MATH 201, Algebra II, Exercises (Week 1, due Jan. 11)

(Winter 1999, R. Boltje)

1. Determine $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/5\mathbb{Z})$ and $\mathrm{Hom}(\mathbb{Z}/5\mathbb{Z}, \mathbb{Z})$.

2. Show that $\mathrm{End}(\mathbb{Q}) \cong \mathbb{Q}$ as rings.

3. Is $\mathrm{End}(F) \cong F$ as rings for every field F?

4. For which $n \in \mathbb{Z}$ is $\mathbb{Z}/n\mathbb{Z}$ an irreducible $\mathbb{Z}$-module?

5. Let R be a ring and let M be an R-module. Show that if M is simple, then $\mathrm{End}_R(M)$ is a division ring. Is the converse true?

6. Let R be a ring. A left ideal I of a ring R is called a *maximal* left ideal, if $I \neq R$ and if R and I are the only left ideals of R containing I. Show that for any R-module M the following are equivalent:
(i) M is an irreducible R-module.
(ii) M is isomorphic to R/I for some maximal left ideal I of R.

7. Let R be a ring and let M be an R-module. Show that the following are equivalent:
(i) M is a cyclic R-module.
(ii) There exists a left ideal I of R such that $M \cong R/I$ as R-modules.

MATH 201, Algebra II, Exercises (Week 2, due Jan. 19)

(Winter 1999, R. Boltje)

8. Let R be a commutative ring and let M be a finitely generated free R-module of R-rank n. Show that if $x_1, \ldots, x_n \in M$ generate M as an R-module, then $x_1, \ldots, x_n$ is an R-basis of M.

9. An *idempotent* of a ring is an element e with the property $e^2 = e$. Let R be a ring and let M be an R-module. Show that M can be decomposed into an internal direct sum $M = U_1 \oplus U_2$ of R-submodules $U_1 \neq 0 \neq U_2$ if and only if there exists an idempotent in $\mathrm{End}_R(M)$ which is different from 0 and 1.

10. Are the elements $x_1 = (0, 1, 2), x_2 = (3, -1, 1), x_3 = (2, 0, -2) \in \mathbb{Z}^{(3)}$
(a) a $\mathbb{Q}$-basis of $\mathbb{Q}^{(3)}$?
(b) a $\mathbb{Z}$-basis of $\mathbb{Z}^{(3)}$?

11. Let V be the $\mathbb{R}$-vector space of all polynomials

$$a_0 + a_1\lambda + \ldots a_4\lambda^4 \in \mathbb{R}[\lambda]$$

of degree at most 4 and let $f\colon V \to V$ be the $\mathbb{R}$-linear map given by taking the derivative of a polynomial. Find an $\mathbb{R}$-basis X of V such that the matrix representing f with respect to X has the form

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

12. Assume that R is a commutative ring, that M is a free R-module with R-basis $X = (x_1, \ldots, x_5)$, and that $f\colon M \to M$ is an R-linear map which is represented by the matrix A in Exercise 11 with respect to X. Show that $U_1 := Rx_1$ is an f-stable submodule of M but that there is no f-stable R-submodule U_2 of M such that $M = U_1 \oplus U_2$.

13. Let $M = \mathbb{Z}[\lambda]/(1 + \lambda + \lambda^2)\mathbb{Z}[\lambda]$.
(a) Is M free as $\mathbb{Z}$-module?
(b) Is M free as $\mathbb{Z}[\lambda]$-module?

14. Let $f\colon \mathbb{R}^{(3)} \to \mathbb{R}^{(3)}$ be given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a - b \\ b - c \\ c - a \end{pmatrix}.$$

Determine the matrix representing f
(a) with respect to the canonical $\mathbb{R}$-basis (e_1, e_2, e_3).
(b) with respect to the $\mathbb{R}$-basis $(e_1, e_1 + e_2, e_1 + e_2 + e_3)$.

MATH 201, Algebra II, Exercises (Week 3, due Jan. 25)

(Winter 1999, R. Boltje)

15. let R be a domain and let M be an R-module.

(a) Show that $\mathrm{tor}_R(M)$ is an R-submodule of M.

(b) Show that for each R-module N and each R-linear map $f\colon M \to N$ one has $f(\mathrm{tor}_R(M)) \subseteq \mathrm{tor}_R(N)$.

(c) Show that if M is R-free then M is R-torsion free.

16. Let $R = \mathbb{Z}[\lambda]/(\lambda^2+1)\mathbb{Z}[\lambda]$ and let $M := R$. We can consider M as $\mathbb{Z}[\lambda]$-module, as R-module, and as $\mathbb{Z}$-module in the usual way. Determine $\mathrm{tor}_{\mathbb{Z}[\lambda]}(M)$, $\mathrm{tor}_R(M)$ and $\mathrm{tor}(M)$.

17. Give an example of a finitely generated free R-module M over a principal ideal domain R and an R-submodule U of M with $\mathrm{rk}_R(U) = \mathrm{rk}_R(M)$ but $U \neq M$.

