Mackey Functors and Related Structures in Representation Theory and Number Theory

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Let $G$ be a finite group and $H$ a subgroup of $G$. In many areas we have the notion of restriction of a $G$-action to an $H$-action and induction of an object with $H$-action which results in an object with $G$-action. The objects we talk about may be finite sets, abelian groups, vector spaces, fields, manifolds, topological spaces, vector bundles, etc. Besides the usual transitivity properties of restriction and induction, G. Mackey discovered a formula (the Mackey-decomposition formula) which allows to express the composition of restriction after induction as a combination of compositions of inductions after restrictions. By the construction of Grothendieck groups, the objects we consider can be regarded as elements of an abelian group, or sometimes even as elements of a commutative ring. The functors of restriction and induction are reflected in maps between these groups, and what we obtain is a Mackey functor (see Definition I.1.1), named after G. Mackey, since the Mackey axiom (see I.1.1 (M4)) reflects the Mackey decomposition theorem. Note that in the definition of a Mackey functor, there is a third structure map called conjugation, which is always so obviously present that it tends to be overlooked.

A precise definition of a Mackey functor is due to Green (cf. [Gr71]) and to Dress in the early seventies. Particularly Dress developed a theory of Mackey functors in the most general and abstract setup (cf. [Dr73]), which made it difficult to be applied by others not willing to follow through a dense jungle of abstractness and notation. The definition of a Mackey functor given here is only a special case of Dress’ general definition, and understandable without algebraic background. In contrast to this definition the rest of the language we introduce in Chapter I deserves again to be compared with a jungle of notation. But there is no remedy for this dilemma. Once we have seen that certain complicated structures arise naturally in a vast variety of situations the mere self-understanding of mathematics forces us to give a name to these structures and to prove theorems about them, independently from a special situation. The alternative would be to repeat more or less identical proofs in different situations, or to prove theorems in special situations by a mixture of general arguments and arguments only possible in this situation, which would hide the real reasons for the proven result, namely the presence of a general structure.

The general study of Mackey functors was continued by Yoshida in the early eighties and by Webb and Thévenaz from the late eighties to now. Other names that should be mentioned here are tom Dieck (cf. [tD87]) and Sasaki.

Let us indicate the importance of Mackey functors in two different areas. One of the most fascinating conjectures in modular representation theory is Alperin’s conjecture (cf. [Al87]). Thévenaz and Webb showed in [TW90] that this conjecture
is equivalent to the existence of a virtual Mackey functor with certain properties. In number theory Kato developed a class field theory for local fields of higher dimension. His complicated proofs were simplified considerably by Fesenko who replaced the groups Kato used by groups carrying the right additional (Mackey functor) structure, cf. [Ne92, IV.6, Exercise 20]. Note that the usual class field theory can be formulated and proved very clearly in the language of Mackey functors, cf. [Ne92, IV.6, Exercise 9 ff.].

In this thesis the language of Mackey functors will be used for two purposes. In Chapter VI we apply a theorem of Yoshida on Mackey functors to derive explicit relations for the class groups of intermediate fields of a finite Galois extension of number fields. The amazing aspect is that number theory is only used for the (almost obvious) fact that the class groups form a Mackey functor. The rest follows from the general theory of Mackey functors, the particular role the Burnside ring plays for Mackey functors, and explicit formulae for the idempotents of the Burnside ring. This should be compared with a classical result by Brauer (cf. [Bra51]) who obtains relations between class numbers (not class groups) up to a rational factor which may assume only finitely many values. Brauer used comparatively deep number theoretic arguments, as for example the theory of Artin $L$-functions and the analytic class number formula. In the meanwhile we learnt that Roggenkamp and Scott used in [RS82] a very similar approach to the one we give in Chapter VI. One of the consequences we obtain in Chapter VI is the following: Let $L/K$ be a Galois extension of number fields with finite Galois group $G$. Then the class group of $L$ is uniquely determined by the class groups of the other intermediate fields through an explicit relation which depends only on $G$, provided that $G$ is not hypo-elementary, i.e. $G/O_p(G)$ is not cyclic for any prime $p$, and provided that the reduced Euler characteristic of the poset of proper non-trivial subgroups of $G$ is not zero. This applies for example for the symmetric group on four letters and the dihedral group of order 12.

The other and principal aim of this thesis is to construct canonical induction formulae in various representation theoretic situations.

Recall that Brauer showed in [Bra47] that each character of a finite group is an integral linear combination of induced one-dimensional characters. In [Bo89] we gave a canonical version of Brauer’s induction theorem. Let us explain what this means. Note that the character rings $R(H)$ for the subgroups $H$ of a finite group $G$ form a Mackey functor with the usual conjugation, restriction, and induction maps. We constructed another Mackey functor $R^{ab}_+(H)$, $H \leq G$, which may be thought of the set of all integral linear combinations of induced one-dimensional characters, where we consider two such linear combinations as the same, if one arises from the other by replacing subgroups and one-dimensional characters with conjugate subgroups and conjugate characters. Note that the induction of such a linear combination and also the restriction (by the Mackey-axiom) results again in a certain integral linear combination of induced one-dimensional characters. Interpreting such a linear combination again as a virtual character yields a (surjective) map

$$b_G : R^{ab}_+(G) \rightarrow R(G)$$

such that the collection of maps $b_H$, $H \leq G$, is a morphism of Mackey functors,
i.e. commutes with conjugation, restriction, and induction. In [Bo89] we constructed a map

\[ a_G : R(G) \longrightarrow R_{+}^{ab}(G) \]

such that the composition

\[ R(G) \xrightarrow{a_G} R_{+}^{ab}(G) \xrightarrow{b_G} R(G) \]

is the identity map. This justifies to call \( a_G \) a Brauer induction formula. Moreover, the maps \( a_H, H \leq G \), were defined in a way that they commute with restriction and conjugation, what induced us to call them a canonical induction formula. It can be shown (cf. Remark II.1.2) that there can’t exist a collection \( a_H, H \leq G \), of sections for \( b_H, H \leq G \), which commute with conjugation, restriction, and induction. So we obtained the best we could hope for. Another canonical but non-additive Brauer induction formula on the semiring of genuine characters was given by Snaith in [Sn88]. A comparison of these two formulae can be found in [BSS92]. For a geometric interpretation of the canonical induction formula in [Bo89] given by Symonds consult [Sy91].

The notation \( R_{+}^{ab}(G) \) needs some explanation. Let \( R^{ab}(G) \) denote the subring of the character ring \( R(G) \) which is generated by one-dimensional characters. Then the groups \( R_{+}^{ab}(H), H \leq G \), do no longer form a Mackey functor, since induced one-dimensional characters don’t generally decompose into one-dimensional characters. However, the groups \( R^{ab}(H), H \leq G \), are stable under conjugation and restriction, so that they form what we call a restriction functor, a Mackey functor without induction. The construction of the groups \( R_{+}^{ab}(H), H \leq G \), from the groups \( R^{ab}(H), H \leq G \), is in fact a functor from the category of restriction functors to the category of Mackey functors. More precisely, the functor \(-_+\) is the left adjoint of the forgetful functor from the category of Mackey functors to the category of restriction functors. Hence we might call it ‘Mackeyfication’ of a restriction functor.

One can show that an arbitrary morphism \( a_H : R(H) \rightarrow R_{+}^{ab}(H), H \leq G \), of restriction functors, is uniquely determined by its residue \( \pi_H \circ a_H : R(H) \rightarrow R^{ab}(H), H \leq G \), where \( \pi_H : R_{+}^{ab}(H) \rightarrow R^{ab}(H) \) maps the linear combination of induced one-dimensional characters to the part which induces from \( H \), i.e. induces trivially. The residue of our canonical induction formula consists of the maps \( p_H : R(H) \rightarrow R^{ab}(H), H \leq G \), which are induced by taking fixed points under the commutator subgroup, i.e. \( p_H \) is the identity on one-dimensional characters and trivial on other irreducible characters. Since the maps \( p_H, H \leq G \), do not commute with restriction, but still with conjugations, we define a third category whose objects we call conjugation functors, namely restriction functors without restriction. This category is very important for the theory of Mackey functors, since there is another functor \(-_+\) from the category of conjugation functors to the category of Mackey functors, which has useful adjointness properties.

Most of Chapter I is dedicated to the definition of the three categories of Mackey functors, restriction functors, and conjugation functors, and to the functors \(-_+\) and \(-_+\) between them. There is also a variety of natural transformations involved. So, for instance, if \( A \) is a restriction functor, then there are the so-called mark homomorphisms \( \rho_H : A_+ \rightarrow A^+, H \leq G \), which generalize the mark homomorphism on the Burnside ring, cf. Example I.2.3.
In Chapter II we define the notion of a canonical induction formula for a Mackey functor $M$ from a restriction subfunctor $A \subseteq M$. The reader should think of $M$ replacing the character ring $R$ and $A$ replacing the abelianized character ring $R^{ab}$. A canonical induction formula is a morphism

$$a: M \to A_+$$

of restriction functors such that composition with a certain natural morphism

$$b^{M,A}: A_+ \to M$$

of Mackey functors is the identity. The idea to generalize the canonical Brauer induction formula was already explained in [Bo89, Chapt. III] and incorporated in the notion of an induction triple. Since then we obtained examples which do not fit into the framework of an induction triple so that we present a slightly different definition of a canonical induction formula in Section II.1. In Section II.2 we study the case where the abelian groups $M(H)$ and $A(H)$, $H \leq G$, carry the structure of a $k$-module, where $|G|$ is invertible in $k$. In most of the applications we may think of $k$ being $\mathbb{Q}$. In this situation the morphisms $a: M \to A_+$ of $k$-restriction functors are in bijective correspondence to the morphisms $p: M \to A$ of $k$-conjugation functors by associating to $a$ its residue $p = \pi^A \circ a$ in a similar way as explained for the character ring. We can relate properties of $a$ to those of $p$. In Section II.3 we restrict the situation further to the case where $A(H)$, $H \leq G$, has a $\mathbb{Z}$-basis which is stable under conjugations, and whose positive span is stable under restriction. This is the case in all examples we will consider. In this standard situation also $A_+(H)$, $H \leq G$, is free and we may embed the whole setup injectively into the $\mathbb{Q}$-tensored version and apply the results of Section II.2. Now the main difficulty arises: The morphism

$$a^{M,A,p}: \mathbb{Q} \otimes M \to \mathbb{Q} \otimes A_+$$

of restriction functors corresponding to $p: M \to A$ is in general not integral, i.e. $M(H)$ is in general not mapped to $A_+(H)$ but only to $\mathbb{Q} \otimes A_+(H)$ for $H \leq G$. For a long time we had no criterion to decide for given $p$ whether $a^{M,A,p}$ is integral, i.e. $a^{M,A,p}(M) \subseteq A_+$. In Section II.4 we state a condition $(*)_{\pi}$ for $p$ which implies that $|H|_{\pi'} \cdot a^{M,A,p}_H(M(H)) \subseteq A_+(H)$ for $H \leq G$, where $\pi$ is any set of primes (see Theorem II.4.5). In particular, if $p$ satisfies condition $(*)_{\pi}$ for the set $\pi$ of all primes, then $a^{M,A,p}$ is integral.

The condition $(*)_{\pi}$ is applied in Chapter III, where we give various examples for $M$, $A$, and $p$. More precisely we consider the situations where $M$ is the character ring, the Brauer character ring, and the group of $p$-projective characters or $\pi$-projective characters for a set $\pi$ of primes. In these situations there is not very much to do except referring to the general theory developed in Chapter II, and integral canonical induction formulae drop like ripe fruit in Sections III.1–III.3. In Section III.4 we consider the Mackey functor of trivial source modules and linear source modules over a complete discrete valuation ring $\mathcal{O}$. We show that there is an integral canonical induction formula in both cases which induces only modules of rank one. Here the proof of the integrality is divided into two steps. In a first step we show that condition $(*)_{\pi}$ holds for the set $\pi$ of primes different from the residue
characteristic $p$ of $O$. In the second step we define a different canonical induction formula whose residue satisfies condition $(\ast_p)$ and we show that the first formula is the composition of the second one with an integral morphism. Having first only considered the trivial source ring after a suggestion of Külshammer, we realized that almost all proofs work without any changes also for the linear source ring, a ring which hasn’t been studied as extensively in the literature as the trivial source ring, although it shares many nice properties. For example it is semisimple after tensoring with $\mathbb{C}$ so that there is a kind of character theory for it. Moreover tensoring with the quotient field and the residue field of $O$ yields a surjection onto the character ring and the Brauer character ring. In Section II.5 we consider the Green ring as a candidate for $M$. But here we have to leave things in a most unsatisfactory state. In particular we have no integral canonical induction formula except one where we allow to induce all modules from all solvable subgroups. Section III.6 gives a summary of the different induction formulae and the relations between them. Finally we calculate in Section III.7 the canonical Brauer induction formula for the case of an extraspecial $p$-group of odd exponent $p$.

Chapter IV is devoted to applications of canonical induction formulae. Assume that we have two Mackey functors $M$ and $N$ together with restriction subfunctors $A \subseteq M$ and $B \subseteq N$. Assume further that we have a canonical induction formula $a: M \rightarrow A_+$ for the first pair. In Section IV.1 we show that each morphism $f: A \rightarrow B$ of restriction functors can be extended to a morphism $F: M \rightarrow N$ of restriction functors

$$
\begin{array}{ccc}
M & \xrightarrow{F} & N \\
\cup & \xrightarrow{f} & \cup \\
A & \xrightarrow{f} & B,
\end{array}
$$

and we give conditions on $M$, $A$, and $N$ which imply that $F$ is unique. For instance, in the case where $M = N = R$ is the character ring Mackey functor and $A = B = R^{ab}$ is the abelianized character ring restriction functor, it is clear that $F$ extends uniquely. In general $F$ is given by the composition

$$
M \xrightarrow{a} A_+ \xrightarrow{f_+} B_+ \xrightarrow{b_{N,B}} N,
$$

in other words, if

$$
\chi = \sum_{H \leq G} \text{ind}_H^G(a_H)
$$

describes the canonical induction formula for $\chi \in M(G)$ with $a_H \in A(H)$ for $H \leq G$, then $F_G(\chi)$ is defined by

$$
F(\chi) = \sum_{H \leq G} \text{ind}_H^G(f_H(a_H)).
$$

This is surprising, since it means that

$$
\sum_{H \leq G} \text{ind}_H^G(F_H(a_H)) = \sum_{H \leq G} F_G(\text{ind}_H^G(a_H))
$$

even if $F$ does not commute with induction.
In Sections IV.2–IV.4 we apply this extension theorem in the cases where $f$ is the determinant, the Adams operation, and the total Chern class, and where $M$ is the character ring, the Brauer character ring, the $(\pi)$-projective character group, the linear source ring and the trivial source ring Mackey functor. Note that in every combined case for $M$ and $f$, it is very easy to define $f$ on $A$, but it is not always clear how to extend $f$ to $M$. In particular, we don’t know about Chern classes defined on the Brauer character ring, the projective character group, the trivial source ring or the linear source ring. Note that for these applications it is crucial to have an integral canonical induction formula, since $N$ may be a torsion group, as for example in the cases of cohomology groups.

A speculation should be allowed here. It is straightforward to extend everything we mentioned so far to profinite groups $G$, if we only consider open subgroups. Fix a prime $p$ and consider the absolute Galois group $G_{\mathbb{Q}_p}$ of the the $p$-adic completion $\mathbb{Q}_p$ of $\mathbb{Q}$. For a finite extension $K/\mathbb{Q}_p$ we define $M(G_K) = R(G_K)$ as the direct limit of all $R(G_K/G_L)$, where $L$ runs over all finite Galois extensions of $K$, and we define $A(G_K) = R^{\text{ab}}(G_K)$ as the direct limit of all $R^{\text{ab}}(G_K/G_L)$, both limits taken with respect to the inflation maps. This defines a Mackey functor $M$ and a restriction subfunctor $A \subseteq M$. Moreover, for a finite extension $K/\mathbb{Q}_p$, we define $B(G_K)$ as the free abelian group on the set of homomorphisms from $K^\times$ to $\mathbb{C}^\times$ with kernel of finite index. Then $B$ is a restriction functor with restrictions given by composition with the norm maps. Local class field theory establishes an isomorphism $f : A \to B$ of restriction functors. One special case of the Langlands conjecture predicts a ‘canonical’ isomorphism between $M(G_K)$ and certain representations of the various groups $GL_n(K)$ for each finite extension $K$ of $\mathbb{Q}_p$. Unfortunately it is not clear how these representations give rise to a Mackey functor structure. At this state the extension theorem of Section IV.1 just waits for someone to define the right Mackey functor $N$ in order to give a canonical Langlands correspondence between $M$ and $N$.

Let $G$ and $S$ be finite groups of coprime order and assume that $S$ acts on $G$ so that we can form the semidirect product $GS$. In Section IV.5 we use the canonical induction formula in the various situations of Chapter III in order to define maps which associate to an $S$-fixed $G$-character or $G$-module a virtual $GS$-character or $GS$-module which extends the original character or module. Note that at least in the situations for linear source rings and trivial source rings this seems to give new results. Following a suggestion of Puig we define in Section IV.6 a map $gl^{G,S}$ from the $S$-fixed points $R(G)^S$ of the character ring $R(G)$ to the character ring $R(G^S)$ of the $S$-fixed points of $G$. We show that if $S$ is a $p$-group, then the map $gl^{G,S}$ is related to the Glauberman correspondence (cf. [Gla68]). It would be very interesting to see if this map could give a direct definition for the Glauberman correspondence without going through a composition series of $S$ in the case of general solvable $S$, which is necessary if we use the descriptions which exist at the moment. Moreover, it might be that the map $gl^{G,S}$ gives a unified definition of the Glauberman correspondence, which is defined for solvable $S$, and a correspondence of Isaacs, cf. [Is73], which is defined for odd (and hence solvable) $G$. Moreover, the definition of $gl^{G,S}$ is also possible for the other Grothendieck groups we considered in Chapter III.

for a character $\chi \in R(G)$ we define the $p$-projectification $pr(\chi)$ as the character $\chi$ multiplied with the characteristic function of the set of $p'$-elements of $G$. In
Section IV.7 we construct a canonical induction formula for \( pr(\chi) \) which is a \( \mathbb{Q} \)-linear combination of induced one-dimensional characters on \( p' \)-groups with the same identification we explained at the beginning for \( R^\text{ab}_+(G) \). For an irreducible character \( \chi \) of \( G \) the rational coefficients of the corresponding linear combination have \( p \)-power denominators, and the biggest occurring denominator is precisely \( p^{d(\chi)} \) where \( d(\chi) \) is the \( p \)-defect of \( \chi \), i.e. it satisfies \( |G|_p = p^{d(\chi)} \chi(1) \).

The Cartan map \( c_G \) is an injective map from the \( p \)-projective character group to the group of Brauer characters of \( G \), which becomes an isomorphism after tensoring with \( \mathbb{Q} \). For a Brauer character \( \chi \) we define a canonical induction formula for \( c_G^{-1}(\chi) \) which is a \( \mathbb{Q} \)-linear combination of induced one-dimensional Brauer characters on \( p' \)-subgroups. The residue of this induction formula satisfies condition \((*)_{p'} \) of Theorem II.4.5 so that the rational coefficients in the above linear combination have only \( p \)-power denominators. This implies that the cokernel of \( c_G \) is a finite \( p \)-group.

The projective resolution of a module is a familiar construction and leads to many interesting invariants. In the case of the category \( kG-\text{mod} \), of modules over a group algebra \( kG \), there is a variety of other interesting classes besides the projective modules and one may also ask whether there is a reasonable theory of a resolution in terms of such modules. Brauer’s famous induction theorem emphasizes the significance of one-dimensional modules for the representation theory of finite groups over the field \( \mathbb{C} \) of complex numbers. Also in number theory, one-dimensional Galois representations have invariants (L-functions, conductors, etc.) which are much better understood (by using class field theory) than the general case. Since these number theoretical invariants behave well under induction and taking direct sums, it is natural to consider the class of monomial modules, i.e. modules arising as sums of induced one-dimensional modules.

