MATH 201, Algebra II, Midterm (February 5, 1999)

(Winter 1999, R. Boltje)

1. Let A be a finitely generated abelian torsion group with elementary divisors $2, 4, 4, 3, 3, 5, 7, 49$. What are the invariant factors of A?

2. Determine the Jordan normal form of the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & -1 & -2 \end{pmatrix}.$$

3. Let R be a ring, let M and N be R-modules, and let $f \in \mathrm{Hom}_R(M, N)$, $g \in \mathrm{Hom}_R(N, M)$ such that $g \circ f = \mathrm{id}_M$. Show that $N = \mathrm{im}(f) \oplus \ker(g)$.

4. Let R be a ring and let M and N be R-modules. Consider the abelian group $F := \mathrm{Hom}_R(M, N)$. Does

$$(rf)(m) := r(f(m)), \qquad \text{for } r \in R,\ f \in F,\ m \in M,$$

define an R-module structure on F? (Give a proof or a counterexample.) If the answer is "no", try to impose a condition on R such that the answer is "yes".

5. Let G be a finite abelian group of order n. Show that for each divisor m of n there exists a subgroup H of G of order m.

Solutions

1. The invariant factors of A are $d_3 = 4 \cdot 3 \cdot 5 \cdot 49$, $d_2 = 4 \cdot 3 \cdot 7$, $d_1 = 2$, since $d_1 \mid d_2 \mid d_3$ and the prime power factors of d_1, d_2, d_3 are the given numbers.

2. We transform the matrix $\lambda 1 - A$ using the elementary row and column operations:

$$\begin{pmatrix} \lambda+1 & 0 & 0 \\ -1 & \lambda & -1 \\ 1 & 1 & \lambda+2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & \lambda+2 \\ -1 & \lambda & -1 \\ \lambda+1 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & \lambda+2 \\ 0 & \lambda+1 & \lambda+1 \\ \lambda+1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & \lambda+2 \\ 0 & \lambda+1 & \lambda+1 \\ 0 & -\lambda-1 & -\lambda^2-3\lambda-2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & \lambda+2 \\ 0 & \lambda+1 & \lambda+1 \\ 0 & -\lambda-1 & -\lambda^2-3\lambda-2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda+1 & \lambda+1 \\ 0 & -\lambda-1 & -\lambda^2-3\lambda-2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda+1 & \lambda+1 \\ 0 & 0 & -\lambda^2-2\lambda-1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda+1 & 0 \\ 0 & 0 & -(\lambda+1)^2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda+1 & 0 \\ 0 & 0 & (\lambda+1)^2 \end{pmatrix}.$$

Therefore, the Jordan normal form of A consists of 2 Jordan blocks, one of size 1, one of size 2, both with -1's on the diagonal:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} .$$

3. Let $n \in N$, then $n = f(g(n)) + [n - f(g(n))]$ with $f(g(n)) \in \mathrm{im}(f)$ and $n - f(g(n)) \in \ker(g)$, since $g(n) - g(f(g(n))) = g(n) - g(n) = 0$. Moreover, if $n \in \mathrm{im}(f) \cap \ker(g)$, then there exists $m \in M$ with $n = f(m)$. But since $n \in \ker(g)$, we have $m = g(f(m)) = g(n) = 0$, and so also $n = f(m) = 0$.

4. First note that for $f \in F$, $r \in R$ and $m \in M$ we have $r(f(m)) = f(rm)$, since f is R-linear. The definition in the text does not give an R-module structure on F, since rf is in general not again in F. In fact, we need to show that $(rf)(sm) = s((rf)(m))$ for all $r, s \in R$, $f \in F$, and $m \in M$. The left hand side is now given by

$$(rf)(sm) = f(r(sm)) = f((rs)m) = (rs)(f(m)) ,$$

and the right hand side is given by

$$s((rf)(m)) = s(f(rm)) = s(r(f(m))) = (sr)(f(m)) .$$

If we take $M = N = R$ for a non-commutative ring R and if we take $r, s \in R$ with $rs \neq sr$, then with $f = \mathrm{id}_R$ and $m = 1$ the above equation does not hold. But it does hold, if R is commutative. Finally, we check that $(r+s)f = rf + sf$, $r(f+g) = rf + fg$, and $1f = f$ for all $r, s \in R$ and $f, g \in F$. Let $m \in M$, then

$$\begin{gathered} ((r+s)f)(m) = f((r+s)m) = f(rm + sm) \\ = f(rm) + f(sm) = (rf)(m) + (sf)(m) = (rf + sf)(m) , \\ (r(f+g))(m) = (f+g)(rm) = f(rm) + g(rm) \\ = (rf)(m) + (rg)(m) = (rf + rg)(m) , \\ (1f)(m) = f(1m) = f(m) . \end{gathered}$$

5. We proof the assertion by induction on the number of primes divisors in a decomposition of n/m. If $n/m = 1$, we can take $G = H$. Assume next that n/m is a prime p. We write $G \cong \mathbb{Z}/p_1^{\alpha_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_t^{\alpha_t}\mathbb{Z}$ with prime numbers $p_1, \ldots, p_t$. Then one of the p_i, say p_1 is equal to p, and the subgroup H of G corresponding to $p\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_t^{\alpha_t}\mathbb{Z}$ has index p in H, since $(\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z})/(p\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. Therefore, $|H| = |G|/[G : H] = n/p = m$. Now assume that $n/m = p_1 \cdots p_k$ is a product of k primes with $k > 1$. By induction there exists a subgroup U of G of order mp_k, and since U is again an abelian torsion group, we can use the argument for the case $k = 1$ again and obtain a subgroup H of U of index p_k which then has order m.