18. Find the invariant factors of the matrix

$$A = \begin{pmatrix} 5 & -1 & 4 \\ 3 & -2 & 8 \\ 1 & 1 & -4 \end{pmatrix} \in M_3(\mathbb{Z}) .$$

19. Find the invariant factors of the matrix

$$A = \begin{pmatrix} \lambda^2-\lambda & \lambda^3-1 & \lambda^2+\lambda+1 \\ 2 & \lambda-1 & -\lambda \\ 3\lambda & \lambda^2-1 & \lambda^2 \end{pmatrix} \in M_3(\mathbb{Q}[\lambda]) .$$

20. Let $M = \mathbb{Z}^{(3)}$ and let $U \subset M$ be the subgroup given by

$$U = \{(x,y,z) \in \mathbb{Z}^{(3)} \mid x - 2y + 3z = 0\} .$$

Find a $\mathbb{Z}$-basis x_1, x_2, x_3 of M and numbers $d_1, d_2 \in \mathbb{Z}$ with $d_1 \mid d_2$ such that d_1x_1, d_2x_2 is a $\mathbb{Z}$-basis of U.

MATH 201, Algebra II, Exercises (Week 4, due Feb. 2)

(Winter 1999, R. Boltje)

21. Let R be a principal ideal domain and let $n \in \mathbb{N}$. Show that every matrix $A \in \mathrm{GL}_n(R)$ can be written as a product of matrices of the form $T_{ij}(a)$, $D_i(u)$, and P_{ij} as described in Remark 2.4 (c).

22. Let R be a ring, let M be an R-module, and suppose that $M = U_1 \oplus \cdots \oplus U_n$ for R-submodules $U_1, \ldots, U_n$ of M. Show that, if $V_1 \subseteq U_1, \ldots, V_n \subseteq U_n$ are further R-submodules and $N := V_1 + \cdots + V_n$, then $N = V_1 \oplus \cdots \oplus V_n$ and

$$M/N \cong U_1/V_1 \oplus \cdots \oplus U_n/V_n$$

as R-modules.

23. Let $U \subseteq \mathbb{Z}^{(n)}$ be a subgroup generated by elements $f_1, \ldots, f_n$ with

$$f_i = \sum_{j=1}^{n} a_{ji} e_i \,, \quad a_{ji} \in \mathbb{Z} \,,$$

where $e_1, \ldots, e_n$ denote the canonical $\mathbb{Z}$-basis of $\mathbb{Z}^{(n)}$. Assume that $d := \det(a_{ij}) \neq 0$. Show that $|\mathbb{Z}^{(n)}/U| = |d|$.

24. Let M be an R-module over a principal ideal domain R such that $dM = 0$ for some $d \in R$. Let $p \in R$ be a prime element such that $M_p \neq 0$. Show that then $p \mid d$.

25. Let $R = \mathbb{C}[\lambda]$. How many R-modules M with $(\lambda^2 - 1)M = 0$ and $\dim_{\mathbb{C}} M = 3$ do there exist (up to isomorphism)?

26. Determine all abelian groups of order 500 up to isomorphism.

27. Let V be a 3-dimensional $\mathbb{C}$-vector space and let $f \in \mathrm{End}_{\mathbb{C}}(V)$. Assume that with respect to some ordered $\mathbb{C}$-basis (v_1, v_2, v_3) the map f is represented by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 3 & 3 & 2 \end{pmatrix} .$$

Determine a Jordan normal form J of the matrix A and an ordered $\mathbb{C}$-basis (w_1, w_2, w_3) of V such that J represents f with respect to (w_1, w_2, w_3).

MATH 201, Algebra II, Exercises (Week 5, due Feb. 8)

(Winter 1999, R. Boltje)

28. Let E/F be a field extension and let $x \in E$ be transcendental. Show that $F[x] \cong F[X]$ and $F(x) \cong F(X)$ as rings.

29. Let F be a field. Show that F contains a smallest subfield P (this means, it is contained in every subfield of F). Show that either $P \cong \mathbb{Z}/p\mathbb{Z}$ for some prime p or $P \cong \mathbb{Q}$. Show that in the first case F has characteristic p and in the second case F has characteristic 0.

30. Find the minimal polynomial of $x = \sqrt{2}+i$ over $\mathbb{Q}$. Is $\mathbb{Q}(\sqrt{2}i)$ contained in $\mathbb{Q}(x)$? Show that $\mathbb{Q}(x) = \mathbb{Q}(\sqrt{2}, i)$. Find the minimal polynomial of x over $\mathbb{Q}(\sqrt{2}i)$. Try to find as many intermediate fields of $\mathbb{Q}(x)/\mathbb{Q}$ as possible.

MATH 201, Algebra II, Exercises (Week 6, due Feb. 16)

(Winter 1999, R. Boltje)

31. Let E/F and E'/E be algebraic field extensions. Show that also E'/F is algebraic.

32. Let E/F be a field extension and let S and T be subsets of E.
(a) Show that if $S \subseteq T$, then $F(S) \subseteq F(T)$.
(b) Show that $(F(S))(T) = F(S \cup T)$.
(c) Does $F(S \cap T) = F(S) \cap F(T)$ hold?