Since Brauer’s induction theorem states that (over \( \mathbb{C} \)) each module is a direct summand of a monomial module with a monomial complement, we can’t expect any interesting theory of resolutions by monomial modules in the classical sense. Therefore we consider in Chapter V the category \( kG-\text{mon} \) of \( kG \)-monomial modules which consists of monomial \( kG \)-modules with the additional structure of a decomposition into submodules satisfying certain properties with respect to the group action. Here \( k \) may be any noetherian integral domain. A slightly different variant of this category has been considered in [Re71] and [Bo90] for \( k = \mathbb{C} \) resp. \( k \) a field.

We construct for each finitely generated \( kG \)-module a chain complex in \( kG-\text{mon} \) with certain natural properties, which form the axioms of a \( kG \)-monomial resolution. The notion of a \( kG \)-monomial resolution has all the nice properties we know from projective resolutions: It is unique up to homotopy equivalence and morphisms in \( kG-\text{mod} \) extend to morphisms between the \( kG \)-monomial resolutions which are unique up to homotopy. Hence, the \( kG \)-monomial resolution may be considered as an embedding of \( kG-\text{mod} \) into the homotopy category of chain complexes in \( kG-\text{mon} \). If we view the objects in a \( kG \)-monomial resolution of some \( V \in kG-\text{mod} \) only as \( kG \)-modules, then we obtain a chain complex of monomial modules whose homology is concentrated in degree zero and is isomorphic to \( V \).

Interestingly the \( kG \)-monomial resolution yields non-trivial invariants even for \( k = \mathbb{C} \). We show that each \( \mathbb{C}G \)-module \( V \) has a finite \( \mathbb{C}G \)-monomial resolution and that the Lefschetz invariant of this resolution as element in the Grothendieck group
of \( \mathbb{C}G\)-mon (which coincides with the ring \( R_{\mathbb{C}}^\text{ab}(G) \) introduced in Section III.1) is just the canonical Brauer induction formula of Section III.1 applied to the character of \( V \). Hence, the functor of taking \( \mathbb{C}G \)-monomial resolutions can be regarded as a lift of the canonical Brauer induction formula on the level of Grothendieck groups to a functor on the level of the underlying categories (cf. Remark V.2.17). The minimal length of a \( \mathbb{C}G \)-monomial resolution of a \( \mathbb{C}G \)-module \( V \) is studied, and we give upper bounds in terms of the rank of a finite poset associated to \( V \). Using again properties of the canonical Brauer induction formula it is shown that \( V \) has a \( \mathbb{C}G \)-monomial resolution of length zero if and only if \( V \) is a direct sum of one-dimensional modules.

Chapter V is arranged as follows. In Section 1 we introduce the category \( kG\text{-mon} \) of \( kG \)-monomial modules and study some basic properties. There is a full classification of isomorphism classes of objects and a description of the Grothendieck group in the case where \( k \) is local or a principal ideal domain. In Section 2 we define the notion of a \( kG \)-monomial resolution and prove the existence, uniqueness and other properties mentioned above. Section 3 gives an interpretation of the monomial resolution as a projective resolution in a bigger category \( \mathcal{C} \) by embedding both \( kG\text{-mod} \) and \( kG\text{-mon} \) into \( \mathcal{C} \) such that the objects of \( kG\text{-mon} \) become projective in \( \mathcal{C} \). We also describe \( \mathcal{C} \) (by constructing equivalences of categories) as a module category and a category of sheaves on a poset.

It happens that the homotopy category of \( kG\text{-mon} \) shares many interesting properties with the derived category of \( kG\text{-mod} \), which in turn plays an important role in connection with recent conjectures in modular representation theory. Moreover, it seems that this category takes much more into account that we consider modules over group rings not just over arbitrary rings. But unfortunately we have no block decomposition for \( kG\text{-mon} \).

In Appendix A we collect the basic properties of the Burnside ring and fix the notation used in connection with the Burnside ring. Appendix B provides the facts about posets and Möbius functions which we need in this thesis.

References to items of other chapters are by triples. Theorem II.4.5 means the theorem with the number 4.5 in Chapter II. Within Chapter II we would only refer to Theorem 4.5. Equation are numbered consecutively in each chapter. Equation (3.1) means the first equation in Chapter III. There is no indication of the section which contains the equation.

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Notation

We assume the following notations throughout all chapters.

Numbers

- **N**: positive integers
- **N₀**: non-negative integers
- **Z**: integers
- **Q**: rational numbers
- **R**: real numbers
- **C**: complex numbers
- **F_p**: finite field with \( p \) elements
- **Z_p**: \( p \)-adic completion of **Z** at a prime \( p \)
- **Q_p**: \( p \)-adic completion of **Q** at a prime \( p \)

Groups

- **G**: a group
- **H ≤ G**: subgroup
- **H < G**: proper subgroup
- **H ≤ G**: normal subgroup
- **H ∯ G**: proper normal subgroup
- **gH**: \( gHg^{-1} \)
- \((G : H)\): index of a subgroup \( H \leq G \)
- **H^g**: \( g^{-1}Hg \)
- **G_π**: set of \( π \)-elements of **G** for a set \( π \) of primes
- **|G|**: order of **G**
- **G^{ab}**: commutator factor group
- **O_p(G)**: biggest normal \( p \)-subgroup for a prime \( p \)
- **K ≤ G H**: \( K \leq gKg^{-1} \) for some \( g \in G \) for \( K, H \leq G \)
- **K = G H**: \( K = gHg^{-1} \) for some \( g \in G \), for \( K, H \leq G \)
- **s = G t**: \( s = gt \) for some \( g \in G \), for elements \( s, t \) of a **G**-set
- **stab_G(s)**: the stabilizer of an element \( s \) in a **G**-set
- **R(G)**: character ring of **G**
- **Irr(G)**: set of irreducible characters of **G**

Categories

- Assume that **A** is a ring and **G** is a finite group.
- **A–Mod**: category of left **A**-modules
- **A–mod**: category of finitely generated left **A**-modules
- **G–set**: category of finite left **G**-sets
- **ab**: category of finite abelian groups
Rings  Let $k$ be a commutative ring and let $G$ be a finite group.

- $kG$ or $k[G]$ group ring
- $k^\times$ multiplicative group of units of $k$
- $G(k)$ $\text{Hom}(G, k^\times)$
- $\text{rk}_k V$ $k$-rank of a $k$-free $k$-module $V$
- $\text{dim}_k V$ $k$-dimension of a $k$-vector space $V$, if $k$ is a field

Miscellaneous

- $f|_B : B \to C$ restriction of a map $f : A \to C$ to a subset $B \subseteq A$
- $n_\pi$ $\pi$-part of $n \in \mathbb{N}$ for a set $\pi$ of primes
- $\#X$ or $|X|$ cardinality of a finite set $X$
- $M \mid N$ $M$ is isomorphic to a direct summand of a module $N$
Chapter 1

The Language:
Mackey Functors,
Restriction Functors, and
Conjugation Functors

Throughout this chapter let $G$ be a finite group and let $k$ be a commutative ring with unity. Unadorned tensor products will be taken over the ring $\mathbb{Z}$ of integers.

This chapter is supposed to serve as a dictionary for the language we use in later chapters. From a logical point of view it can be read from Section 1 to Section 6 continuously, but it seems more advisable to start out reading Chapter II and skip back whenever it is necessary to become acquainted with another part of the language.

In Section 1 we introduce the notions of $k$-Mackey functors, $k$-restriction functors, and $k$-conjugation functors on $G$. Each of these notions has less structure than its predecessor and consequently there are forgetful functors from each of the resulting categories to the next one. If $G$ is the trivial group, the three categories are equivalent to the category of $k$-modules. Each of the three categories has a $k$-algebra variant, namely $k$-Green functors, $k$-algebra restriction functors, and $k$-algebra conjugation functors on $G$, and if $G$ is the trivial group, they all coincied with the category of $k$-algebras. For the sake of consistency we might call a Green functor also an algebra Mackey functor, but for historical reasons we stick to the non-consistent terminology. The standard example of a $k$-Green functor on $G$ consists of the Burnside rings of all subgroups of $G$ tensored with $k$. Other examples are provided by character rings, cohomology groups, ideal class groups, etc.

Mackey functors were introduced by Dress in [Dr73] and Green in [Gr71]. Some papers of Thévenaz, Webb an Yoshida, who studied Mackey functors further, should also be mentioned here: [Th88], [TW91], [We91], [Yo80], [Yo83b]. Some of these authors prefer the name ‘$G$-functor’ instead of ‘Green functor’ or ‘Mackey functor’. The category of restriction functors was introduced in [Bo89, III.1] where they were called ‘$k$-sheaves’. We introduce the notion of conjugation functors here, since many Mackey functors (even all of them, if the order of $G$ is invertible in $k$) arise from conjugation functors by the functor $-^+$, which is introduced in Section 2. Also, the
construction of the twin functor (see [Th88] or 2.8) factorizes through the category of conjugation functors. The category of conjugation functors is indispensable for the analysis of certain morphisms of restriction functors via their residues in Section 4.

In Section 2 we define various functors between the categories defined in Section 1. The most important ones of them for the construction of canonical induction formulae are the functors $-^+$ and $-_$, The Burnside ring functor arises by the functor $-^+$, and the mark homomorphism on the Burnside ring generalizes to a natural transformation $\rho^A: A_+ \to A^+$ for a $k$-restriction functor $A$ which is studied in Section 3. The mark morphism $\rho^A$ is injective in many cases (in fact in all cases that we consider in Chapter II) and it is an isomorphism if the order of $G$ is invertible in $k$ (cf. Proposition 3.2). Since we can give an explicit inverse for $\rho^A$ in this case, we obtain explicit descriptions for the canonical induction formulae considered in Chapter II.

In Section 4 we prove some adjointness properties of the functors $-^+$ and $-_-$. The functor $-_+$ itself and the functor $-^+$ followed by a forgetful functor are adjoint functors to the two forgetful functors between the categories considered in Section 1 (cf. Proposition 4.1). In Chapter II a canonical induction formula will be a morphism $B \to A_+$ of restriction functors for certain restriction functors $A$ and $B$. The adjointness properties and the mark morphism allow us to set up a bijection between such morphisms and morphisms $B \to A$ of the underlying conjugation functors, provided that the order of $G$ is invertible in $k$ (cf. Corollary 4.2). In more general cases we still obtain an injective map.

In order to place ourselves in this situation we need to know how the functors $-^+$ and $-_-$ behave under scalar extension. This is examined in Section 5. Finally, Section 6 summarizes miscellaneous results in the case, where the order of $G$ is invertible in $k$. In this case the categories of Mackey functors and conjugation functors are equivalent. Moreover there is a ‘split’ embedding of the category of conjugation functors into the category of restriction functors in general, which induces an isomorphism between the Grothendieck groups of these two categories under certain finiteness conditions.

1.1 The categories

The following definition of a Mackey functor is the most convenient one, if one wants to verify that a certain object is a Mackey functor. There are equivalent and more general definitions which explain that a Mackey functor is indeed a functor or also a bifunctor. There is also an $k$-algebra (depending on $G$) whose module category is equivalent to the category of $k$-Mackey functors on $G$. See [Dr73] and [TW91, Sections 2 and 3] for such definitions.

1.1 Definition ([Dr73, Gr71]) A $k$-Mackey functor on $G$ is a quadruple $(M, c, \text{res}, \text{ind})$ consisting of

- a family of $k$-modules $M(H)$, one for each $H \leq G$,
- a family of $k$-module homomorphisms $c_{g,H}: M(H) \to M(\langle g \rangle H)$, one for each $g \in G$ and $H \leq G$, called conjugation maps,
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- a family of \( k \)-module homomorphisms \( \text{res}^H_K : M(H) \to M(K) \), one for each pair \( K \leq H \) of subgroups of \( G \), called restriction maps, and

- a family of \( k \)-module homomorphisms \( \text{ind}^H_K : M(K) \to M(H) \), one for each pair \( K \leq H \) of subgroups of \( G \), called induction maps,

which satisfy the following axioms:

(M1) (Triviality) \( c_{h,H} = \text{res}^H_H = \text{ind}^H_H = \text{id}_{M(H)} \) for \( H \leq G, \ h \in H \).

(M2) (Transitivity) \( \begin{align*}
  c_{g'g,H} &= c_{g',g_H} \circ c_{g,H} \\
  \text{res}^L_H &= \text{res}^K_L \circ \text{res}^H_K \\
  \text{ind}^L_H &= \text{ind}^K_L \circ \text{ind}^H_K
\end{align*} \) for \( L \leq K \leq H \leq G, \ g, g' \in G \).

(M3) (\( G \)-equivariance of restriction and induction) \( \begin{align*}
  c_{g,K} \circ \text{res}^H_K &= \text{res}^{g_H}_K \circ c_{g,H} \\
  c_{g,H} \circ \text{ind}^H_K &= \text{ind}^{g_H}_K \circ c_{g,K}
\end{align*} \) for \( K \leq H \leq G \).

(M4) (Mackey axiom) \( \begin{align*}
  \text{res}^U_H \circ \text{ind}^H_K &= \sum_{h \in U \setminus H/K} \text{ind}^U_{U \cap K} \circ \text{res}^K_{U \cap K} \circ c_{h,K}
\end{align*} \) for \( U, K \leq H \leq G \).

We will often write \( M \) instead of \((M,c,\text{res},\text{ind})\) when there is no need for specifying conjugation, restriction and induction, and for \( H \leq G, \ m \in M(H), \ g \in G \) we will abbreviate \( c_{g,H}(m) \) by \( gm \) when there is no need for specifying \( H \).

A morphism \( f : M \to N \) between two \( k \)-Mackey functors \( M \) and \( N \) on \( G \) is a family \( f = (f_H : M(H) \to N(H))_{H \leq G} \) of \( k \)-module homomorphisms which commute with conjugation, restriction and induction. The class of \( k \)-Mackey functors on \( G \) together with their morphisms form a category \( k-\text{Mack}(G) \).

A \( k \)-Mackey functor \( M \) on \( G \) is called cohomological, if the following axiom holds:

(M5) (Cohomologicality axiom) \( \begin{align*}
  \text{ind}^H_K \circ \text{res}^H_K &= (H : K) \cdot \text{id}_{M(H)}
\end{align*} \) for \( K \leq H \leq G \).

The cohomological \( k \)-Mackey functors on \( G \) form a full subcategory \( k-\text{Mack}^c(G) \) of \( k-\text{Mack}(G) \).

A \( k \)-Green functor on \( G \) is a \( k \)-Mackey functor \( M \) on \( G \) together with a unitary \( k \)-algebra structure on each \( M(H), \ H \leq G \), such that conjugations and restrictions are unitary \( k \)-algebra homomorphisms and the following axiom is satisfied:
A morphism \( f : M \to N \) between \( k \)-Green functors \( M, N \) on \( G \) is a morphism of the underlying Mackey functors such that each \( f_H, \ H \leq G \), is a unitary \( k \)-algebra homomorphism. The category of \( k \)-morphic \( \text{functors} \) such that each \( k \leq H \leq G \) by \( k \)-\textit{Mackey} functors. The category of \( k \)-Green functors on \( G \) will be denoted by \( k-\text{Mack}_{\text{alg}}(G) \) and the full subcategory of cohomological \( k \)-Green functors on \( G \) by \( k-\text{Mack}^c_{\text{alg}}(G) \).

\[ (M6) \text{ (Frobenius axiom)} \]

\[
m \cdot \text{ind}_H^K(n) = \text{ind}_H^K(\text{res}_K^H(m) \cdot n) \]

\[
\text{ind}_H^K(n) \cdot m = \text{ind}_H^K(n \cdot \text{res}_K^H(m))
\]

for all \( K \leq H \leq G, \ m \in M(H), \ n \in M(K) \).

\textbf{1.2 Remark} \quad (i) If \( M \in k-\text{Mack}(G) \) (resp. \( M \in k-\text{Mack}_{\text{alg}}(G) \)), and \( N(H) \subseteq M(H) \) is a \( k \)-submodule (resp. ideal) for each \( H \leq G \) such that the family \( (N(H))_{H \leq G} \) is stable under conjugation, restriction and induction, then \( N \) is called a \textit{subfunctor} (resp. \textit{ideal}) of \( M \) and we write \( N \subseteq M \) (resp. \( N \leq M \)). In this case, the \textit{quotient functor} \( M/N \), with \( (M/N)(H) := M(H)/N(H) \) for \( H \leq G \), and conjugation, restriction and induction inherited from \( M \), is again a \( k \)-Mackey functor (resp. \( k \)-Green functor) on \( G \). By a \textit{subfunctor} \( N \subseteq M \) of a \( k \)-Green functor on \( G \) we mean a family \( N(H) \subseteq M(H), \ H \leq G \), of unitary \( k \)-subalgebras which are stable under conjugation, restriction and induction. It is clear that subfunctors and quotient functors of cohomological Mackey functors and cohomological Green functors are again cohomological.

(ii) Let \( M, N \in k-\text{Mack}(G) \) and \( f \in k-\text{Mack}(G)(M, N) \). The constructions \( (M \oplus N)(H) := M(H) \oplus N(H), \ker(f)(H) := \ker(f_H) \subseteq M(H), \text{im}(f)(H) := \text{im}(f_H) \subseteq N(H) \) and \( \text{coker}(f)(H) := \text{coker}(f_H) = N(H)/\text{im}(f_H), \) for \( H \leq G \), which we call \textit{direct sum} of \( M \) and \( N \), \textit{kernel} of \( f \), \textit{image} of \( f \) and \textit{cokernel} of \( f \), inherit the structure of a \( k \)-Mackey functor on \( G \). The direct sum \( M \oplus N \) is both a categorical product and coproduct of \( M \) and \( N \), and the categories \( k-\text{Mack}(G) \) and \( k-\text{Mack}^c(G) \) are abelian. Moreover, \( k-\text{Mack}(G)(M, N) \) is a \( k \)-module and composition of morphisms is \( k \)-bilinear. In case that \( M, N \in k-\text{Mack}_{\text{alg}}(G) \) and \( f \in k-\text{Mack}_{\text{alg}}(G)(M, N) \), \( \ker(f) \) is an ideal in \( M \), and \( M/\ker(f) \) is again a \( k \)-Green functor on \( G \).

(iii) If \( k \) is a subring of the commutative ring \( k' \) (with the same unity element), then the functor \( k' \otimes_k - : k-\text{Mod} \to k'-\text{Mod} \) of scalar extensions of \( k \)-modules induces functors

\[
k' \otimes_k - : k-\text{Mack}(G) \to k'-\text{Mack}(G)
\]

and

\[
k' \otimes_k - : k-\text{Mack}_{\text{alg}}(G) \to k'-\text{Mack}_{\text{alg}}(G),
\]

which map cohomological \( k \)-Mackey functors (resp. \( k \)-Green functors) to cohomological \( k' \)-Mackey functors (resp. \( k' \)-Green functors).

\textbf{1.3 Definition} ([Dr73, Yo80, Th88]) \quad (i) Let \( X, Y, Z \in k-\text{Mack}(G) \). A \textit{pairing} \( X \otimes Y \to Z \) is a family of \( k \)-module homomorphisms

\[
X(H) \otimes_k Y(H) \to Z(H), \quad x \otimes_k y \mapsto x \cdot y, \quad H \leq G,
\]

such that the following axioms hold (cf. [Th88, Yo83a]):
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(P1) (Compatibility with conjugation)

\[ g(x \cdot y) = g_gx \cdot gy \]

for \( H \leq G \), \( x \in X(H) \), \( y \in Y(H) \), \( g \in G \).

(P2) (Compatibility with restriction)

\[ \text{res}^H_K(x \cdot y) = \text{res}^H_K(x) \cdot \text{res}^H_K(y) \]

for \( K \leq H \leq G \), \( x \in X(H) \), \( y \in Y(H) \).

(P3) (Frobenius axioms)

\[ (\text{ind}^H_K(x)) \cdot y = \text{ind}^H_K(x \cdot \text{res}^H_K(y)) \]
\[ x' \cdot \text{ind}^H_K(y') = \text{ind}^H_K(\text{res}^H_K(x') \cdot y') \]

for \( K \leq H \leq G \), \( x \in X(K) \), \( y \in Y(H) \), \( x' \in X(H) \), \( y' \in Y(K) \).

