if $a, b \in f^{-1}(V)$ then $ab \in f^{-1}(V)$, and
   (iii) if $a \in f^{-1}(V)$ then $a^{-1} \in f^{-1}(V)$.

First recall that the preimage $f^{-1}(V)$ is defined as $f^{-1}(V) = \{a \in G \mid f(a) \in V\}$. (A common mistake was to assume that there is a function $f^{-1}: H \to G$ which plays the role of the inverse of $f$. Such a function only exists if $f$ is bijective!)

Proof of (i): Since $f(1_G) = 1_H \in V$, we have $1_G \in f^{-1}(V)$, by definition of $f^{-1}(V)$.

Proof of (ii): Let $a, b \in f^{-1}(V)$. By definition of $f^{-1}(V)$, this means that $f(a) \in V$ and $f(b) \in V$. Since $V \leq H$ and $f$ is a homomorphism, we obtain $f(ab) = f(a)f(b) \in V$. By the definition of $f^{-1}(V)$ this implies that $ab \in f^{-1}(V)$.

Proof of (iii): Let $a \in f^{-1}(V)$. By definition of $f^{-1}(V)$, this means that $f(a) \in V$. Since $V \leq H$, we have $f(a)^{-1} \in V$. Since $f$ is a homomorphism we obtain $f(a^{-1}) = f(a)^{-1} \in V$. By definition of $f^{-1}(V)$, this implies that $a^{-1} \in f^{-1}(V)$.

2. Let $G := \text{Sym}(5)$ and $H := \langle (2,3,4), (2,3) \rangle$. Determine $[G : H]$. (5 points)

Solution: By Lagrange’s Theorem, we have $[G : H] = \frac{|G|}{|H|}$. We know that $|G| = 5! = 120$ and still need to find $|H|$. Set $\sigma = (2,3,4)$ and $\tau = (2,3)$. Thus, $H = \langle \sigma, \tau \rangle$. Note first that, since $\sigma \in H$ and $o(\sigma) = 3$, we know that $|H|$ is a multiple of 3. Similarly, since $\tau \in H$ and $o(\tau) = 2$, $|H|$ is a multiple of 2. Thus, $|H|$ is a multiple of 6. On the other hand, since $\sigma(1) = 1 = \tau(1)$ and $\sigma(5) = 5 = \tau(5)$, we obtain $\pi(1) = 1$ and $\pi(5) = 5$ for all $\pi \in H$ (in fact,
every element in \( H \) is a product of factors each of which is equal \( \sigma \) or \( \tau \). But there are only 6 permutations \( \pi \in \text{Sym}(5) \) which satisfy \( \pi(1) = 1 \) and \( \pi(5) = 5 \). Therefore, we now know that \( |H| = 6 \). (There are other ways to show that \( |H| = 6 \), for example by direct computation of products of \( \sigma \) and \( \tau \).) Altogether we obtain \( [G : H] = 120/6 = 20 \).

Common mistake was a wrong argument like: \( o(\sigma) = 3 \) and \( o(\tau) = 2 \) implies that \( \langle \sigma, \tau \rangle = \text{lcm}(2, 3) = 6 \). Another common mistake: \( H = \{\sigma, \tau\} \) and \( |H| = 2 \).

3. (a) Define what it means for a group \( G \) to be cyclic. (2 points)
(b) Formulate Lagrange’s Theorem. (2 points)
(c) Define what a left coset of a subgroup \( H \) in a group \( G \) is. (2 points)

Solution: (a) \( G \) is called cyclic if there exists \( a \in G \) with \( G = \langle a \rangle = \{a^n \mid n \in \mathbb{Z}\} \).
(b) If \( G \) is a group and \( H \) is a subgroup of \( G \) then \( |G| = |H| \cdot [G : H] \).
(c) A left coset of \( H \) in \( G \) is a subset of \( G \) which is of the form \( aH = \{ah \mid h \in H\} \) for some \( a \in G \).

4. Find (with proof) an example of a group \( G \) and subgroups \( H \) and \( K \) of \( G \) such that \( HK \) is not a subgroup of \( G \). (4 points)

Solution: Let \( G = \text{Sym}(3) \) and let \( H = \langle (1,2) \rangle = \{\text{id}, (1,2)\} \) and \( K = \langle (2,3) \rangle = \{\text{id}, (2,3)\} \). Then clearly \( G \) is a group, and \( H \) and \( K \) are subgroups of \( G \). But \( HK = \{hk \mid h \in H, k \in K\} = \{\text{id}, (1,2), (2,3), (1,2,3)\} \). The latter can’t be a subgroup of \( G \), since it has 4 elements and \( G \) has order 6.

(There are plenty of other examples of this type. However, if \( G \) is abelian then \( HK \) is always a subgroup of \( G \).)

5. Decide for each of the following statements if it is true. Answer with "YES" or "NO". You don’t have to justify your answer. For each part (a), (b), (c), (d), a correct answer is worth 2 points, a wrong answer counts as -2 points, no answer counts as 0 points. If your overall score for this problem is negative, it will be counted as 0 points. (8 points)
(a) If \( n \in \mathbb{N} \) and \( \sigma \in \text{Sym}(n) \) then \( \sigma^2 \in \text{Alt}(n) \).
(b) There exists an element of order 6 in \( \text{Sym}(5) \).
(c) The dihedral group \( D_8 \) of order 8 has an element \( g \neq 1 \) such that \( gx = xg \) for all \( x \in D_8 \).
(d) Every group of order 4 has an odd number of subgroups of order 2.

Solution: (a) YES. (The product of two even permutations is even, and the product of two odd permutations is even.)
(b) YES, (namely \( (1,2,3)(4,5) \), or any other element that is the product of a 3-cycle and a disjoint 2-cycle.)
(c) YES. (The rotation by 180° is such an element: the matrix is the negative of the identity matrix! Alternatively, if you use our standard notation from class, you will see that \( \sigma^2 \) commutes with \( \sigma \) and with \( \tau \), and therefore with every element of \( D_8 \).)
(d) YES. (Isomorphic groups have the same number of subgroups of order 2. Up to isomorphism there are only two groups of order 4: the cyclic group of order 4 and the Klein 4-group. The first has one subgroup of order 2, the latter has three subgroups of order 2. You find this explicitly in the class notes.)