33. Let F be a field, let $f(X) \in F[X]$ have degree $n > 0$, and let E/F be a splitting field of $f(X)$ over F. Show that $[E : F] \mid n!$.

34. Find a splitting field of $f(X) = X^4 - 2$ over $\mathbb{Q}$ and determine its degree over $\mathbb{Q}$.

35. Find all $\mathbb{Q}$-algebra homomorphisms from $\mathbb{Q}(\sqrt[4]{2})$ to $\mathbb{Q}(\sqrt[4]{2})$ and to $\mathbb{Q}(\sqrt[4]{2}, i)$.

36. Show that each field extension of degree 2 is normal.

37. Let E/F be an algebraic field extension and let K, L be intermediate fields such that K/F and L/F are normal. Show that then also $K \cap L/F$ and KL/F are normal. (Here, KL is the smallest subfield of E containing K and L. Also $KL = K(L) = L(K) = F(K \cup L)$.)

MATH 201, Algebra II, Exercises (Week 7, due Feb. 22)

(Winter 1999, R. Boltje)

38. Let F be a field of characteristic $p > 0$. Show that F contains only one p-th root of unity (i.e., an element $\zeta \in F$ with $\zeta^p = 1$), namely 1.

39. Let E/F be a field extension and let $f(X), g(X) \in F[X]$. Let $p(X) \in F[X]$ (resp. $q(X) \in E[X]$) be a monic gcd of $f(X)$ and $g(X)$, considered as elements in $F[X]$ (resp. $E[X]$). Show that $p(X) = q(X)$.

40. Let E/F be an algebraic field extension and let E' be an intermediate field. Show that E/F is separable if and only if E/E' and E'/F are separable.

41. Let E/F ba an algebraic field extension. Show that the set

$$E' := \{a \in E \mid \ a \text{ is separable over } F\}$$

is a subfield of E containing F.

42. Let E/F be a finite field extension. Show that the E-vector space $\mathrm{Hom}_F(E, E)$ as defined in Remark 6.2 has dimension $[E : F]$.

MATH 201, Algebra II, Exercises (Week 8, due March 1)

(Winter 1999, R. Boltje)

43. Let E/F be a finite Galois extension with Galois group G and let E_1 and E_2 be intermediate fields corresponding to subgroups H_1 and H_2 of G. Show that then $H_1 \cap H_2$ corresponds to $E_1 E_2$ and $< H_1 \cup H_2 >$ corresponds to $E_1 \cap E_2$.

44. Let $F \subseteq E \subseteq E_1$ be fields such that E_1/F is a finite normal field extension and E/F is a separable field extension. Show that there exists a smallest Galois extension E'/F with $E \subseteq E' \subseteq E_1$.

45. Let E/F be a finite separable field extension. Show that there exists a Galois extension E'/F such that $E \subseteq E'$.

46. (a) Show that $\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}$ is a Galois extension.
(b) Compute its Galois group G in terms of the images of $\sqrt[4]{2}$ and i.
(c) Find its action on the roots of $f(X) = X^4 - 2$.
(d) Compute all the intermediate fields.

47. Let $\zeta := e^{2\pi i/5}$.
(a)Show that $\mathbb{Q}(\zeta + \zeta^{-1}) = \mathbb{Q}(\sqrt{5})$.
(b) Find the minimal polynomial of ζ over $\mathbb{Q}$.
(c) Show that $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a Galois extension.
(d) Find all the intermediate fields of $\mathbb{Q}(\zeta)/\mathbb{Q}$.

MATH 201, Algebra II, Exercises (Week 9, due March 8)

(Winter 1999, R. Boltje)

48. Let $E \subset \mathbb{C}$ be the splitting field of $f(X) = X^{10} - 5$ over $\mathbb{Q}$.
(a) Determine $[E : \mathbb{Q}]$.
(b) Determine the Galois automorphisms of $E/\mathbb{Q}$ in terms of images of $\zeta := e^{2\pi i/5}$ and $\sqrt[10]{5}$.
(c) Determine $\mathrm{Gal}(E/\mathbb{Q})$ as a subgroup of the symmetric group of the roots of $f(X)$.

49. Let F be a field and let $f \in F[X] \smallsetminus F$, such that f and f' are coprime. Show that f has no multiple roots in any extension field E of F.

50. Compute the cyclotomic polynomial $\Phi_{12} \in \mathbb{Z}[X]$.

51. Construct a field with 8 elements and find a primitive 7-th root of unity in it.

52. Let E/F be a field extension and let F', E_1, E_2 be intermediate fields of E/F such that $E_1 \subseteq E_2$ and such that E_2/E_1 is a Galois extension. Show that then also $F'E_2/F'E_1$ is a Galois extension and that $\mathrm{Gal}(F'E_2/F'E_1)$ is isomorphic to the subgroup $\mathrm{Gal}(E_2/E_2 \cap F'E_1)$ of $\mathrm{Gal}(E_2/E_1)$.