We can define a \( k \)-Green functor on \( G \) alternatively as a \( k \)-Mackey functor \( M \) on \( G \) together with a pairing \( M \otimes_k M \to M \) which furnishes each \( M(H) \), \( H \leq G \) with the structure of a unitary \( k \)-algebra such that conjugations and restrictions preserve the unity elements.

(ii) Let \( A \in k-\text{Mack}_\text{alg}(G) \). A \( k \)-Mackey functor \( M \) on \( G \) together with a pairing \( A \otimes_k M \to M \) is called a left \( A \)-module, if the pairing furnishes each \( M(H) \), \( H \leq G \), with the structure of a left \( A(H) \)-module. Let \( M \) and \( N \) be two left \( A \)-modules. An \( A \)-module homomorphism \( M \to N \) is an element \( f \in k-\text{Mack}(G)(M,N) \) with the property that each \( f_H : M(H) \to N(H) \), \( H \leq G \), is an \( A(H) \)-module homomorphism. The category of left \( A \)-modules will be denoted by \( A-\text{Mod} \).

(iii) Let \( A \in k-\text{Mack}_\text{alg}(G) \) such that \( A(H) \) is commutative for all \( H \leq G \). A \( k \)-Green functor \( B \) on \( G \) together with a morphism \( i \in k-\text{Mack}_\text{alg}(G)(A,B) \) is called an \( A \)-algebra, if each \( i_H : A(H) \to B(H) \), \( H \leq G \) furnishes \( B(H) \) with the structure of an \( A(H) \)-algebra, i.e. \( i_H(A(H)) \) is contained in the center of \( B(H) \). If \( i \) : \( A \to B \) and \( j \) : \( A \to C \) define two \( A \)-algebras, then an \( A \)-algebra homomorphism between \( B \) and \( C \) is an element \( f \in k-\text{Mack}_\text{alg}(G)(B,C) \) with \( f \circ i = j \), i.e. each \( f_H : B(H) \to C(H) \), \( H \leq G \), is a unitary \( A(H) \)-algebra homomorphism. The category of \( A \)-algebras will be denoted by \( A-\text{Alg} \).

The standard example of a \( k \)-Green functor on \( G \) is the Burnside ring functor \( k \otimes \Omega \) (see Appendix A for a definition and the basic properties). It is easily checked that the commutative \( k \)-algebras \( k \otimes \Omega(H) \), \( H \leq G \), with conjugations, restrictions, inductions and multiplications defined as in Appendix A satisfy the axioms (M1)–(M4) and (M6). The Burnside ring functor plays a distinguished role for the categories \( k-\text{Mack}(G) \) and \( k-\text{Mack}_\text{alg}(G) \), namely the role that \( \mathbb{Z} \) plays for the category of abelian groups and the category of rings, as the next proposition shows.

1.4 Proposition (cf. [Dr73, Prop. 4.2], [Th88, Prop. 6.1], [Yo80, Ex. 2.11])

(i) Every \( k \)-Mackey functor \( M \) on \( G \) is in a unique way a \( k \otimes \Omega \)-module, namely by the pairing defined by

\[ (k \otimes \Omega(H)) \otimes_k M(H) \to M(H), \quad [H/K] \otimes_k m \mapsto \text{ind}^H_K(\text{res}^H_K(m)), \]
for $H \leq G$ and $K \leq H$. This pairing will be referred to as the \textbf{canonical pairing}. Moreover, each morphism of $k$-Mackey functors is a morphism of $k \otimes \Omega$-modules.

(ii) For every $k$-Green functor $A$ on $G$ there is a unique morphism $i : k \otimes \Omega \to A$ of $k$-Green functors on $G$, namely

$$i_H : k \otimes \Omega(H) \to A(H), \quad [H/K] \mapsto \text{ind}_K^H(1_{A(K)}),$$

for $H \leq G$ and $K \leq H$, i.e. $k \otimes \Omega$ is an initial object in the category $k\text{-Mack}_{\text{alg}}(G)$. This morphism provides a unique $k \otimes \Omega$-algebra structure and the unique $k \otimes \Omega$-module structure on $A$. Moreover, each morphism of $k$-Green functors is a morphism of $k \otimes \Omega$-algebras.

\textbf{Proof} Both parts can be proved by straightforward verifications. \hfill \Box

An example of a cohomological $k$-Mackey functor on $G$ is provided by the cohomology groups $H^n(H,V)$, $H \leq G$, for a fixed $kG$-module $V$ and a fixed non-negative integer $n \in \mathbb{N}_0$. Conjugation and restriction maps are the obvious ones, and induction maps are defined as the corestriction maps of cohomology.

When we will work on induction formulae in Chapter II, our standard example will be the character ring Green functor $R$ on $G$. There the subrings generated by all one-dimensional characters will be of importance. These rings, however, do not form a subfunctor, since stability under induction fails. But still we have stability under conjugation and restriction. This leads to the following definition of a restriction functor, a ‘Mackey functor without induction’.

\begin{itemize}
    \item \textbf{1.5 Definition (cf. [Bo89, III.2])} A \textbf{$k$-restriction functor} on $G$ is a triple $(A,c,\text{res})$ consisting of a family $A(H)$, $H \leq G$, of $k$-modules and $k$-module homomorphisms $c_{g,H} : A(H) \to A(g^H)$ (the \textbf{conjugations}) and $\text{res}_L^H : A(H) \to A(K)$ (the \textbf{restrictions}) for $K \leq H \leq G$, $g, g' \in G$, satisfying the following axioms:
    \begin{enumerate}
        \item \textbf{(R1) (Triviality)} $c_{e,H} = \text{res}_H^H = \text{id}_{A(H)}$ for $H \leq G$.
        \item \textbf{(R2) (Transitivity)} $c_{g'g,H} = c_{g',g^H} \circ c_{g,H}$
            $\text{res}_L^H = \text{res}_L^K \circ \text{res}_K^H$
            for $L \leq K \leq H \leq G$, $g, g' \in G$.
        \item \textbf{(R3) (G-equivariance of restriction)} $c_{g,K} \circ \text{res}_K^H = \text{res}_K^g \circ c_{g,H}$
            for $K \leq H \leq G$, $g \in G$.
    \end{enumerate}
\end{itemize}

As in the case of Mackey functors we often write just $A$ instead of $(A,c,\text{res})$ and also just $%$ instead of $c_{g,H}(a)$ for $g \in G$, $H \leq G$, $a \in A(H)$. A \textbf{morphism} $f : A \to B$ between $k$-restriction functors $A, B$ on $G$ is a family $(f_H : A(H) \to B(H))_{H \leq G}$ of $k$-module homomorphisms which commute with conjugations and restrictions. We denote the category of $k$-restriction functors on $G$ by $k\text{-Res}(G)$.
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An \( k \)-algebra restriction functor on \( G \) is a \( k \)-restriction functor \( A \) on \( G \) together with the structure of a unitary \( k \)-algebra on each \( A(H) \), \( H \leq G \), such that conjugations and restrictions are unitary \( k \)-algebra homomorphisms. A morphism between two \( k \)-algebra restriction functors on \( G \) is a morphism of the underlying \( k \)-restriction functors on \( G \) which consists of unitary \( k \)-algebra homomorphisms. The category of \( k \)-algebra restriction functors on \( G \) is denoted by \( k \text{-Res}_{\text{alg}}(G) \).

1.6 Remark Analogous to Remark 1.2 we may speak of subfunctors of restriction functors, subfunctors and ideals of algebra restriction functors, quotient functors and kernels, images and cokernels of morphisms of (algebra) restriction functors, subfunctors and ideals of algebra restriction functors, quotient functors, conjugations and restrictions are unitary \( k \)-algebra homomorphisms. This leads to the notion of a \( k \)-module such that composition is \( k \)-bilinear. If \( k \) is a subring of the commutative ring \( k' \), we have scalar extension functors

\[
k' \otimes_k - : k \text{-Res}(G) \to k' \text{-Res}(G), \quad k' \otimes_k - : k \text{-Res}_{\text{alg}}(G) \to k' \text{-Res}_{\text{alg}}(G).
\]

In our standard example, the character rings \( R(H) \), \( H \leq G \), the spans of one-dimensional characters form a subfunctor \( R^{ab} \) of the underlying \( \mathbb{Z} \)-algebra restriction functor of \( R \) on \( G \). The notation is explained by the obvious isomorphism of \( \mathbb{Z} \)-algebra restriction functors \( R^{ab}(H) \cong R(H^{ab}) \), \( H \leq G \), where \( H^{ab} \) denotes the commutator factor group of \( H \). We will have to consider the family of maps \( p_H : R(H) \to R^{ab}(H) \), \( H \leq G \), which are induced by taking the \( H^{ab} \)-fixed points of a \( CH \)-module. Thus, \( p_H \) is the identity on one-dimensional characters and zero on all other irreducible characters. Since an irreducible character of degree higher than one may have one-dimensional constituents after restriction to a subgroup, the family \( (p_H)_{H \leq G} \) is in general not a morphism of \( \mathbb{Z} \)-restriction functors on \( G \). But still this family commutes with the conjugation maps. This leads to the notion of a conjugation functor, a ‘restriction functor without restriction’.

1.7 Definition A \( k \)-conjugation functor on \( G \) (resp. \( k \)-algebra conjugation functor on \( G \)) is a pair \((X, c)\) consisting of \( k \)-modules (resp. unitary \( k \)-algebras) \( X(H) \), \( H \leq G \), and \( k \)-module homomorphisms (resp. unitary \( k \)-algebra homomorphisms) \( c_{g,H} : X(H) \to X(gH) \), the conjugations, for \( H \leq G \), \( g \in G \), satisfying the following axioms:

(C1) (Triviality) \( c_{h,H} = \text{id}_{X(H)} \)

for \( H \leq G \), \( h \in H \).

(C2) (Transitivity) \( c_{g',g,H} = c_{g',gH} \circ c_{g,H} \)

for \( H \leq G \), \( g, g' \in G \).

We will often write \( X \) instead of \((X, c)\) and also \( gx \) instead of \( c_{g,H}(x) \) for \( g \in G \), \( H \leq G \), \( x \in X(H) \).

A morphism \( f : X \to Y \) between \( k \)-conjugation functors (resp. \( k \)-algebra conjugation functors) \( X, Y \) on \( G \) is a family \((f_H : X(H) \to Y(H))_{H \leq G} \) of \( k \)-module homomorphisms (resp. unitary \( k \)-algebra homomorphisms) which commute with
conjugation maps. The category of $k$-conjugation functors (resp. $k$-algebra conjugation functors) on $G$ is denoted by $k\text{-Con}(G)$ (resp. $k\text{-Con}_{\text{alg}}(G)$).

1.8 Remark

(i) The statements in Remark 1.2 hold also for the categories $k\text{-Con}(G)$ and $k\text{-Con}_{\text{alg}}(G)$.
(ii) There is an obvious commutative diagram of forgetful functors:

\[
\begin{array}{ccc}
k\text{-Mack}_{\text{alg}}(G) & \longrightarrow & k\text{-Res}_{\text{alg}}(G) \\
\downarrow & & \downarrow \\
k\text{-Mack}(G) & \longrightarrow & k\text{-Res}(G)
\end{array}
\]

In particular we will assume that each definition which will be made for one of the categories in the above diagram is automatically a definition for all categories in the same row on the left of this one. So we define for example $f_H: X \to Y$ to be injective (resp. surjective), if $f_H: X \to Y$ is injective (resp. surjective) for all $H \leq G$.

(iii) Let $X$ be an object in one of the six categories in the above diagram and let $H \leq G$. Then, using the conjugation maps $c_{n,H}$, the $k$-module $X(H)$ is provided with the structure of a $kN_G(H)$-module.

(iv) In a complete treatment of the sort of categories we have introduced by now one should also define the notion of a $k$-induction functor on $G$ as a ‘$k$-Mackey functor without restriction’. While taking $k$-duals yields functors $k\text{-Mack}(G) \to k\text{-Mack}(G)$ and $k\text{-Con}(G) \to k\text{-Con}(G)$, the $k$-dual of a restriction functor would be an induction functor and vice-versa. We don’t introduce this notion here, since it doesn’t play a role in the considerations of the following chapters. Moreover, it is not clear what the definition of an algebra induction functor should be. With this remark we want to explain the lack of symmetry when certain constructions will result in restriction functors while the dual constructions result ‘only’ in conjugation functors. The truth is that these constructions would result in induction functors.

1.2 The functors

In this section we will define several functors between the categories defined in the last section. Among them, the functors

\[{-}^+: k\text{-Con}(G) \to k\text{-Mack}(G)\]

and

\[{-}_{+}: k\text{-Res}(G) \to k\text{-Mack}(G)\]

will be of particular interest in Chapter II.

2.1 The functor ${-}^+: k\text{-Con}(G) \to k\text{-Mack}(G)$ is part of the construction of the ‘twin functor’ of a Mackey functor which was introduced by Thévenaz (cf. [Th88, Sec. 4]). Let $(X,c)$ be a $k$-conjugation functor on $G$. We set

\[P_X(H) := \prod_{K \leq H} X(K),\]
1.2. THE FUNCTORS

for $H \leq G$, and observe that $N_G(H)$ acts on $P_X(H)$ by the conjugation maps $c$, i.e. for $n \in N_G(H)$ and $(x_K)_{K \leq H} \in P_X(H)$ we set

$$n((x_K)_{K \leq H}) := (n(x_K))^H.$$

Thus, $n \in N_G(H)$ maps the $K$-component in $P_X(H)$ to the $^nK$-component in $P_X(^nH) = P_X(H)$ via $c_{n,K}$. In particular we may consider the $H$-fixed points $P_X(H)^H$ of $H$ on $P_X(H)$, and define

$$X^+(H) := P_X(H)^H = \left\{ (x_K)_{K \leq H} \in \prod_{K \leq H} X(K) \mid ^h(x_K) = x_{(h)K} \text{ for } K \leq H, \ h \in H \right\}.$$

For $K \leq H \leq G$ and $g \in G$ we define a conjugation map

$$c_{g,H} : X^+(H) \to X^+(gH), \quad (x_L)_{L \leq H} \mapsto (g(x_L^g))_{L \leq gH},$$

a restriction map

$$\text{res}^+_K : X^+(H) \to X^+(K), \quad (x_L)_{L \leq H} \mapsto (x_L)_{L \leq K},$$

as the projection, and an induction map

$$\text{ind}^+_K : X^+(K) \to X^+(H), \quad (x_L)_{L \leq K} \mapsto \sum_{h \in H/K} c^+_{h,K}((x_L)_{L \leq K}),$$

as relative trace, where the last summands are considered as elements in the $k$-module $X^+(H)$ by filling the missing components with zero, and $x_L$ always denotes an element in $X(L)$. It is easy to verify that these three maps are well defined and satisfy the axioms (M1)–(M4) of Definition 1.1. Hence, $(X^+, c^+, \text{res}^+, \text{ind}^+)$ is a $k$-Mackey functor on $G$.

For a morphism $f : X \to Y$ of $k$-conjugation functors on $G$ we define

$$f_H^+ : X^+(H) \to Y^+(H), \quad (x_K)_{K \leq H} \mapsto (f_K(x_K))_{K \leq H},$$

for $H \leq G$. Obviously $f_H^+$ is well defined and $f^+ = (f_H^+)_{H \leq G}$ is a morphism of $k$-Mackey functors on $G$. Thus, the definition of the functor $-^+ : k\text{-Con}(G) \to k\text{-Mack}(G)$ is complete.

We note that if $X \in k\text{-Con}_{\text{alg}}(G)$, then $X^+(H)$ is a $k$-subalgebra of the direct product $k$-algebra $\prod_{K \leq H} X(K)$ for all $H \leq G$, and $X^+$ is a $k$-Green functor on $G$. Moreover, if $f : X \to Y$ is a morphism of $k$-algebra conjugation functors on $G$, then $f^+$ is a morphism of $k$-Green functors on $G$. Hence we also obtain a functor $-^+ : k\text{-Con}_{\text{alg}}(G) \to k\text{-Mack}_{\text{alg}}(G)$ such that the diagram

$$
\begin{array}{ccc}
\text{k-Con}_{\text{alg}}(G) & \xrightarrow{-^+} & \text{k-Mack}_{\text{alg}}(G) \\
\downarrow & & \downarrow \\
\text{k-Con}(G) & \xrightarrow{-^+} & \text{k-Mack}(G)
\end{array}
$$
is commutative, where the vertical arrows are the forgetful functors.

2.2 The construction of the functor \(-_+ : k\text{-Res}(G) \to k\text{-Mack}(G)\) is to some extent dual to the construction of \(-^+\), although one needs a restriction functor to start with. This construction was introduced in [Bo89, III.2] where proofs for the statements below can be found.

For \((A, c, \text{res}) \in k\text{-Res}(G)\) we set

\[ S_A(H) := \bigoplus_{K \leq H} A(K) \]

for \(H \leq G\). Note that, as \(k\)-modules, \(S_A(H)\) and \(P_A(H)\) are isomorphic. Nevertheless we prefer this formal distinction between direct sum and direct product, because of the dual nature of the construction, and also because in the algebra case we will construct a multiplication which does not come from the direct product of the algebras \(A(K)\) as in (2.1). As before, the normalizer \(N_G(H)\) acts on \(S_A(H)\) by

\[ ^n(a_K) := c_{n,K}(a) \in A(\nK) \]

for \(n \in N_G(H), K \leq H, a_K \in A(K)\). In contrast to the previous construction we now consider the \(H\)-cofixed points \(S_A(H)_H\) of the \(kH\)-module \(S_A(H)\), i.e. the biggest quotient of \(S_A(H)\) by a \(kH\)-submodule that admits trivial \(H\)-action. More explicitly, we define

\[ A_+(H) := S_A(H)_H := S_A(H)/I(kH) \cdot S_A(H), \quad H \leq G, \]

where \(I(kH)\) is the augmentation ideal \(\{\sum_{h \in H} \alpha_h h \mid \sum_{h \in H} \alpha_h = 0\}\) of \(kG\). Then \(I(kH) \cdot S_A(H)\) is the \(k\)-submodule of \(S_A(H)_H\) generated by the elements \(a_K - ^b(a_K)\) for \(K \leq H, a_K \in A(K), h \in H\). Clearly \(A_+(H)\) is generated as a \(k\)-module, even as abelian group, by elements of the form \(a_K + I(kH) \cdot S_A(H)\), with \(K \leq H, a_K \in A(K)\), and we introduce the following notation:

\[ [K, a]_H := a + I(kH) \cdot S_A(H) \quad \text{for} \quad K \leq H, a \in A(K). \]

In particular the symbol \([K, a]_H\) indicates that \(K \leq H\) and \(a \in A(K)\). Obviously we have \([^bK, ^b\alpha]_H = [K, a]_H\) for \(K \leq H, a \in A(K), h \in H\). If \(\mathcal{R}_H\) is a set of representatives for the conjugacy classes of subgroups of \(H\), we can write each element of \(A_+(H)\) in the form

\[ \sum_{K \in \mathcal{R}_H} [K, a_K]_H, \quad a_K \in A(K), \]

with the ambiguity

\[ \sum_{K \in \mathcal{R}_H} [K, a_K]_H = \sum_{K \in \mathcal{R}_H} [K, a'_K]_H \iff a'_K = ^{n_K}(a_K) \text{ for all } K \in \mathcal{R}_H \text{ and some } n_K \in N_H(K). \]

The conjugation maps of \(A\) induce conjugation maps on \(A_+\):

\[ c_{+g,H} : A_+(H) \to A_+(gH), \quad [K, a]_H \mapsto [^gK, ^g\alpha]_gH, \]
for $H \leq G$, $g \in G$. For $K \leq H \leq G$ we define a restriction map
\[
\text{res}_+^H_K: A_+(H) \to A_+(K), \quad [L,a]_H \mapsto \sum_{h \in K \setminus H/L} [K \cap hL, \text{res}_+^H_K hL(a)]_K,
\]
and an induction map
\[
\text{ind}_+^H_K: A_+(K) \to A_+(H), \quad [L,a]_K \mapsto [L,a]_H.
\]
A straightforward (but not very short) verification shows that $(A_+, c_+, \text{res}_+, \text{ind}_+)$ is a $k$-Mackey functor on $G$.

For a morphism $f: A \to B$ of two $k$-restriction functors on $G$ we define
\[
f_+: A_+(H) \to B_+(H), \quad [K,a]_H \mapsto [K, f_K(a)]_H, \quad H \leq G.
\]
The family $f_+ = (f_+^H)_{H \leq G}$ then forms a morphism of $k$-Mackey functors on $G$ and this completes the definition of the functor $\_+ : k\text{-Res}(G) \to k\text{-Mack}(G)$.

If $A$ is in $k\text{-Res}_{\text{alg}}(G)$, then we can define a $k$-algebra structure on $A_+(H)$ for $H \leq G$ by
\[
[K, a]_H \cdot [L, b]_H := \sum_{h \in K \setminus H/L} [K \cap hL, \text{res}_+^H_K hL(a) \cdot \text{res}_+^H_K hL(b)]_H.
\]
The element $[H, 1_{A(H)}]_H$ is the unity in $A_+(H)$. It is easy to show that $A_+$ is a $k$-Green functor on $G$. If $A(H)$ is commutative for all $H \leq G$, then the same is true for $A_+(H)$, $H \leq G$. Furthermore, if $f: A \to B$ is a morphism of $k$-algebra restriction functors on $G$, then $f_+ : A_+ \to B_+$ is a morphism of $k$-Green functors on $G$, and we obtain a functor $\_+ : k\text{-Res}_{\text{alg}}(G) \to k\text{-Mack}_{\text{alg}}(G)$ which renders the diagram
\[
\begin{array}{ccc}
k\text{-Res}_{\text{alg}}(G) & \xrightarrow{\_+} & k\text{-Mack}_{\text{alg}}(G) \\
\downarrow & & \downarrow \\
k\text{-Res}(G) & \xrightarrow{\_+} & k\text{-Mack}(G)
\end{array}
\]
commutative, where the vertical arrows are the forgetful functors.

If $A \in k\text{-Res}_{\text{alg}}(G)$ is commutative, then $A_+(H)$ is an $A(H)$-algebra and in particular an $A(H)$-module via the ring homomorphism $A(H) \to A_+(H)$, $a \mapsto [H,a]_H$, for $H \leq G$.

### 2.3 Example
We consider the constant $\mathbb{Z}$-algebra restriction functor $\mathbb{Z}$ on $G$ with $\mathbb{Z}(H) := \mathbb{Z}$ for $H \leq G$, and with conjugation and restriction maps being the identity. Then $\mathbb{Z}_+$ and the Burnside ring functor $\Omega$ are isomorphic as $\mathbb{Z}$-Green functors on $G$ by the maps
\[
f_H : \Omega(H) \xrightarrow{\sim} \mathbb{Z}_+(H), \quad [H/K] \mapsto [K,1]_H, \quad H \leq G.
\]
On the other hand we have the $\mathbb{Z}$-Green functor $\mathbb{Z}^+$ on $G$ with $\mathbb{Z}(H) = (\prod_{K \leq H} \mathbb{Z})^H$ for $H \leq G$, and there are the well-known mark homomorphisms (cf. Appendix A) $\rho_H : \Omega(H) \to \mathbb{Z}^+(H), H \leq G$, which form a morphism of $\mathbb{Z}$-Green functors on $G$, as
one can check easily. Identifying $\Omega(H)$ with $\mathbb{Z}_+(H)$ for $H \leq G$ by the isomorphism $f_H$ we obtain the morphism

$$\rho_H : \mathbb{Z}_+(H) \to \mathbb{Z}_+(H), \quad [K,1]_H \mapsto (|(H/K)^U|)_{U \leq H}, \quad H \leq G,$$

of $\mathbb{Z}$-Green functors on $G$.

In the same way we obtain from the constant $k$-algebra restriction functor $k$ the Burnside ring functor $k \otimes \Omega$ over $k$ with the same identification, and also the maps $\rho_H$, $H \leq G$, generalize to a morphism $\rho_H : \mathbb{k}_+ \to \mathbb{k}_+$ of $k$-Green functors. In Section 3 we will generalize this ‘mark homomorphism’ $\rho_H$, which is a main tool for the study of the Burnside ring, to a morphism $\rho^A : A_+ \to A^+$ of $k$-Mackey functors (resp. $k$-Green functors) on $G$ for an arbitrary $k$-restriction functor (resp. $k$-algebra restriction functor) $A$ on $G$.

In Chapter II we will have to restrict our attention to $k$-restriction functors $A$ on $G$ with the property that $A_+(H)$ is $k$-free for all $H \leq G$, since then certain maps related to scalar extensions are injective (cf. Section 5). The following proposition gives a sufficient condition for this to hold.

We say that $A \in k-\text{Res}(G)$ has a conjugation-stable basis (or just a stable basis), if there exist $k$-bases $B(H)$ of $A(H)$, for $H \leq G$, such that $c_{g,H}(B(H)) = B(h^gH)$ for $H \leq G$, $g \in G$. Note that this is equivalent to the existence of a $k$-basis $B(H)$ of $A(H)$, where $H$ runs through a set of representatives of the conjugacy classes of subgroups of $G$, such that $c_{n,H}(B(H)) = B(H)$ for $n \in N_G(H)$, since such bases can be transported to all $A(H)$, $H \leq G$, by the conjugation maps. For a stable basis $B(H) \subseteq A(H)$, $H \leq G$, and a given subgroup $H$ of $G$ the disjoint union $B_H := \bigcup_{K \leq H} B(K)$ is an $H$-set by the definition $b_h := c_{h,K}(b)$ for $h \in H$, $K \leq H$, $b \in B(K)$. Instead of $B_H$ we might as well consider the isomorphic $H$-set consisting of all pairs $(K,b)$ with $K \leq H$, $b \in B(K)$, where $H$ acts on the second component as before and on the first component by conjugation.

**2.4 Proposition** Let $A \in k-\text{Res}(G)$ with stable basis $B(H) \subseteq A(H)$ for $H \leq G$. For $H \leq G$ let $B_H := \bigcup_{K \leq H} B(K)$. Then any set $\mathcal{R}_H$ of representatives for the $H$-orbits of the $H$-set of pairs $(K,b)$, $K \leq H$, $b \in B(K)$, gives rise to a $k$-basis $\{[K,b]_H \mid (K,b) \in \mathcal{R}_H\}$ of $A_+(H)$.

**Proof** For $H \leq G$ let $B'_H \subseteq B_H$ be the set of representatives for the $H$-orbits of $B_H$ which corresponds to $\mathcal{R}_H$. Clearly $B_H$ is a $k$-basis of the $k$-module $S_A(H) = \bigoplus_{K \leq H} A(K)$ and we have the following decomposition into $k$-submodules

$$S_A(H) = kB'_H \oplus k\{b - h b \mid b \in B'_H, h \in H\}.$$  

It is obvious that the second summand equals $I(kH) \cdot S_A(H)$, and therefore $A_+(H)$ is free with basis $[K,b]_H$, $(K,b) \in \mathcal{R}_H$.

2.5 For a $k$-Mackey functor $M$ on $G$ and a subgroup $H \leq G$ we consider the $k$-submodule

$$\mathcal{I}(M)(H) := \sum_{K < H} \text{ind}_K^H(M(K)) = \sum_{K < H} \text{im}(\text{ind}_K^H : M(K) \to M(H))$$
of $M(H)$. The $G$-equivariance of induction, cf. (M3), implies that $\mathcal{I}(M)$ is a $k$-conjugation subfunctor of $M$ on $G$. Since morphisms of $k$-Mackey functors commute with induction, these submodules are preserved under such morphisms, and we obtain a functor

$$ \mathcal{I}: k\text{-}\text{Mack}(G) \to k\text{-}\text{Con}(G). $$

Since the restriction maps don’t play a role in this construction, we could have started with an induction functor instead of a Mackey functor (cf. Remark 1.8 (iv)). Note that if $M \in k\text{-}\text{Mack}_{\text{alg}}(G)$, then $\mathcal{I}(M)(H)$ is an ideal in $M(H)$ for $H \leq G$ by the Frobenius axiom (M6).

Let $M \in k\text{-}\text{Mack}(G)$. Following Thévenaz (cf. [Th88]) we call a subgroup $H$ of $G$ primordial for $M$, if $\mathcal{I}(M)(H) \neq M(H)$, i.e. $H$ is not primordial for $M$, if each element of $M(H)$ can be obtained as a sum of properly induced elements. For the character ring functor $R$ for instance, the primordial subgroups are precisely the elementary subgroups (cf. [Se78, 11.3]). We denote the set of primordial subgroups for $M$ by $\mathcal{P}(M)$. More about primordial subgroups can be found in [Th88]. Note that we have

$$ M(H) = \sum_{K \leq H \atop K \in \mathcal{P}(M)} \text{ind}_K^H(M(K)) $$

for $H \leq G$.

2.6 We dualize the construction in 2.5. For a $k$-restriction functor $A$ on $G$ and a subgroup $H \leq G$ we consider the $k$-submodule

$$ \mathcal{K}(A)(H) := \bigcap_{K < H} \ker(\text{res}_K^H: A(H) \to A(K)) $$

of $A(H)$. The $k$-submodules $\mathcal{K}(A)(H)$, $H \leq G$, form a $k$-conjugation subfunctor of $A$ and they are preserved under morphisms of $k$-restriction functors on $G$. Hence, we obtain a functor

$$ \mathcal{K}: k\text{-}\text{Res}(G) \to k\text{-}\text{Con}(G). $$

Note that if $A \in k\text{-}\text{Res}_{\text{alg}}(G)$, then $\mathcal{K}(A)(H)$ is an ideal of $A(H)$ for $H \leq G$, since the restriction maps are $k$-algebra homomorphisms. Note also that $\mathcal{K}(A)$ is even a $k$-restriction subfunctor of $A$ on $G$ with trivial restrictions $\text{res}_K^H$, $K < H \leq G$, but because of the triviality of the restrictions we prefer to view $\mathcal{K}$ only as a $k$-conjugation functor on $G$.

Let $A \in k\text{-}\text{Res}(G)$. A subgroup $H$ of $G$ is called coprimordial for $A$ (cf. [Bo89, II.1.9]), if $\mathcal{K}(A)(H) \neq 0$, i.e. $H$ is not coprimordial for $A$, if the elements of $A(H)$ are separated by proper restrictions. For the character ring functor for instance, the coprimordial subgroups are precisely the cyclic subgroups. We denote the set of coprimordial subgroups for $A$ by $\mathcal{C}(A)$. More about coprimordial subgroups and the connection to primordial subgroups can be found in [Bo89, III.1.11-1.18]. Note that for $H \leq G$ two elements $x, y \in A(H)$ are equal if and only if $\text{res}_K^H(x) = \text{res}_K^H(y)$ for all $K \leq H$ with $K \leq \mathcal{C}(A)$.

2.7 For a $k$-Mackey functor $M$ on $G$ and $H \leq G$ the $k$-module

$$ \overline{M}(H) := M(H)/\mathcal{I}(M)(H) = M(H)/\sum_{K < H} \text{ind}_K^H(M(K)) $$

is called the residue of $M$ at $H$. This notion was studied extensively by Puig in [Pu88] in the case of various representation rings and by Thévenaz in [Th88] in the general case. Since morphisms of $k$-Mackey functors on $G$ induce maps on the residues, we obtain a functor $\overline{\tau}: k-\text{Mack}(G) \to k-\text{Con}(G)$. Since $\mathcal{I}(M)(H)$ is an ideal of $M(H)$ for $M \in k-\text{Mack}_{\text{alg}}(G)$ and $H \leq G$, we also obtain a functor $\overline{\tau}: k-\text{Mack}_{\text{alg}}(G) \to k-\text{Con}_{\text{alg}}(G)$ which renders the diagram

\[
\begin{array}{ccc}
k-\text{Mack}_{\text{alg}}(G) & \xrightarrow{\overline{\tau}} & k-\text{Con}_{\text{alg}}(G) \\
\downarrow & & \downarrow \\
k-\text{Mack}(G) & \xrightarrow{\overline{\tau}} & k-\text{Con}(G)
\end{array}
\]

commutative, where the vertical arrows denote the forgetful functors.

2.8 In [Th88] Thévenaz defined the twin functor $\mathcal{T}(M)$ of a $k$-Mackey functor $M$ on $G$ by $\mathcal{T}(M) := \overline{M}^{+} \in k-\text{Mack}(G)$. This construction defines functors $\mathcal{T}$ which are given as the composite functors

\[
\begin{array}{ccc}
k-\text{Mack}(G) & \xrightarrow{\overline{\tau}} & k-\text{Con}(G) \\
\downarrow & & \downarrow \\
k-\text{Mack}_{\text{alg}}(G) & \xrightarrow{\overline{\tau}} & k-\text{Con}_{\text{alg}}(G)
\end{array}
\]

and

\[
\begin{array}{ccc}
k-\text{Mack}_{\text{alg}}(G) & \xrightarrow{\overline{\tau}} & k-\text{Con}_{\text{alg}}(G) \\
\downarrow & & \downarrow \\
k-\text{Mack}(G) & \xrightarrow{\overline{\tau}} & k-\text{Con}(G)
\end{array}
\]

of the functors defined in (2.1) and (2.7) such that the diagram

\[
\begin{array}{ccc}
k-\text{Mack}_{\text{alg}}(G) & \xrightarrow{\mathcal{T}} & k-\text{Mack}_{\text{alg}}(G) \\
\downarrow & & \downarrow \\
k-\text{Mack}(G) & \xrightarrow{\mathcal{T}} & k-\text{Mack}(G)
\end{array}
\]

is commutative, where the vertical arrows are the forgetful functors. Furthermore Thévenaz defined for $M \in k-\text{Mack}(G)$ (resp. $M \in k-\text{Mack}_{\text{alg}}(G)$) morphisms $\beta^M: M \to \mathcal{T}(M)$ of $k$-Mackey functors (resp. $k$-Green functors) on $G$ by

$\beta^M_H: M(H) \to \overline{M}^{+}(H), \ m \mapsto (\text{res}_K^H(m) + \mathcal{I}(M)(K))_{K \leq H}$,

for $H \leq G$, and showed that $\ker(\beta^M_H)$ and $\coker(\beta^M_H)$ are annihilated by a power of $|H|$ (cf. [Th88, 3.2 and 12.2]). Moreover, the morphisms $\beta^M_H$, for $M \in k-\text{Mack}(G)$ (resp. $M \in k-\text{Mack}_{\text{alg}}(G)$), form a natural transformation between the functors $\text{Id}, \mathcal{T}: k-\text{Mack}(G) \to k-\text{Mack}(G)$ (resp. the functors $\text{Id}, \mathcal{T}: k-\text{Mack}_{\text{alg}}(G) \to k-\text{Mack}_{\text{alg}}(G)$).

1.3 The mark morphism

In this section we are going to generalize the mark homomorphism $\rho_G: \Omega(G) \cong \mathbb{Z}_+(G) \to \mathbb{Z}^{+}(G)$ of the Burnside ring (cf. Appendix A and Example 2.3).
1.3. THE MARK MORPHISM

3.1 For $A \in k-\text{Res}(G)$ (resp. $A \in k-\text{Res}_{\text{alg}}(G)$) and $H \leq G$, the inclusion $A(H) \to \bigoplus_{K \leq H} A(K)$ and the projection $\bigoplus_{K \leq H} A(K) \to A(H)$ induce maps

$$\iota^A_H: A(H) \to A_+(H), \quad a \mapsto [H,a]_H,$$

and

$$\pi^A_H: A_+(H) \to A(H), \quad [K,a]_H \mapsto \begin{cases} a, & \text{if } K = H, \\ 0, & \text{if } K < H. \end{cases}$$

The maps $\iota^A_H$, $H \leq G$, are injective by axiom (R1) and form a morphism $\iota^A: A \to A_+$ of $k$-restriction functors (resp. $k$-algebra restriction functors) on $G$, which is natural in $A$. The maps $\pi^A_H$, $H \leq G$, are well-defined by axiom (R1). They will be called the Brauer maps (cf. [Th88, Section 1]), since $\pi^A_H$ is surjective and vanishes exactly on elements which are induced from proper subgroups of $H$ (cf. the definition of induction for $A_+$). The Brauer maps form a morphism $\pi^A: A_+ \to A$ of $k$-conjugation functors (resp. $k$-algebra conjugation functors) on $G$, which is natural in $A$ and a splitting morphism for $\iota^A: A \to A_+$, i.e. $\pi^A \circ \iota^A = \text{id}_A$. Moreover the Brauer maps induce an isomorphism $\overline{A_+} \to A$ of $k$-conjugation functors (resp. $k$-algebra conjugation functors) which is natural in $A$. Therefore (cf. 2.7) we call $\pi^A_H(x) \in A(H)$ the residue of $x$ for $x \in A_+(H)$, $H \leq G$. Note that $T(A_+) = \overline{A_+} = A^+$, i.e. $A^+$ is the twin functor of $A_+$, cf. 2.8.

Using the Brauer maps we define the mark homomorphism:

$$\rho^A_H := (\pi^A_K \circ \text{res}_{+K})_{K \leq H}: A_+(H) \to A^+(H),$$

for $H \leq G$. Note that $\rho^A_H$ takes values in the $H$-fixed points, since $\pi^A_K$ and $\text{res}_{+K}$ commute with conjugations for $K \leq H \leq G$. Again it is easy to verify that the mark homomorphisms $\rho^A_H$, $H \leq G$, form a morphism $\rho^A: A_+ \to A^+$ of $k$-Mackey functors (resp. $k$-Green functors) on $G$, which we call the mark morphism. With the above identification $T(A_+) \cong A^+$, $\rho^A: A_+ \to A^+$ coincides with Thévenaz’ morphism $\beta^A$: (cf. [Th88, Section 3] and (2.8)). Note that $\rho^A$ is natural in $A$, i.e. for $A,B \in k-\text{Res}(G)$ (resp. $A,B \in k-\text{Res}_{\text{alg}}(G)$) and $f \in k-\text{Res}(G)(A,B)$ (resp. $f \in k-\text{Res}_{\text{alg}}(G)(A,B)$) the following diagram is commutative:

$$
\begin{array}{ccc}
A_+ & \xrightarrow{f} & B_+ \\
\rho^A \downarrow & & \downarrow \rho^B \\
A^+ & \xrightarrow{f^+} & B^+.
\end{array}
$$

If in particular, $A = k$ as in Example 2.3, and $f: k \to B$ is the unique morphism of $k$-algebra restriction functors, namely $f_H(x) = x \cdot 1_{B(H)}$ for $H \leq G$ and $x \in k$, then $\rho^A$ is just the family of mark homomorphisms of the Burnside rings $k \otimes \Omega(H)$, $H \leq G$, $f_+$ is the unique element in $k-\text{Mack}_{\text{alg}}(G)(k \otimes \Omega, B_+)$, and the mark morphism $\rho^B$ extends the mark morphism of the Burnside ring.

Next we investigate in which circumstances $\rho^A$ is injective or even an isomorphism. For $H \leq G$ we define a map

$$\sigma^A_H: A^+(H) \to A_+(H), \quad (a_K)_{K \leq H} \mapsto \sum_{L \leq K \leq H} |L| \mu(L,K) [L, \text{res}^K_L(a_K)]_H,$$
where \( \mu \) denotes the Möbius function on the poset of subgroups of \( G \) (cf. Appendix B).

The following proposition shows that \( \sigma_A^H \) is almost an inverse of \( \rho_A^H \) for \( H \leq G \). This property together with the explicit definition of \( \sigma_A^H \) will be used in Chapter II to give explicit induction formulae. For the reader’s convenience we give a reformulated proof from [Bo89, Prop. II.2.13] in our notation.

**3.2 Proposition** For \( A \in \text{k−Res}(G) \) (resp. \( A \in \text{k−Res}_{\text{alg}}(G) \)) and \( H \leq G \) we have
\[
\sigma_A^H \circ \rho_A^H = |H| \cdot \text{id}_{A_+(H)} \quad \text{and} \quad \rho_A^H \circ \sigma_A^H = |H| \cdot \text{id}_{A_+(H)}.
\]
In particular the kernels and cokernels of \( \rho_A^H \) and \( \sigma_A^H \) are annihilated by \( |H| \). If \( |G| \) is invertible in \( \text{k} \), then \( \rho_A^H \) is an isomorphism of \( \text{k}-\text{Mackey functors} \) (resp. \( \text{k}-\text{Green functors} \)) on \( G \) with inverse \((|H|^{-1} \cdot \sigma_A^H)_{H \leq G}\). If, for \( H \leq G \), \( A_+(H) \) has trivial \( |H| \)-torsion, then \( \rho_A^H \) is injective.

**Proof** For \( U \leq H \) and \( a \in A(U) \) we have
\[
\rho_A^H([U,a]_H) = \left( (\pi_K^A \circ \text{res}_+^H)([U,a]_H) \right)_{K \leq H} = \left( \sum_{h \in K \cap H/U} \pi_K^A([K \cap hU, \text{res}_K^h(U)(h)]_K) \right)_{K \leq H} = \left( \sum_{h \in H/U} \text{res}_K^h(U)(h) \right)_{K \leq H},
\]
since for \( K \leq H \) the relation \( K \leq hU \) implies \( KhU = hU \). Applying \( \sigma_A^H \) to this family we obtain
\[
(\sigma_A^H \circ \rho_A^H)([U,a]_H) = \sum_{L \leq K \leq H} |L| \mu(L,K) \sum_{h \in H/U \atop K \leq hU} [L, \text{res}_L^h(U)(h)]_H = \frac{1}{|U|} \sum_{h \in H} \sum_{L \leq K \leq hU} \mu(L,K) |L| [L, \text{res}_L^h(U)(h)]_H,
\]
and considering for \( h \in H \) the map
\[
f : \{L \leq hU \} \to A_+(H), \quad L \mapsto |L| [L, \text{res}_L^h(U)(h)]_H,
\]
the inner sum collapses by Möbius inversion (see Proposition B.2 (i) in Appendix B) to \( |hU|[[hU, hU]_H = |U|[U,a]_H \) which gives
\[
(\sigma_A^H \circ \rho_A^H)([U,a]_H) = \frac{1}{|U|} \sum_{h \in H} |U|[U,a]_H = |H|[U,a]_H.
\]

Conversely let \((a_K)_{K \leq H} \in A^+(H)\). Then the \( U \)-component, \( U \leq H \), of the
element \((\rho_H^A \circ \sigma_H^A)((a_K)_{K \leq H})\) is given by
\[
\left(\pi_U^A \circ \text{res}^H_U \circ \sigma_H^A\right)((a_K)_{K \leq H})
= \sum_{L \leq K \leq H} |L| \mu(L, K) \pi_U^A\left(\text{res}^H_U\left(\left[\text{res}_L^K(a_K)\right]_H\right)\right)
= \sum_{L \leq K \leq H} |L| \mu(L, K) \sum_{h \in H/L} \text{res}_U^K\left(h(a_K)\right)\]
\[
= \sum_{h \in H} \mu(L, K) \sum_{U \leq hL} \text{res}_U^K\left(h(a_K)\right).
\]

Considering for each \(h \in H\) the map \(f: \{U^h \leq K \leq H\} \to A(U), K \mapsto \text{res}_U^K(h(a_K))\), the inner sum collapses by Möbius inversion (see Proposition B.2 (ii) in Appendix B) to \(h(a_K)_U = a_U\), and we have
\[
\left(\pi_U^A \circ \text{res}^H_U \circ \sigma_H^A\right)((a_K)_{K \leq H}) = \sum_{h \in H} a_U = |H| a_U,
\]
which completes the proof of the two equations. The other assertions of the proposition now follow immediately. 

3.3 Remark

(i) Note that the above proposition reproves the formula of Gluck (cf. Appendix A) for the primitive idempotents of \(\mathbb{Q} \otimes \Omega(G)\) (and also of \(k \otimes \Omega(G)\), if \(|G|\) is invertible in \(k\)),
\[
e(G)_H = \frac{1}{|N_G(H)|} \sum_{K \leq H} |K| \mu(K, H) [G/K],
\]
for \(H \leq G\).

(ii) If \(A_+(H)\) has no \(|H|\)-torsion for all \(H \leq G\), then it follows immediately from Proposition 3.2 that with \(\rho^A\) also \(((G : H) \cdot \sigma_H^A)_{H \leq G}\) is a morphism of \(k\)-Mackey functors (resp. \(k\)-Green functors) on \(G\). We are convinced that this is true in general, but we don’t see any short proof. Since we will never need this result, it does no harm to leave the general case open.

1.4 Adjointness properties

The following adjointness properties illustrate the significance of the constructions \(-_+\) and \(-^+\). They will both be used in Chapter II for the construction of canonical induction formulae.

4.1 Proposition

(i) The forgetful functor \(k-\text{Mack}(G) \to k-\text{Res}(G)\) is right adjoint to \(-_+\): \(k-\text{Res}(G) \to k-\text{Mack}(G)\). More precisely, for \(A \in k-\text{Res}(G)\),
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$M \in k\text{-}\text{Mack}(G)$ we have inverse $k$-linear isomorphisms

$$\kappa_{A,M} : k\text{-}\text{Mack}(G)(A_+, M) \to k\text{-}\text{Res}(G)(A, M),$$

$$(f_H)_{h \leq G} \mapsto (f_H \circ \epsilon_H^A)_{h \leq G},$$

$$\left( [K, a]_H \mapsto \text{ind}_K^H(g_K(a)) \right)_{h \leq G} \leftrightarrow (g_H)_{h \leq G},$$

of $k$-modules which are natural in $A$ and $M$. The same maps yield inverse natural $k$-linear isomorphisms between $k\text{-}\text{Mack}_{\text{alg}}(G)(A_+, M)$ and $k\text{-}\text{Res}_{\text{alg}}(G)(A, M)$ for $A \in k\text{-}\text{Res}_{\text{alg}}(G)$ and $M \in k\text{-}\text{Mack}_{\text{alg}}(G)$.

(ii) The forgetful functor $k\text{-}\text{Res}(G) \to k\text{-}\text{Con}(G)$ is left adjoint to the composition of the functor $-^+ : k\text{-}\text{Con}(G) \to k\text{-}\text{Mack}(G)$ and the forgetful functor $k\text{-}\text{Mack}(G) \to k\text{-}\text{Res}(G)$. More precisely, for $A \in k\text{-}\text{Res}(G)$ and $X \in k\text{-}\text{Con}(G)$ we have $k$-linear inverse isomorphisms

$$\lambda_{A,X} : k\text{-}\text{Res}(G)(A, X^+) \to k\text{-}\text{Con}(G)(A, X),$$

$$(f_H)_{h \leq G} \mapsto (pr_H^X \circ f_H)_{h \leq G},$$

$$\left( A(H) \ni a \mapsto (g_K(\text{res}_K^H(a)))_{k \leq H} \right)_{h \leq G} \leftrightarrow (g_H)_{h \leq G},$$

of $k$-modules which are natural in $A$ and $X$, and where $pr_H^X : (\prod_{k \leq H} X(K))^H = X^+(H) \to X(H)$ denotes the projection map onto the $H$-component. The same maps yield inverse natural $k$-linear isomorphisms between $k\text{-}\text{Res}_{\text{alg}}(G)(A, X^+)$ and $k\text{-}\text{Con}_{\text{alg}}(G)(A, X)$ for $A \in k\text{-}\text{Res}_{\text{alg}}(G)$ and $X \in k\text{-}\text{Con}_{\text{alg}}(G)$.

**Proof** All assertions in (i) and (ii) are immediate consequences of the very definitions of the categories and functors involved. The verifications are without exception straightforward.

Let $A, B \in k\text{-}\text{Res}(G)$ and assume that for all $H \leq G$, $B_+(H)$ has trivial $|H|$-torsion. Then $((G : H) \cdot \sigma_H^B)_{h \leq G} : B^+ \to B_+$ is a morphism of $k$-Mackey functors on $G$ (cf. Remark 3.3 (ii)), and we obtain $k$-homomorphisms

$$\sigma_{A,B} : k\text{-}\text{Res}(G)(A, B^+) \to k\text{-}\text{Res}(G)(A, B_+),$$

$$(f_H)_{h \leq G} \mapsto ((G : H) \cdot \sigma_H^B \circ f_H)_{h \leq G},$$

and

$$\rho_{A,B} : k\text{-}\text{Res}(G)(A, B_+) \to k\text{-}\text{Res}(G)(A, B^+),$$

$f \mapsto \rho^B \circ f,$

with $\sigma_{A,B} \circ \rho_{A,B} = \text{id}$ and $\rho_{A,B} \circ \sigma_{A,B} = \text{id}$ by Proposition 3.2. The composition $\theta_{A,B} := \lambda_{A,B} \circ \rho_{A,B}$ with $\lambda_{A,B}$ from Proposition 4.1 and $X$ replaced with $B$ is the $k$-homomorphism

$$\theta_{A,B} : k\text{-}\text{Res}(G)(A, B_+) \to k\text{-}\text{Con}(G)(A, B),$$

$f \mapsto \pi^B \circ f,$

since $pr_H^B \circ \rho_{H}^B = \pi_H^B$ for $H \leq G$. We call $\theta_{A,B}(f) = \pi^B \circ f$ the **residue** of $f \in k\text{-}\text{Res}(G)(A, B_+)$; cf. 2.6 and 3.1 for the terminology.

Note that throughout the preceding paragraph we can replace $k\text{-}\text{Res}(G)$ by $k\text{-}\text{Res}_{\text{alg}}(G)$ and $k\text{-}\text{Con}(G)$ by $k\text{-}\text{Con}_{\text{alg}}(G)$.
In Chapter II we will be interested in elements of \(k-\text{Res}(G)(A, B_+)\) for \(A, B \in k-\text{Res}(G)\). The following corollary allows us to analyze such elements by their residues.

4.2 Corollary  
Let \(A, B \in k-\text{Res}(G)\).
(i) Let \(f \in k-\text{Res}(G)(A, B_+)\) and let \(g := \theta_{A,B}(f) = \pi^B \circ f \in k-\text{Con}(G)(A, B)\) be the residue of \(f\). Then the diagrams

\[
\begin{align*}
A(H) & \xrightarrow{f_H} B_+(H) \\
(g_{K \circ \text{res}_K^H)_{K \leq H} & \xrightarrow{} \rho_H^B = (\pi_K^B \circ \text{res}_K^H)_{K \leq H} B^+(H),
\end{align*}
\]

\(H \leq G\), are commutative.

(ii) If \(B_+(H)\) has trivial \(|G|\)-torsion for all \(H \leq G\), then \(\theta_{A,B}\) is injective and for \(f \in k-\text{Res}(G)(A, B_+)\) we have

\[|G| \cdot f = (\sigma_{A,B} \circ \lambda_{A,B})^{-1}(\theta_{A,B}(f)).\]

In particular, for \(H \leq G\) and \(a \in A(H)\), we have

\[|H| \cdot f_H(a) = \sum_{L \leq K \leq H} |L| \mu(L, K) \left[ L, \text{res}_K^H \left( \theta_{A,B}(f)(\text{res}_K^H(a)) \right) \right]_H,\]

where \(\mu\) denotes the Möbius function of the poset of subgroups of \(G\).

(iii) If \(|G|^{-1} \in k\), then \(\theta_{A,B}\) is an isomorphism with inverse \(|G|^{-1} \cdot (\sigma_{A,B} \circ \lambda_{A,B})^{-1}\).

In particular, for \(g \in k-\text{Con}(G)(A, B)\), \(H \leq G\), and \(a \in A(H)\), we have

\[\theta_{A,B}^{-1}(g)_H(a) = \frac{1}{|H|} \sum_{L \leq K \leq H} |L| \mu(L, K) \left[ L, \text{res}_K^H \left( g(\text{res}_K^H(a)) \right) \right]_H,\]

where \(\mu\) denotes the Möbius function of the poset of subgroups of \(G\).

(iv) Assume that \(A\) and \(B\) are \(k\)-algebra restriction functors on \(G\) and that \(\rho^B\) is injective. Let \(f \in k-\text{Res}(G)(A, B_+)\) and let \(g := \theta_{A,B}(f) = \pi^B \circ f \in k-\text{Con}(G)(A, B)\) be the residue of \(f\). Then \(f \in k-\text{Res}_{\text{alg}}(G)(A, B_+)\) if and only if \(g \in k-\text{Con}_{\text{alg}}(G)(A, B)\).

**Proof**  
(i) For \(K \leq H \leq G\) we have \(\pi_K^B \circ \text{res}_K^H \circ f_H = \pi_K^B \circ f_K \circ \text{res}_K^H\).

(ii) \(\theta_{A,B} = \lambda_{A,B} \circ \rho_{A,B}\) is injective, since \(\lambda_{A,B}\) is an isomorphism by Proposition 4.1 (ii) and \(\rho_{A,B}\) is injective by the relation \(\sigma_{A,B} \circ \rho_{A,B} = |G| \cdot \text{id}\). The two equations follow again from \(\sigma_{A,B} \circ \rho_{A,B} = |G| \cdot \text{id}\) and the definition of \(\sigma_H^B, H \leq G\), from (3.1). (iii) \(\theta_{A,B}\) is an isomorphism, since \(\lambda_{A,B}\) and \(\rho_{A,B}\) are isomorphisms by Proposition 4.1 (ii) and the relations \(\rho_{A,B} \circ \sigma_{A,B} = |G| \cdot \text{id}\) and \(\sigma_{A,B} \circ \rho_{A,B} = |G| \cdot \text{id}\). The explicit equation for \(\theta_{A,B}^{-1}\) is immediate from part (i).

(iv) Let \(f \in k-\text{Res}_{\text{alg}}(G)(A, B_+)\). Since \(\pi_H^B\) is a \(k\)-algebra homomorphism for \(H \leq G\), so is \(g_H = \pi_H^B \circ f_H\). Conversely, let \(g \in k-\text{Con}_{\text{alg}}(G)(A, B)\). Since \(\rho_H^B\) is injective and a \(k\)-algebra homomorphism for \(H \leq G\), it suffices to show that \(\rho_H^B \circ f_H\) is a \(k\)-algebra homomorphism. But this follows from the diagram in part (i), since restrictions are \(k\)-algebra homomorphisms. \(\Box\)
1.5 Scalar extension

In Proposition 3.2 and Corollary 4.2 we have seen that it is of advantage to have \(|G|\) invertible in \(k\). We can place ourselves in this situation by considering the localization \(k[[G^{-1}]]\) of \(k\) at the multiplicative set \(\{1, |G|, |G|^2, \ldots\} \subseteq k\) and assuming that \(|G|\) is not a zero divisor in \(k\), so that we also may consider \(k\) as a subring of \(k[[G^{-1}]]\). Therefore we will have a closer look on the functor of scalar extension. Let \(k\) be a subring (with the same unity) of the commutative ring \(k'\) throughout this section.

For \(X \in k-\text{Con}(G)\) (resp. \(X \in k-\text{Con}_{\text{alg}}(G)\)) there is a canonical morphism (i.e. natural in \(X\)) of \(k'\)-Mackey functors (resp. \(k'\)-Green functors) on \(G\),

\[
k' \otimes_k X^+ \to (k' \otimes_k X)^+,
\]

which maps \(\alpha \otimes_k (x_K)_{K \leq H} \in k' \otimes_k X^+(H)\) to \((\alpha \otimes_k x_K)_{K \leq H} \in (k' \otimes_k X)^+(H)\), for \(H \leq G\). Moreover there is a canonical morphism of \(k\)-Mackey functors (resp. \(k\)-Green functors) on \(G\), \(X^+ \to k' \otimes_k X^+\), which maps \((x_K)_{K \leq H} \in X^+\) for \(H \leq G\) to \(1 \otimes_k (x_K)_{K \leq H} \in k' \otimes_k X^+(H)\).

For \(A \in k-\text{Res}(G)\) (resp. \(A \in k-\text{Res}_{\text{alg}}(G)\)) there is a canonical morphism (i.e. natural in \(A\)) of \(k'\)-Mackey functors (resp. \(k'\)-Green functors) on \(G\),

\[
k' \otimes_k A_+ \to (k' \otimes_k A)_+,
\]

which maps \(\alpha \otimes_k (a+I(kH) \cdot S_A(H)) \in k' \otimes_k A_+(H)\) to \((\alpha \otimes_k a)+I(kH) \cdot S_{k'} \otimes_A(H) \in (k' \otimes_k A)_+(H)\), for \(H \leq G\). Moreover there is a canonical morphism (natural in \(A\)) of \(k\)-Mackey functors (resp. \(k\)-Green functors) on \(G\), \(A_+ \to k' \otimes_k A_+\), which maps \(a + I(kH) \cdot S_A(H) \in A_+(H)\) for \(H \leq G\) to \(1 \otimes_k (a + I(kH) \cdot S_A(H)) \in k' \otimes_k A_+(H)\).

It is easy to see that the diagram

\[
\begin{array}{ccc}
A_+ & \longrightarrow & k' \otimes_k A_+ \\
\rho^A \downarrow & & \downarrow \rho^{k' \otimes_k A} \\
A^+ & \longrightarrow & k' \otimes_k A^+ \\
\end{array}
\]

is commutative.

5.1 Lemma Let \(k \subseteq k'\) be commutative rings with the same unity, \(X \in k-\text{Con}(G)\), and \(A \in k-\text{Res}(G)\).

(i) If \(k'\) is \(k\)-flat, then the canonical morphism \(k' \otimes_k X^+ \to (k' \otimes_k X)^+\) is an isomorphism.

(ii) The canonical morphism \(k' \otimes_k A_+ \to (k' \otimes_k A)_+\) is always an isomorphism.

(iii) If \(k'\) is \(k\)-flat and \(X(H)\) is \(k\)-free for all \(H \leq G\), then the canonical morphism \(X^+ \to k' \otimes_k X^+\) is injective.

(iv) If \(A_+(H)\) is \(k\)-free for all \(H \leq G\), then the canonical morphism \(A_+ \to k' \otimes_k A_+\) is injective. In particular, if \(A\) has a stable basis, then \(A_+ \to k' \otimes_k A_+\) is injective.

Proof (i) Let \(M\) be any \(kG\)-module. The \(k\)-module \(M^G\) of \(G\)-fixed points is naturally isomorphic to \(\text{Hom}_{kG}(k, M)\), where \(k\) is the trivial \(kG\)-module. Moreover,
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1.6. Invertible group order

Let $k' \cong \text{Hom}_k(k', k')$, and we obtain a sequence of natural isomorphisms

$$k' \otimes_k M^G \cong \text{Hom}_k(k', k') \otimes_k \text{Hom}_k(k, M) \cong \text{Hom}_{k' \otimes_k k'}(k', k' \otimes_k M) \cong \text{Hom}_{k' \otimes_k k'}(k', k' \otimes_k M)^G,$$

where the second map, given by $f \otimes g \mapsto f \otimes g$ for $f \in \text{Hom}_k(k', k')$, $g \in \text{Hom}_k(k, M)$, is an isomorphism, since $k'$ is $k$-flat and $k$ is finitely presented as $kG$-module (cf. [KO74, Lemma I.4.1 (b)]). The resulting isomorphism maps $\alpha \otimes_k m \in k' \otimes_k M^G$ to $\alpha \otimes_k m \in (k' \otimes_k M)^G$. If we set $M = P_X(H)$ and replace $G$ with $H$, for $H \leq G$, these isomorphisms form the canonical morphism $k' \otimes_k X^+ \rightarrow (k' \otimes_k X)^+$. 

(i) Let $M$ be any $kG$-module. The $k$-module $M_G$ of $G$-cofixed points (which are defined as $M_G := M/I_M$, where the $I_M := I(kG) \cdot M = \{m - gm \mid m \in M, g \in G\}$) is naturally isomorphic to $k \otimes_k G M$, where $k$ is the trivial $kG$-module. We obtain a sequence of natural isomorphisms

$$k' \otimes_k M_G \cong k' \otimes_k (k \otimes_k G M) \cong k' \otimes_{k'G} (k' \otimes_k M) \cong (k' \otimes_k M)_G,$$

where the middle isomorphism is given by $\alpha' \otimes_k (\alpha \otimes_k G m) \mapsto \alpha' \otimes_{k'G} (\alpha \otimes_k k')$ with inverse $\alpha' \otimes_k G (\beta' \otimes_k m) \mapsto \alpha' \beta' \otimes_k (1 \otimes_k G m)$ for $\alpha \in k$, $\alpha', \beta' \in k'$, $m \in M$. The resulting isomorphism maps $\alpha' \otimes_k (m + I_M)$ to $(\alpha' \otimes_k m) + I_{k' \otimes_k M}$ for $\alpha' \in k'$, $m \in M$. If we set $M = S_A(H)$ and replace $G$ with $H$, for $H \leq G$, these isomorphisms form the canonical morphism $k' \otimes_k A_+ \rightarrow (A' \otimes_k A)_+$. 

(ii) This follows from the commutativity of the diagram

$$
\begin{array}{ccc}
X^+(H) & \cong & k \otimes_k \left( \prod_{K \leq H} X(K) \right)^H \\
\downarrow & & \downarrow \\
K \otimes_k \prod_{K \leq H} X(K) & \longrightarrow & k' \otimes_k \prod_{K \leq H} X(K)
\end{array}
$$

by the injectivity of the vertical maps ($k'$ is $k$-flat) and the injectivity of the lower horizontal map ($\prod_{K \leq H} X(K)$ is $k$-flat and $k \rightarrow k'$ is injective), for $H \leq G$.

(iv) The first statement is obvious and the second follows from Proposition 2.4. □

5.2 Corollary Let $k \subseteq k'$ be as in Lemma 5.1 and let $A \in k - \text{Res}(G)$. If $A$ has a stable basis and $k'$ is $k$-flat, then the left horizontal morphisms in diagram (1.1) are injective, the right horizontal morphisms are isomorphisms, and the vertical morphisms are injective (cf. Propositions 2.4 and 3.2.) □

1.6 Invertible group order

In this section we prove miscellaneous results about $k$-Mackey functors on $G$ in the case that $|G|$ is invertible in $k$.

6.1 Let $|G|$ be invertible in $k$. We identify the Burnside ring functor $k \otimes \Omega$ with $k_+$ as in Example 2.3. Since $|G|$ is invertible in $k$ we know from Proposition 3.2 that
the mark morphism $\rho^k_K: \mathbb{k}_+ \rightarrow \mathbb{k}_+$ is an isomorphism of $k$-Green functors on $G$. For $H \leq G$ and for any set $\mathcal{R}_H$ of representatives for the conjugacy classes of subgroups of $H$, the set $\{e^{(H)}_K | K \in \mathcal{R}_H\}$ of mutually orthogonal primitive idempotents of $k \otimes \Omega(H)$, whose sum is the unity element, is also a $k$-basis of $k \otimes \Omega(H)$. From Remark 3.3 (i) we recall the explicit formula
\[
e^{(H)}_K = \frac{1}{|N_H(K)|} \sum_{L \leq K} |L| \mu(L, K) [H/L],
\]for $K \leq H$ (see also Appendix A).

In the sequel we will use for any $k$-Mackey functor $M$ the canonical pairing
\[(k \otimes \Omega) \otimes_k M \rightarrow M, \quad [H/K] \otimes_k m \mapsto \text{ind}^H_K(\text{res}^H_K(m)),\]
for $K \leq H \leq G$, $m \in M(H)$, which was defined in Proposition 1.4.

6.2 Proposition ([Dr73, Theorem 2], [Yo83a, Theorem 4.1']) Let $G$ be invertible in $k$ and let $M \in k$-Mack$(G)$ (resp. $M \in k$-Mack$_{\text{alg}}(G)$). For $H \leq G$ we have a decomposition into $k$-submodules (resp. ideals)
\[M(H) = e^{(H)}_H \cdot M(H) \oplus (1 - e^{(H)}_H) \cdot M(H),\]
and the summands are given by (cf. 2.5 and 2.6)
\[e^{(H)}_H \cdot M(H) = \mathcal{K}(M)(H) \quad \left(= \bigcap_{K < H} \ker(\text{res}^H_K: M(H) \rightarrow M(K)) \right)\]
and
\[(1 - e^{(H)}_H) \cdot M(H) = \mathcal{I}(M)(H) \quad \left(= \sum_{K < H} \text{im}(\text{ind}^H_K: M(K) \rightarrow M(H)) \right).\]

In particular, $M$ decomposes as $k$-conjugation functor (resp. $k$-algebra conjugation functor) on $G$ into $M = \mathcal{K}(M) \oplus \mathcal{I}(M)$, and $\mathcal{C}(M) = \mathcal{P}(M)$.

Moreover, for $H \leq G$ we have $H \in \mathcal{C}(M)$ if and only if $e^{(H)}_H \cdot M(H) \neq 0$, and for an inclusion $M \subseteq N$ of $k$-Green functors on $G$ we have $\mathcal{C}(M) = \mathcal{C}(N)$.

Proof Since the pairing $(k \otimes \Omega) \otimes_k M \rightarrow M$ provides $M(H)$ with a $k \otimes \Omega(H)$-module structure, we clearly have the decomposition $M(H) = e^{(H)}_H \cdot M(H) \oplus (1 - e^{(H)}_H) \cdot M(H)$ for $H \leq G$, and the summands are ideals in $M(H)$, if $M$ is a $k$-Green functor on $G$.

For the proof of the remaining equations we first claim that for $K < H \leq G$ we have $\text{res}^H_K(e^{(H)}_H) = 0$. In fact, it is enough to show that $(\rho^k_K \circ \text{res}^H_K)(e^{(H)}_H) \in \mathbb{k}_+(K)$ is trivial. But $\rho^k_K$ commutes with restrictions, and the definition of $e^{(H)}_H$ via its image under $\rho^k_K$ together with the definition of restriction for $\mathbb{k}_+$ proves the claim.

Using this we see that $\text{res}^H_K(e^{(H)}_H \cdot M(H)) = \text{res}^H_K(e^{(H)}_H) \cdot \text{res}^H_K(M(H)) = 0$ for all $K < H \leq G$. Conversely, the explicit formula (1.2) yields
\[1 - e^{(H)}_H = - \frac{1}{|H|} \sum_{K < H} |K| \mu(K, H) [H/K],\]
and the definition of the canonical pairing shows that $\bigcap_{K < H} \ker(\res_K^H : M(H) \to M(K))$ is annihilated by $1 - e_H^{(H)}$, and therefore contained in $e_H^{(H)} \cdot M(H)$ by the decomposition of $M(H)$. This proves the first remaining equation.

Again the definition of the canonical pairing and the above formula for $1 - e_H^{(H)}$ shows that $(1 - e_H^{(H)}) \cdot M(H)$ is contained in $\mathcal{I}(M)(H)$ for $H \leq G$. Conversely, for the inclusion $\mathcal{I}(M)(H) \subseteq (1 - e_H^{(H)}) \cdot M(H)$, it suffices to show that $e_H^{(H)} \cdot \ind_K^H(M(K)) = 0$ for $K < H \leq G$. But this is immediate from the Frobenius axiom (P3) in Definition 1.3 and the claim above:

$$e_H^{(H)} \cdot \ind_K^H(M(K)) = \ind_K^H(\res_K^H(e_H^{(H)}) \cdot M(K)) = 0.$$ 

Now it follows immediately that $M$ decomposes into $M = \mathcal{K}(M) \oplus \mathcal{I}(M)$ as $k$-conjugation functor on $G$, that $\mathcal{C}(M) = \mathcal{P}(M)$, and that $H \leq \mathcal{C}(M)$ if and only if $e_H^{(H)} \cdot M(H) \neq 0$.

If $M \subseteq N$ is an inclusion of $k$-Green functors on $G$ and $H \leq G$, then

$$H \in \mathcal{C}(M) \iff e_H^{(H)} \cdot M(H) \neq 0 \iff e_H^{(H)} \cdot 1_{M(H)} \neq 0 \iff e_H^{(H)} \cdot 1_{N(H)} \neq 0 \iff e_H^{(H)} \cdot N(H) \neq 0 \iff H \in \mathcal{C}(N).$$

This completes the proof.

**6.3 Corollary** Let $|G|$ be invertible in $k$, $M \in k-\text{Mack}(G)$, $H \leq G$, and $m \in M(H)$. Then $m = 0$ if and only if $e_H^{(H)} \cdot m = 0$ and $\res_K^H(m) = 0$ for all $K < H$.

**Proof** One implication is trivial. So we assume that $e_H^{(H)} \cdot m = 0$ and $\res_K^H(m) = 0$ for all $K < H$. Then by Proposition 6.2, the element $m$ lies in $(1 - e_H^{(H)}) \cdot M(H)$, since $m$ is annihilated by $e_H^{(H)}$, and simultaneously in $\mathcal{K}(M)(H) = e_H^{(H)} \cdot M(H)$. This implies $m = 0$.

**6.4 Proposition** Let $|G|$ be invertible in $k$, and let $M$ be a $k$-Green functor on $G$. Furthermore, let $i \in k-\text{Mack}_{\text{alg}}(G)(k \otimes \Omega, M)$ be the unique morphism of $k$-Green functors on $G$ (cf. Proposition 1.4).

(i) Assume that $H \leq G$ is not contained in any $H' \in \mathcal{C}(M)$. Then $i_G(e_H^{(G)}) = 0$.

(ii) Assume that $M(H)$ is $k$-torsion free for $H \leq G$. Then, the elements $i_G(e_H^{(G)})$, where $H$ runs through a set of representatives for the $G$-conjugacy classes of $\mathcal{C}(M)$, are $k$-linearly independent in $M(G)$.

**Proof** Note that by Proposition 1.4 we have

$$i_H(x) \cdot m = x \cdot m,$$

for $H \leq G$, $x \in k \otimes \Omega(H)$, and $m \in M(H)$, where the first dot denotes multiplication in $M(H)$, and the second dot denotes the canonical pairing.

(i) Let $H \leq G$. Since $\mathcal{P}(M) = \mathcal{C}(M)$ by Proposition 6.2, we have

$$1_{M(G)} \in \sum_{K \in \mathcal{C}(M)} \ind_K^G(M(K)).$$
The Frobenius axiom (P3) in Definition 1.3 implies
\[
    i_G(e_H^{(G)}) = e_H^{(G)} \cdot 1_{M(G)} \leq \sum_{K \in \mathcal{C}(M)} e_H^{(G)} \cdot \text{ind}^G_K(M(K))
\]
\[
    = \sum_{K \in \mathcal{C}(M)} \text{ind}^G_K(\text{res}^G_K(e_H^{(G)}) \cdot M(K))
\]
\[
    = 0,
\]
since \( \text{res}^G_K(e_H^{(G)}) = 0 \) by (A.4) in Appendix A.

(ii) Let \( \mathcal{C}_G \) be a set of representatives for the \( G \)-conjugacy classes of coprimordial subgroups for \( M \), and assume that
\[
    \sum_{H \in \mathcal{C}_G} \alpha_H \cdot i_G(e_H^{(G)}) = 0
\]
for elements \( \alpha_H \in k \), \( H \in \mathcal{C}_G \). If not all elements \( \alpha_H \), \( H \in \mathcal{C}_G \), are zero, we choose \( H_0 \in \mathcal{C}_G \) of minimal order such that \( \alpha_{H_0} \neq 0 \). Applying \( \text{res}^G_{H_0} \) to the above equation, we obtain
\[
    0 = \sum_{H \in \mathcal{C}_G} \alpha_H \cdot i_{H_0}(\text{res}^G_{H_0}(e_H^{(G)})) = \alpha_{H_0} \cdot i_{H_0}(e_{H_0}^{(H_0)})
\]
by (A.4) in Appendix A and by the minimality of \( |H_0| \). Since \( M(H_0) \) is \( k \)-torsion free, this implies \( i_{H_0}(e_{H_0}^{(H_0)}) = 0 \), and we obtain by Proposition 6.2 that
\[
    K(M)(H_0) = e_{H_0}^{(H_0)} \cdot M(H_0) = i_{H_0}(e_{H_0}^{(H_0)}) \cdot M(H_0) = 0,
\]
which contradicts \( H_0 \in \mathcal{C}(M) \).

\[\] 6.5 Proposition \ Let \( |G|^{-1} \in k \). Then the functors \( -^+ : k{-}\text{Con}(G) \rightarrow k{-}\text{Mack}(G) \) and \( \overline{7} : k{-}\text{Mack}(G) \rightarrow k{-}\text{Con}(G) \) are inverse equivalences. The same functors are inverse equivalences between \( k{-}\text{Con}_{\text{alg}}(G) \) and \( k{-}\text{Mack}_{\text{alg}}(G) \).

**Proof** \ It has already been mentioned in 2.8 that Thévenaz proved in [Th88] that there is a natural isomorphism \( \beta^M : M \rightarrow \overline{M}^+ \) of \( k \)-Mackey functors (resp. \( k \)-Green functors) on \( G \) for \( M \in k{-}\text{Mack}(G) \) (resp. \( M \in k{-}\text{Mack}_{\text{alg}}(G) \)).

Conversely, let \( X \in k{-}\text{Con}(G) \) (resp. \( X \in k{-}\text{Con}_{\text{alg}}(G) \)). By Proposition 6.2 we have a natural isomorphism \( \overline{X}^+ = X^+/\overline{X}(X^+) \cong K(X^+) \) of \( k \)-conjugation functors (resp. \( k \)-algebra conjugation functors) on \( G \). Moreover, the definition of the restriction maps for \( M^+ \) implies that
\[
    K(X^+)(H) = \{(x_K)_{K \leq H} \in \prod_{K \leq H} X(K) | x_K = 0 \text{ for } K < H\}
\]
for \( H \leq G \). Hence, \( K(X^+)(H) \cong X(H) \), and \( K(X^+) \) is naturally isomorphic to \( X \) as \( k \)-conjugation functor (resp. \( k \)-algebra conjugation functor) on \( G \). Altogether \( \overline{X}^+ \) is naturally isomorphic to \( X \) in \( k{-}\text{Con}(G) \) (resp. \( k{-}\text{Con}_{\text{alg}}(G) \)).

\[\] 6.6 Remark \ Let \( k \) be an arbitrary commutative ring (without assumption on \( |G| \)).
(i) Let $R_G$ be a set of representatives for the conjugacy classes of subgroups of $G$. Then the functor

$$k^{-\text{Con}}(G) \to \prod_{H \in R_G} k[N_G(H)/H]-\text{Mod}$$

which associates to $X \in k^{-\text{Con}}(G)$ the family $X(H)$, $H \in R_G$, endowed with $k[N_G(H)/H]$-module structures via the conjugation maps, is obviously an equivalence. Hence, the category $k^{-\text{Mack}}(G)$ is semisimple by Proposition 6.4 in the case that $|G|$ is invertible in $k$.

Similarly the category $k^{-\text{Con}_{\text{alg}}}(G)$ is equivalent to the direct product of the categories of $N_G(H)/H$-algebras over $k$, $H \leq R_G$, i.e. $k$-algebras $A$ together with a group homomorphism $N_G(H)/H \to \text{Aut}(A)$. Hence, in the case of invertible group order, the category of $k$-Green functors on $G$ is exactly as complicated as the various categories of $N_G(H)/H$-algebras over $k$.

(ii) There is a functor $F: k^{-\text{Con}}(G) \to k^{-\text{Res}}(G)$ which defines restriction maps on a $k$-conjugation functor $X$ on $G$ by $\text{res}^H_K X = \text{id}_{X(H)}$ and $\text{res}^H_K = 0$ for $K < H \leq G$. The composition of $F$ with the forgetful functor $k^{-\text{Res}}(G) \to k^{-\text{Con}}(G)$ is equal to the identity functor. But in general these functors are not inverse equivalences. If we restrict our attention to the full subcategories of $k^{-\text{Con}}(G)$ (resp. $k^{-\text{Res}}(G)$) consisting of objects $X$ (resp. $A$) with the property that $X(H)$ (resp. $A(H)$) has a finite composition series as $k$-module, for $H \leq G$, then the Grothendieck groups of these two subcategories are isomorphic via the maps induced by the functors defined above. In fact, they induce a bijection between the isomorphism classes of simple objects. Note that $X \in k^{-\text{Con}}(G)$ (resp. $A \in k^{-\text{Res}}(G)$) is simple if and only if there is some $H \leq G$ such the $X(H)$ (resp. $A(H)$) is a simple $kN_G(H)/H$-module and $X(H') = 0$ (resp. $A(H') = 0$) for all $H' \leq G$ which are not conjugate to $H$. 
Chapter 2

Canonical Induction Formulae

This chapter is arranged as follows. In Section 1 we define the general notion of a canonical induction formula \( a : M \to A_+ \) for a Mackey functor \( M \) on a finite group \( G \) from a restriction subfunctor \( A \subseteq M \). Moreover the residue \( p := \pi^A \circ a \) of a canonical induction formula \( a \) is defined. In Section 2 we consider the case where \( |G| \) is invertible in the base ring \( k \). In this case we have a bijection between the set of morphisms \( a : M \to A_+ \) of \( k \)-restriction functors and the set of morphisms \( p : M \to A \) of \( k \)-conjugation functors which is given by \( a \mapsto p := \pi^A \circ a \). Note that it is much easier to define a morphism \( p : M \to A \) of conjugation functors than a morphism \( a : M \to A_+ \) of restriction functors. We relate the properties of \( a \) to properties of \( p \). In Section 3 we consider for \( k = \mathbb{Z} \) in Hypothesis 3.1 the situation where \( M \) is \( \mathbb{Z} \)-free and \( A \) has a stable basis \( \mathcal{B} \) whose positive span is stable under restriction. This condition which we also call the standard situation will always be satisfied in the examples we are interested in, and it implies that \( A_+(H) \) is \( \mathbb{Z} \)-free for \( H \leq G \). Hence, we may imbed \( M, A, \) and \( A_+ \) into \( \mathbb{Q} \otimes M, \mathbb{Q} \otimes A, \) and \( \mathbb{Q} \otimes A_+ \) and apply the results obtained in Section 2 to the morphism \( a^{M,A,p} : \mathbb{Q} \otimes M \to A_+ \) of \( \mathbb{Q} \)-restriction functors which is associated to a morphism \( p : \mathbb{Q} \otimes M \to \mathbb{Q} \otimes A \) of \( \mathbb{Q} \)-conjugation functors. Using results from Chapter I we obtain an explicit formula for \( a^{M,A,p} \) in terms of \( p \), cf. Definition 3.4. In general, even if \( p_H(M(H)) \subseteq A(H) \) for \( H \leq G \), the morphism \( a^{M,A,p} \) is not necessarily integral, i.e. \( a^{M,A,p}_H(M(H)) \subseteq A_+(H) \) for \( H \leq G \). Section 4 is devoted to this integrality problem. Starting with the explicit formula for \( a^{M,A,p} \) in terms of \( p \) and applying suitable rearrangements of the summands in this formula, we succeed in deriving an expression for \( a^{M,A,p} \) which allows to state a condition \( (*)_\pi \) on \( p \) in Theorem 4.5 such that this condition implies \( |H|_\pi \cdot a^{M,A,p}_H(M(H)) \subseteq A_+(H) \) for \( H \leq G \) and any set \( \pi \) of primes. Luckily, condition \( (*)_\pi \) can be verified in almost all examples we are interested in. Note that the proof of the integrality of the canonical Brauer induction formula in [Bo89] or [Bo90] was quite elaborate and limited to the situation of the character ring Mackey functor. Theorem 4.5 and the statement of condition \( (*)_\pi \) may be considered of the heart of Chapter I and II.
CHAPTER 2. CANONICAL INDUCTION FORMULAE

2.1 The definition

Throughout this section let \( k \) be a commutative ring, \( G \) a finite group and \( M \) a \( k \)-Mackey functor on \( G \).

1.1 Definition For any \( k \)-restriction subfunctor \( A \) of \( M \) on \( G \) (i.e. \( A(H) \), \( H \leq G \), are stable under conjugation and restriction), the inclusion \( A \subseteq M \) is an element of \( k - \text{Res}(G)(A,M) \) and corresponds via the adjointness in Proposition I.4.1 (i) to a morphism \( b^{M,A} : A_+ \to M \) of \( k \)-Mackey functors on \( G \) which is given for \( H \leq G \) by

\[
b^{M,A}_H : A_+(H) \to M(H), \quad [K,a]_H \mapsto \text{ind}_K^H(a),
\]

where \( K \leq H \) and \( a \in A(K) \). We will call \( b^{M,A} \) the induction morphism of \( M \) from \( A \).

1.2 Remark For \( k = \mathbb{Z} \), \( M \) the character ring functor \( R \), and \( A \) the \( \mathbb{Z} \)-restriction subfunctor \( R_{ab} \) consisting of the \( \mathbb{Z} \)-span of one-dimensional characters, the map \( b^{M,A}_H : A_+(H) \to M(H) \) is surjective for \( H \leq G \) by Brauer’s induction theorem (which explains the choice of the letter \( b \), cf. [Se78, Théorème 20]). For fixed \( \chi \in M(H) \), \( H \leq G \), the ‘different ways’ of writing \( \chi \) as a linear combination of induced one-dimensional characters correspond bijectively to the set of elements in \( A_+(H) \) which are mapped to \( \chi \) under \( b^{M,A}_H \). Here we identify two such linear combinations, if one arises from the other by replacing subgroups and one-dimensional characters with \( H \)-conjugate subgroups and the corresponding \( H \)-conjugate one-dimensional characters (which doesn’t change the induced one-dimensional character).

If we want to specify for each \( \chi \in M(H) \), \( H \leq G \), a preferred way of writing \( \chi \) via Brauer’s induction theorem, this amounts to defining a map \( a_H : M(H) \to A_+(H) \) for \( H \leq G \) such that \( b^{M,A}_H \circ a_H = \text{id}_{M(H)} \) for \( H \leq G \). Knowing that \( A_+ \) and \( M \) are Mackey functors on \( G \) and \( b^{M,A} \) is a morphism of Mackey functors, it is just natural to require that \( a \) is a morphism of Mackey functors on \( G \). But – at least in this example – nature doesn’t allow the existence of such a section \( a \) for \( b^{M,A}_H \) in the category of \( k \)-Mackey functors on \( G \) by the following reason:

Assume that \( a : M \to A_+ \) is a morphism of \( k \)-Mackey functors on \( G \) with \( b^{M,A}_H \circ a = \text{id}_M \). Since \( a \) then commutes with induction maps, Artin’s induction theorem (see [CR81, 15.4]), namely

\[
|H| \cdot M(H) \subseteq \sum_{K \leq H \atop K \text{ cyclic}} \text{ind}_K^H(M(K)),
\]

for \( H \leq G \), implies that

\[
|H| \cdot a_H(M(H)) \subseteq \sum_{K \leq H \atop K \text{ cyclic}} \text{ind}_K^H(A_+(K))
\]

for \( H \leq G \). Recalling the definition of \( \text{ind}_K^H \) for \( K \leq H \leq G \) from I.2.2 we observe from this inclusion that \( |H| \cdot a_H(M(H)) \) is contained in the span of the elements \( [K,\varphi]_H \), where \( K \leq H \) is a cyclic subgroup and \( \varphi \) is a one-dimensional character.
of $K$. Since these elements are part of a $\mathbb{Z}$-basis of $A_+(H)$ (cf. Lemma I.2.4), also $a_H(M(H))$ is contained in this span. But then, $b^{M,A}_H \circ a_H = \text{id}_{M(H)}$ implies that

$$M(H) = \sum_{K \leq H \text{ cyclic}} \text{ind}^H_K(A(K)),$$

which is certainly not true for arbitrary finite groups $H$.

Therefore, in general, we cannot hope for a section of $b^{M,A}$ in the category of Mackey functors, even if $b^{M,A}$ is surjective. One obstruction, as the above example shows, is the commutativity of the section with induction maps. Consequently, we just drop this requirement.

1.3 Definition Let $A \subseteq M$ be a $k$-restriction subfunctor on $G$ and let $b^{M,A}: A_+ \to M$ be the induction morphism of $M$ from $A$ (see Definition 1.1). A morphism $a \in k-\text{Res}(G)(M,A_+)$ with $b^{M,A} \circ a = \text{id}_M$ corresponds by the adjointness in Proposition I.4.1 (b) to a morphism $p := \pi^A \circ a = \theta_{M,A}(a) \in k-\text{Con}(G)(M,A)$ which we call the residue of $a$ (cf. the paragraph following Proposition I.4.1).

2.2 The case of invertible group order

Throughout this section we assume that $k$ is a commutative ring and $G$ is a finite group whose order is invertible in $k$.

2.1 For $M \in k-\text{Mack}(G)$ and a $k$-restriction subfunctor $A \subseteq M$ on $G$ we know from Corollary I.4.2 (iii) that the map $a \mapsto \pi^A \circ a$ induces a bijection between $k-\text{Res}(G)(M,A_+)$ and $k-\text{Con}(G)(M,A)$. If $a$ and $p = \pi^A \circ a$ are in correspondence, then $a$ can be recovered from $p$ by the the explicit formula (cf. Corollary I.4.2 (iii))

$$a_H(m) = \frac{1}{|H|} \sum_{L \leq K \leq H} |L| \mu(L, K) [L, (\text{res}_L^K \circ p_K \circ \text{res}_K^H)(m)]_H,$$

for $H \leq G$ and $m \in M(H)$, or by the commutative diagrams

$$
\begin{array}{ccc}
M(H) & \xrightarrow{a_H} & A_+(H) \\
(p_K \circ \text{res}_K^H)_{K \leq H} \downarrow & \cong & \downarrow \rho_H^A = (\pi^A \circ \text{res}_K^H)_{K \leq H} \\
A_+(H) & \xrightarrow{(p_K \circ \text{res}_K^H)_{K \leq H}} & A_+(H)
\end{array}
$$

(2.2)

for $H \leq G$, since $\rho_H^A$ is an isomorphism by our assumptions on $k$ and $G$ (cf. Proposition I.3.2).

If $a: M \to A_+$ is a canonical induction formula for $M$ from $A$ and $p: M \to A$ is its residue, we obtain from the explicit formula (2.1) in Remark 1.4 by applying $b^{M,A}_H$ the formula

$$m = \frac{1}{|H|} \sum_{L \leq K \leq H} |L| \mu(L, K) \text{ind}^H_L ((\text{res}_L^K \circ p_K \circ \text{res}_K^H)(m))$$

(2.3)
for $H \leq G$ and $m \in M(H)$, where $(\text{res}_L^K \circ p_K \circ \text{res}_K^H)(m)$ is an element of $A(L)$ for $L \leq K \leq H \leq G$, $m \in M(H)$.

The following proposition determines the subset of $k^{-\text{Con}}(G)(M,A)$ which consists of the residues of canonical induction formulae for $M$ from $A$, and in view of the above considerations this also determines the set of canonical induction formulae for $M$ from $A$. Recall the definition of the idempotent $e^{(H)}_H \in k \otimes \Omega(H)$ for $H \leq G$ from Appendix A and the decomposition $M(H) = e^{(H)}_H \cdot M(H) \oplus (1 - e^{(H)}_H) \cdot M(H)$ from Proposition I.6.2 with $e^{(H)}_H \cdot M(H) = \bigcap_{K<H} \ker(\text{res}_K^H \colon M(H) \to M(K))$ and $(1 - e^{(H)}_H) \cdot M(H) = \sum_{K<H} \text{ind}_K^H(M(K))$ for $H \leq G$.

2.2 Proposition let $M \in k^{-\text{Mack}}(G)$, $A \subseteq M$ a $k$-restriction subfunctor on $G$, $a \in k^{-\text{Res}}(G)(M,A)$, and $p := \pi^A \circ a \in k^{-\text{Con}}(G)(M,A)$ the residue of $a$. Then the following are equivalent:

(i) the morphism $a$ is a canonical induction formula, i.e. $b^{M,A} \circ a = \text{id}_M$.
(ii) We have $e^{(H)}_H \cdot (p_H(m) - m) = 0$, i.e. $p_H(m) - m \in \sum_{K<H} \text{ind}_K^H(M(K))$, for $H \leq G$, $m \in M(H)$.

Proof Using Corollary I.4.2 we can express $a$ in terms of $p$ and obtain

\[
e^{(H)}_H \cdot b^{M,A}_H(a_H(m)) = \frac{1}{|H|} e^{(H)}_H \cdot \sum_{L \leq K \leq H} |L| \mu(L,K) (\text{ind}_L^H \circ \text{res}_L^K \circ p_K \circ \text{res}_K^H)(m) \tag{2.4}
\]

for $H \leq G$, $m \in M(H)$, since $e^{(H)}_H$ annihilates all summands with $L < H$.

If $a$ is a canonical induction formula for $M$ from $A$, then Equation (2.4) shows that $e^{(H)}_H \cdot m = e^{(H)}_H \cdot p_H(m)$ for $H \leq G$, $m \in M(H)$.

Conversely, let us assume that $e^{(H)}_H \cdot m = e^{(H)}_H \cdot p_H(m)$ for $H \leq G$, $m \in M(H)$. By Corollary I.6.3 it suffices to show that

\[e_H \cdot b^{M,A}_H(a_H(m)) = e^{(H)}_H \cdot m \quad \text{and} \quad \text{res}_K^H(b^{M,A}_H(a_H(m))) = \text{res}_K^H(m)\]

for all $K < H \leq G$ and $m \in M(H)$. The first equation follows immediately from Equation (2.4) and the assumption, and the second equation follows by induction on $|H|$, since $b^{M,A}$ and $a$ commute with restrictions, and since the induction hypothesis for $K$ together with the first equation for $K$ implies $b^{M,A}_K \circ a_K = \text{id}_{M(K)}$. $\square$

We obtain as an easy corollary a necessary condition on the size of $A(H)$, $H \leq G$, for the existence of a canonical induction formula for $M$ from $A$.

2.3 Corollary Let $M$ and $A$ be as in the above proposition. If there exists a canonical induction formula for $M$ from $A$, then $e^{(H)}_H \cdot M(H) \subseteq e^{(H)}_H \cdot A(H)$ for all $H \leq G$.

We recall from I.(2.6) that the set $\mathcal{C}(M)$ of coprimordial subgroups of $G$ with respect to $M$ consists of those subgroups $H \leq G$ with $e^{(H)}_H \cdot M(H) \neq 0$ which is equivalent to $(1 - e^{(H)}_H) \cdot M(H) \neq M(H)$ by Proposition I.6.2.
2.4 Corollary Let $M \in k\text{-Mack}(G)$, and assume that $A \subseteq M$ is a $k$-restriction subfunctor on $G$. Furthermore, let $p \in k\text{-Con}(G)(M, A)$ with the property that $A(H) = M(H)$ and $p_H = \text{id}_{M(H)}$, for $H \in \mathcal{C}(G)$. Then $p$ is the residue of a canonical induction formula for $M$ from $A$.

Proof Condition (ii) in Proposition 2.2 is satisfied for $H \in \mathcal{C}(M)$, since in this case $p_H(m) - m = 0$ for $m \in M(H)$, and also for $H \notin \mathcal{C}(M)$, since $c_H(H) \cdot M(H) = 0$ in that case.

Next we determine the canonical induction formulae which are morphisms of Mackey functors; cf. Remark 1.2 for the impossibility of such formulae in an example with $k = \mathbb{Z}$.

2.5 Lemma Let $M$ be a $k$-Mackey functor on $G$, $A \subseteq M$ a $k$-restriction subfunctor on $G$, $a \in k\text{-Res}(G)(M, A_+)$, and $p := \pi^A \circ a \in k\text{-Con}(G)(M, A)$ the residue of $a$. Then the following statements are equivalent:

(i) $a \in k\text{-Mack}(G)(M, A_+)$.
(ii) $p_H(\text{ind}^H_K(M(K))) = 0$ for $K < H \leq G$.
(iii) $p(H((1 - e_H^H) \cdot M(H))) = 0$ for $H \leq G$.

Proof The statements (ii) and (iii) are equivalent by Proposition I.6.2. We will show that also (i) and (ii) are equivalent. The morphism $a$ is an element in $k\text{-Mack}(G)(M, A_+)$ if and only if

$$\text{ind}_K^H \circ a_K = a_H \circ \text{ind}_K^H : M(K) \to A_+(H)$$

for $K \leq H \leq G$. Since $\rho_H^A$, $H \leq G$ is injective, this is equivalent to

$$\pi_U^A \circ \text{res}_K^H \circ \text{ind}_K^H \circ a_K = \pi_U^A \circ \text{res}_K^H \circ a_H \circ \text{ind}_K^H : M(K) \to A(U)$$

for $H \leq G$ and $U, K \leq G$. We transform the left hand side by using the Mackey axiom and the commutativity of $a$ with restrictions and conjugations, and we transform the right hand side by using the commutativity of $a$ with restrictions, the Mackey axiom, and the relation $p_U = \pi_U^A \circ a_U$. Thus we obtain the equivalent condition

$$\sum_{h \in U \setminus H/K} \pi_U^A \circ \text{ind}_{U \cap K}^U \circ a_{U \cap K} \circ \text{res}_{U \cap K}^H \circ c_{h,K}$$

$$= \sum_{h \in U \setminus H/K} p_U \circ \text{ind}_{U \cap K}^U \circ \text{res}_{U \cap K}^H \circ c_{h,K} : M(K) \to A(U)$$

for $H \leq G$ and $K, U \leq H$. Since $\pi_U^A \circ \text{ind}_{U \cap K}^U = 0$ unless $U \leq hK$, this is equivalent to

$$\sum_{h \in U \setminus H/K \atop U \leq hK} p_U \circ \text{res}_{U \cap K}^h \circ c_{h,K}$$

$$= \sum_{h \in U \setminus H/K} p_U \circ \text{ind}_{U \cap K}^U \circ \text{res}_{U \cap K}^h \circ c_{h,K} : M(K) \to A(U)$$

(2.5)
for $H \leq G$ and $K, U \leq H$.

Now Equation (2.5) implies $0 = p_H \circ \text{ind}_K^H$ for $K < H \leq G$ by choosing $U = H$. Conversely, if $p_H \circ \text{ind}_K^H = 0$ for $K < H \leq G$, then Equation (2.5) holds, since the right hand side reduces to the left hand side.

2.6 Remark In the proof of the equivalence of (i) and (ii) in Lemma 2.5 we actually don’t need the assumption that $|G|$ is invertible in $k$, but only the fact that $\rho^A$ is injective. Hence the equivalence of (i) and (ii) still holds when $k$ is arbitrary and $A$ has a stable basis (cf. Proposition I.2.4) or even weaker when $A_+(H)$ has no $|H|$-torsion for $H \leq G$.

2.7 Corollary Let $M, A, a,$ and $p$ be as in the above lemma. Then $a$ is a canonical induction formula for $M$ from $A$ in the category of $k$-Mackey functors on $G$ if and only if

$$(p_H - \text{id}_{M(H)})(M(H)) \subseteq (1 - e_H^H) \cdot M(H) \subseteq \ker(p_H)$$

for $H \leq G$.

Proof This is an immediate consequence of Proposition 2.2 and Lemma 2.5.

2.8 Example For each $k$-Mackey functor $M$ on $G$ there is a canonical induction formula for $M$ of a minimal type (cf. Corollary 2.3), namely from $A(H) := e_H^H \cdot M(H)$ for $H \leq G$ (which form a $k$-restriction subfunctor of $M$ on $G$ with trivial restrictions $\text{res}_K^H$, $K < H \leq G$) and with $p_H : M(H) \to A(H)$ given by multiplication with $e_H^H$ for $H \leq G$. In fact, this choice of $M, A, p$ satisfies the condition of the above corollary. The corresponding induction formula $a \in k^{\text{Mack}}(G)(M, A_+)$ can be calculated explicitly from Equation (2.2) as

$$a_H(m) = \frac{1}{|H|} \sum_{K \leq H} |K| [K, e_K^K \cdot \text{res}_K^K(m)]_H$$

for $H \leq G, m \in M(H)$, by observing that $\text{res}_L^K$ is trivial on $\text{im}(p_K)$ for $L < K \leq G$, and $e_K^K \cdot M(K) = 0$ for $K \notin C(M)$. Moreover, from Equation (2.3) and the explicit formula (1.2) in 1.6.1 for $e_K(K), K \leq G$, we derive

$$m = \frac{1}{|H|} \sum_{L \leq K \leq H} |L| \mu(L, K) \text{ind}_L^H(\text{res}_L^K(m))$$

for $H \leq G, m \in M(H)$, which generalizes Brauer’s explicit version (cf. [Bra51] or [CR81, 15.4]) of Artin’s induction theorem for the character ring tensored with $\mathbb{Q}$, where $C(M)$ consists of the cyclic subgroups of $G$, and Conlon’s induction theorem for the Green ring and the ring of $p$-permutation modules tensored with $\mathbb{Q}$, where $C(M)$ consists of the $p$-hypo-elementary, i.e. cyclic mod $p$, subgroups of $G$ ($p$ denoting the prime characteristic of the base field), cf. [CR87, 80.61,81.32].

This kind of minimal example of a canonical induction formula was one of the main tools used in [BCS93] in the case of the character ring Mackey functor tensored with $\mathbb{Q}$. 
2.9 Proposition Let $M \in k\text{-Mack}_{\text{alg}}(G)$, $A \subseteq M$ a $k$-algebra restriction subfunctor on $G$, $a \in k\text{-Res}(G)(M, A_+)$, and $p := \pi^A \circ a \in k\text{-Con}(G)(M, A)$ the residue of $a$.

(i) The maps $a_H$, $H \leq G$, are $k$-algebra homomorphisms if and only if the maps $p_H$, $H \leq G$, are $k$-algebra homomorphisms.

(ii) The maps $a_H$, $H \leq G$, are $A(H)$-module homomorphisms if and only if the maps $p_H$, $H \leq G$, are $A(H)$-module homomorphisms, cf. the last paragraph in I.2.2.

Proof (i) This follows immediately from Corollary I.4.2 (iv).

(ii) The maps $a_H$ are $A(H)$-linear for all $H \leq G$ if and only if $a_H(a \cdot m) = [H, a]_H \cdot a_H(m)$ for all $H \leq G$, $a \in A(H)$, $m \in M(H)$. By the injectivity and multiplicativity of $\rho^A_H$ this is equivalent to $(\rho^A_H \circ a_H)(a \cdot m) = \rho^A_H([H, a]_H) \cdot \rho^A_H(a_H(m))$. Since $\rho^A_H = (\pi^K_+ \circ \text{res}^H_K)_{K \leq H}$, since $a$ commutes with restriction and $\pi^K_+ \circ a_H = p_H$ for $H \leq G$, this is equivalent to

$$p_K(\text{res}^H_K(a) \cdot \text{res}^H_K(m)) = \text{res}^H_K(a) \cdot p_K(\text{res}^H_K(m))$$

for $K \leq H \leq G$, $a \in A(H)$, $m \in M(H)$. But this is equivalent to the $A(H)$-linearity of $p_H$ for all $H \leq G$. \qed

2.3 The standard situation

In this section we will restrict our attention to a less general situation, which will all the same cover the examples we are interested in. Throughout this section we assume the following hypothesis.

3.1 Hypothesis Let $M$ be a $\mathbb{Z}$-Mackey functor on a finite group $G$ and $A \subseteq M$ a $\mathbb{Z}$-restriction subfunctor of $M$ on $G$ such that

(i) $M(H)$ is a free abelian group for $H \leq G$;

(ii) $A$ has a stable basis $\mathcal{B}(H) \subseteq A(H)$, $H \leq G$, such that, for $K \leq H \leq G$ and $\varphi \in \mathcal{B}(H)$, the element $\text{res}^H_K(\varphi)$ is a linear combination of the basis elements in $\mathcal{B}(K)$ with non-negative coefficients.

This hypothesis is clearly designed for the various representation rings, as for example character rings, Brauer character rings, projective character rings, trivial source rings, Green rings, etc. as candidates for $M$, and certain subrings, which we will specify later for the different examples, as candidates for $A$. In our standard example, where $M$ is the character ring and $A$ the span of one-dimensional characters, we clearly stay within the limits of this hypothesis.

As a convention, the letters $\chi$, $\vartheta$, $\xi$, $\zeta$ will always denote elements in $M(H)$, $H \leq G$, and $\varphi$, $\psi$, $\lambda$ will always denote elements in $\mathcal{B}(H) \subseteq A(H)$, $H \leq G$, as already done in Hypothesis 3.1. This should help the reader by referring to the standard example mentioned above.

3.2 Definition A pair $(H, \varphi)$ with $H \leq G$ and $\varphi \in \mathcal{B}(H)$ is called a monomial pair. For $H \leq G$, the set of monomial pairs $(K, \psi)$ with $K \leq H$ will be denoted by $\mathcal{M}(H)$. 
For $K \leq H \leq G$, $\varphi \in \mathcal{B}(H)$, $\psi \in \mathcal{B}(K)$ we define the multiplicity $m^{(H, \varphi)}_{(K, \psi)} \in \mathbb{N}$ to be the coefficient of $\psi$ in $\text{res}^H_K(\varphi)$. For $H \leq G$, we define the structure of an $H$-poset on $\mathcal{M}(H)$ by

$$(L, \lambda) \leq (K, \psi) : \iff L \leq K \text{ and } m^{(K, \psi)}_{(L, \lambda)} \neq 0,$$

$$h(K, \psi) : \iff (hK, h\psi).$$

Note that for the transitivity axiom (P3) in Appendix B we need the positivity condition in Hypothesis 3.1 (ii) We write $(L, \lambda) =_H (K, \psi)$ if $(L, \lambda)$ and $(K, \psi)$ lie in the same $H$-orbit, and we denote by $N_H(L, \lambda)$ the $H$-stabilizer of $(L, \lambda)$. The posets $\mathcal{M}(H)$, $H \leq G$, are subposets of the poset $\mathcal{M}(G)$.

In the following remark we recollect some results that have already been established in the situation of Hypothesis 3.1.

### 3.3 Remark

For $H \leq G$, $A_+(H)$ is a free $\mathbb{Z}$-module with basis $\{[K, \psi]|_H\}$, where $(K, \psi)$ runs through a set of representatives of the $H$-orbits $H \backslash \mathcal{M}(H)$ of $\mathcal{M}(H)$, cf. Proposition I.2.4. Therefore, $\mathbb{Q} \otimes A_+(H)$ has the same $\mathbb{Q}$-basis, when we write again $[K, \psi]|_H \in \mathbb{Q} \otimes A_+(H)$ instead of $1 \otimes [K, \psi]|_H$. We will always identify $(\mathbb{Q} \otimes A_+)(H)$ with $\mathbb{Q} \otimes A_+(H)$ via the natural map, cf. Lemma I.5.1 (ii). These natural maps form an isomorphism between the $\mathbb{Q}$-Mackey functors $(\mathbb{Q} \otimes A)_+$ and $\mathbb{Q} \otimes A_+$ on $G$. Hence, conjugation, restriction and induction maps are identified. Moreover the morphisms $\mathbb{Q} \otimes \pi^A$ and $\pi^{\mathbb{Q} \otimes A}$ are identified under this isomorphism. Similarly we have a natural isomorphism $\mathbb{Q} \otimes A^+ \to (\mathbb{Q} \otimes A)A^+$ of $\mathbb{Q}$-Mackey functors on $G$ such that the mark homomorphisms $\mathbb{Q} \otimes \rho^A$ and $\rho^{\mathbb{Q} \otimes A}$ are identified, cf. Diagram I.(5.1). We will work with the Mackey functors $\mathbb{Q} \otimes A_+$ and $\mathbb{Q} \otimes A^+$ and denote their structure maps again by $c_+$, res$_+$, ind$_+$, $c^+$, res$^+$, ind$^+$, $\pi^A$, $\rho^A$, etc. Note that the mark morphism $\rho^A_+ : A_+ \to A^+$ is injective and the mark morphism $\rho^A : (\mathbb{Q} \otimes A_+ \to \mathbb{Q} \otimes A^+$ is an isomorphism by Proposition I.3.2.

We have a commutative diagram

$$\begin{array}{ccc}
\mathbb{Q} \mathsf{Res}(G)(\mathbb{Q} \otimes M, \mathbb{Q} \otimes A_+) & \xrightarrow{\theta_{\mathbb{Q} \otimes M, \mathbb{Q} \otimes A}} & \mathbb{Q} \mathsf{Con}(G)(\mathbb{Q} \otimes M, \mathbb{Q} \otimes A) \\
\bigcup_{\mathbb{Z} \mathsf{Res}(G)(M, A_+)} & \xrightarrow{\theta_{M,A}} & \mathbb{Z} \mathsf{Con}(G)(M, A),
\end{array}$$

(2.6)

where the vertical inclusions are described by extensions of morphisms, $\mathbb{Q} \otimes A_+$ is identified with $(\mathbb{Q} \otimes A)_+$, and $\theta_{M,A}$, $\theta_{\mathbb{Q} \otimes M, \mathbb{Q} \otimes A}$ are the maps of taking residues (cf. the paragraph preceding Corollary I.4.2). By Corollary I.4.2, $\theta_{M,A}$ is injective and $\theta_{\mathbb{Q} \otimes M, \mathbb{Q} \otimes A}$ is an isomorphism. The set of canonical induction formulae for $M$ from $A$ is a subset of $\mathbb{Z} \mathsf{Res}(G)(M, A_+)$ and can be identified via $\theta_{M,A}$ with a subset of $\mathbb{Z} \mathsf{Con}(G)(M, A)$, namely the set of residues of canonical induction formulae for $M$ from $A$. We are going to find conditions on an arbitrary morphism $p \in \mathbb{Z} \mathsf{Con}(G)(M, A)$ which are equivalent to $p$ being the residue of a canonical induction formula for $M$ from $A$. Since we know an explicit formula for taking preimages under $\theta_{M,A}$ (cf. Corollary I.4.2 (iii)), we are able to construct a canonical induction formula for $M$ from $A$, whenever we find some $p \in \mathbb{Z} \mathsf{Con}(G)(M, A)$ satisfying these conditions.
3.4 Definition  For $p \in \mathbb{Q} - \text{Con}(G)(\mathbb{Q} \otimes M, \mathbb{Q} \otimes A)$ we define

$$a^{M,A,p} := \theta_{\mathbb{Q} \otimes M, \mathbb{Q} \otimes A}(p) \in \mathbb{Q} - \text{Res}(G)(\mathbb{Q} \otimes M, \mathbb{Q} \otimes A_+)$$

(cf. Diagram (2.6)), i.e. $a^{M,A,p}$ makes the diagrams

$$\begin{array}{ccc}
\mathbb{Q} \otimes M(H) & \xrightarrow{a^{M,A,p}} & \mathbb{Q} \otimes A_+(H) \\
(p_K \circ \text{res}_K^H)_{K \leq H} & \downarrow & \mu_H^{\otimes A} \\
\mathbb{Q} \otimes A^+(H) & & \\
\end{array}$$

$H \leq G$, commutative (cf. Diagram (2.2)), or more explicitly (cf. Equation (2.1)),

$$a_H^{M,A,p}(\chi) := \frac{1}{|H|} \sum_{L \leq K \leq H} |L| \mu(L, K) \left[ L, (\text{res}_L^K \circ p_K \circ \text{res}_K^H)(\chi) \right]_H$$

for $H \leq G$, $\chi \in \mathbb{Q} \otimes M(H)$. We call $a^{M,A,p}$ integral, if $a_H^{M,A,p}(\chi) \in A_+(H) \subseteq \mathbb{Q} \otimes A_+(H)$ for $H \leq G$, $\chi \in M(H)$. Since $\pi_A^H \circ a_H^{M,A,p} = p_H$ for $H \leq G$, a necessary condition for $a^{M,A,p}$ to be integral is that $p$ is the $\mathbb{Q}$-extension of a morphism in $\mathbb{Z} - \text{Con}(G)(M, A)$.

3.5 Proposition  Let $p \in \mathbb{Z} - \text{Con}(G)(M, A)$. Then $p$ is the residue of (a unique) canonical induction formula for $M$ from $A$, if and only if the following two conditions are satisfied:

(i) $a^{M,A,p}$ is integral.

(ii) For $H \leq G$ and $\chi \in M(H)$ we have

$$p_H(\chi) - \chi \in \mathbb{Q} \otimes \left( \sum_{K < H} \text{ind}_K^H(M(K)) \right) \subseteq \mathbb{Q} \otimes M(H).$$

If conditions (i) and (ii) are satisfied, then $a^{M,A,p}$ restricted to $M$ is the unique canonical induction formula for $M$ from $A$ with residue $p$.

Proof  In view of Diagram (2.2), condition (i) is equivalent to $p$ being in the image of $\theta_{M,A}$, i.e. $p = \pi_A \circ a$ for some $a \in \mathbb{Z} - \text{Res}(G)(M, A_+)$. This element $a$ now is a canonical induction formula for $M$ from $A$, if and only if $b^{M,A} \circ a = \text{id}_M$, which again is equivalent to $\mathbb{Q} \otimes b^{M,A} \circ \mathbb{Q} \otimes a = \text{id}_{\mathbb{Q} \otimes M}$. But this means that the residue of $\mathbb{Q} \otimes a$, which equals $\mathbb{Q} \otimes p$ by the commutativity of Diagram (2.6), satisfies condition (ii) for $H \leq G$ and $\chi \in \mathbb{Q} \otimes M(H)$ by Proposition 2.2. Since $M(H)$ is $\mathbb{Z}$-free for $H \leq G$, this is equivalent to condition (ii). \[\square\]

3.6 Remark  In the later examples, typically condition (ii) is easy to verify, but the integrality condition (i) is by no means trivial. In order to attack condition (i) we have to go much deeper into the study of the nature of the alternating sum involved. This will be done in the next section.

3.7 Lemma  Let $p \in \mathbb{Q} - \text{Con}(G)(\mathbb{Q} \otimes M, \mathbb{Q} \otimes A)$, $H \leq G$, and $\chi \in \mathbb{Q} \otimes M(H)$ with

$$\text{res}_K^H(p_H(\chi)) = p_K(\text{res}_K^H(\chi))$$
for all $K \leq H$. Then $a_{H}^{M,A,p}(\chi) = [H,p_{H}(\chi)]_{H}$.

**Proof** In view of the explicit definition of $a_{H}^{M,A,p}$ in Definition 3.4 we have to show that

$$\frac{1}{|H|} \sum_{L \leq K \leq H} |L|\mu(L,K)[L,\text{res}^{H}_{L}(p_{H}(\chi))]_{H} = [H,p_{H}(\chi)]_{H}.$$ 

But applying Möbius inversion in the second form of Corollary B.3 (i) to the function

$$\{K \leq H\} \rightarrow A_{+}(H), \quad K \mapsto [K][K,\text{res}^{H}_{K}(p_{H}(\chi))]_{H}$$

shows that the sum above collapses to $[H][H,p_{H}(\chi)]_{H}$, and the proof is complete. □

### 3.8 Definition
Let $A, A' \subseteq M$ be two $k$-restriction subfunctors on $G$ and $p: \mathbb{Q} \otimes M \rightarrow \mathbb{Q} \otimes A$, $p': \mathbb{Q} \otimes M \rightarrow \mathbb{Q} \otimes A'$ morphisms of $k$-conjugation functors on $G$. We call the two canonical induction formulae $a = a_{M,A,p}$ and $a' = a_{M,A',p'}$ **equivalent**, if they coincide as functions taking values in $\mathbb{Q} \otimes M_{+}$, i.e. if the two maps

$$\mathbb{Q} \otimes M \xrightarrow{a} \mathbb{Q} \otimes A_{+} \xrightarrow{i_{+}} \mathbb{Q} \otimes M_{+} \quad \text{and} \quad \mathbb{Q} \otimes M \xrightarrow{a'} \mathbb{Q} \otimes A'_{+} \xrightarrow{i'_{+}} \mathbb{Q} \otimes M_{+}$$

coincide, where $i: A \rightarrow M$ and $i': A' \rightarrow M$ denote the inclusion maps.

### 3.9 Lemma
Let $M$, $A$, $A'$, $p$, $p'$, $i$, $i'$, $a$, $a'$ be as in the above definition and assume that $M$ has a stable basis. Then $a$ and $a'$ are equivalent if and only if $i \circ p: \mathbb{Q} \otimes M \rightarrow \mathbb{Q} \otimes M$ and $i' \circ p': \mathbb{Q} \otimes M \rightarrow \mathbb{Q} \otimes M$ coincide.

**Proof** We first observe that $\pi^{M} \circ i_{+} \circ a = i \circ p$. In fact, since $\pi$ is natural (cf. I.(3.1)), we have $\pi^{M} \circ i_{+} = i \circ \pi^{A}: A_{+} \rightarrow M$, and the equation follows from $\pi^{A} \circ a = p$. Similarly we obtain $\pi^{M} \circ i'_{+} \circ a' = i' \circ p'$. Hence $i_{+} \circ a = i'_{+} \circ a'$ implies $i \circ p = i' \circ p'$. Conversely, if $i \circ p = i' \circ p'$, then $\pi^{M} \circ i_{+} \circ a = \pi^{M} \circ i'_{+} \circ a'$ and therefore $\pi_{K}^{M} \circ i_{+} \circ a_{K} \circ \text{res}^{H}_{K} = \pi_{K}^{M} \circ i'_{+} \circ a'_{K} \circ \text{res}^{H}_{K}$ for all $K \leq H \leq G$. Since $a, a', i_{+}, i'_{+}$ commute with restrictions, we obtain $\rho^{M} \circ i_{+} \circ a = \rho^{M} \circ i'_{+} \circ a'$ and $i_{+} \circ a = i'_{+} \circ a'$ by the injectivity of $\rho^{M}$ (cf. Propositions 2.4 and 3.2). □

### 3.10 Remark
Lemma 3.9 indicates that $A$ may vary without changing the equivalence class of the canonical induction formula. One may always substitute $p: \mathbb{Q} \otimes M \rightarrow \mathbb{Q} \otimes A$ by $p' := i \circ p: \mathbb{Q} \otimes M \rightarrow \mathbb{Q} \otimes A \rightarrow \mathbb{Q} \otimes M$ without changing the equivalence class. Hence, not the choice of $A \subseteq M$ is important, but the choice of $p$, which we can always consider as an endomorphism in $\mathbb{Q} \text{-} \text{Con}(G)(\mathbb{Q} \otimes M, \mathbb{Q} \otimes M)$. The elements which are induced in the resulting canonical induction formula, are contained in the image of $p$. Nevertheless it makes sense to choose a suitable $A$ to indicate a priori the sort of elements which we allow to be induced. Moreover, we will need to choose $A$ as small as possible in Chapter IV, where we will compose $a$ with functions on $\mathbb{Q} \otimes A$ which are not defined on $\mathbb{Q} \otimes M$.

### 3.11 Proposition
Let $M \in \mathbb{Z} \text{-} \text{Mack}(G)$ and let $A' \subseteq M'$ be a $\mathbb{Z}$-restriction subfunctor of $M$ with stable basis $B'$, and assume that $M'$, $A'$ and $B'$ satisfy Hypothesis 3.1. Furthermore let $p: \mathbb{Q} \otimes M \rightarrow \mathbb{Q} \otimes A$ and $p': \mathbb{Q} \otimes M' \rightarrow \mathbb{Q} \otimes A'$ be
morphism of \(Q\)-conjugation functors on \(G\) and denote by \(a: Q \otimes M \to Q \otimes A_+\) and \(a': Q \otimes M' \to Q \otimes A'_+\) the morphisms \(a^{M, A, p}\) and \(a^{M', A', p'}\) respectively. If \(f: Q \otimes M \to Q \otimes M'\) is a morphism of \(Q\)-restriction functors with \(f_H(Q \otimes A(H)) \subseteq Q \otimes A'(H)\) for \(H \leq G\), then the diagram

\[
\begin{array}{ccc}
Q \otimes M & \xrightarrow{a} & Q \otimes A_+ \\
f \downarrow & & \downarrow f_+ \\
Q \otimes M' & \xrightarrow{a'} & Q \otimes A'_+
\end{array}
\]

is commutative if and only if the diagram

\[
\begin{array}{ccc}
Q \otimes M & \xrightarrow{p} & Q \otimes A \\
f \downarrow & & \downarrow f \\
Q \otimes M' & \xrightarrow{p'} & Q \otimes A'
\end{array}
\]

is commutative.

**Proof** First we assume that the first diagram is commutative. Composing \(\pi^{A'}: Q \otimes A'_+ \to Q \otimes A'\) with the first diagram we obtain

\[\pi^{A'} \circ f_+ \circ a = \pi^{A'} \circ a' \circ f.\]

But \(\pi^{A'} \circ a' = p', \pi^{A'} \circ f_+ = f \circ \pi^A\) and \(\pi^A \circ a = p\), so that we obtain \(f \circ p = p' \circ f\).

Now we assume that \(f \circ p = p' \circ f: Q \otimes M \to Q \otimes A'\). By the injectivity of \(\rho^{A'}\), the first diagram commutes if and only if

\[p_K^{A'} \circ \res^{H} \circ f_+ \circ a_H = \pi_K^{A'} \circ \res^{H} \circ a'_H \circ f_H\]

for \(K \leq H \leq G\). But the left hand side is equal to \(\pi_K^{A'} \circ f_+ \circ a_K \circ \res^H = f_K \circ \pi_K^{A'} \circ a_K \circ \res^H\), and the right hand side is equal to \(\pi_K^{A'} \circ a'_K \circ f_K \circ \res^H = p'_K \circ f_K \circ \res^H\). Now the result follows.

### 2.4 Integrality

Throughout this section we assume Hypothesis 3.1 and fix a morphism \(p: M \to A\) of \(\mathbb{Z}\)-conjugation functors on \(G\).

**4.1 Definition** (i) For \(H \leq G\) and \(\chi \in M(H)\) we write

\[p_H(\chi) = \sum_{\varphi \in B(H)} m_\varphi(\chi) \cdot \varphi\]

with \(m_\varphi(\chi) \in \mathbb{Z}\). The number \(m_\varphi(\chi)\) is called the **multiplicity** of \(\varphi\) in \(\chi\). Note that these multiplicities depend on \(p: M \to A\). But in order to avoid overloaded notations we omit to indicate this.

(ii) Recall from Definition 3.2 that for \(H \leq G\), the set \(\mathcal{M}(H)\) of monomial pairs is a poset. For any chain \(\sigma = ((H_0, \varphi_0) < \ldots < (H_n, \varphi_n)) \in \Gamma(\mathcal{M}(H)) \subseteq \Gamma(\mathcal{M}(G))\) we define the **multiplicity** \(m_\sigma\) of \(\sigma\) by

\[m_\sigma := m_{(H_0, \varphi_0)} \cdot \ldots \cdot m_{(H_{n-1}, \varphi_{n-1})} \cdot m_{(H_n, \varphi_n)}.\]
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Of course, this definition is independent of the choice of \( H \).

4.2 Lemma  
For \( H \leq G \) and \( \chi \in M(H) \) we have

\[
ad_H^{M,A,p}(\chi) = \frac{1}{|H|} \sum_{\sigma = (H_0, \varphi_0) < \ldots < (H_n, \varphi_n) \in \Gamma(M(H))} (-1)^n |H_0| m_{\varphi_n} (\text{res}_{H_n}^H(\chi)) [H_0, \varphi_0]_H.
\]

Proof  
We start with the explicit formula for \( a_H^{M,A,p}(\chi) \) given in Definition 3.4 and use the identities

\[
\mu(L, K) = \sum_{L = H_0 < \ldots < H_n = K} (-1)^n
\]

for \( L \leq K \leq H \) from (B.1) in Appendix B. This yields

\[
a_H^{M,A,p}(\chi) = \frac{1}{|H|} \sum_{H_0 < \ldots < H_n \leq H} (-1)^n |H_0| \left[ H_0, (\text{res}_{H_0}^H \circ p_{H_n} \circ \text{res}_{H_n}^H)(\chi) \right]_H
\]

for \( H \leq G \) and \( \chi \in M(H) \). For a fixed chain \( H_0 < \ldots < H_n \) of subgroups of \( H \) and \( \varphi_n \in B(H_n) \) we have

\[
\text{res}_{H_0}^{H_n}(\varphi_n) = (\text{res}_{H_0}^{H_1} \circ \ldots \circ \text{res}_{H_0}^{H_{n-1}})(\varphi_n)
\]

\[
= \sum_{\varphi_{n-1} \in B(H_{n-1})} m_{(H_{n-1}, \varphi_{n-1})} (\text{res}_{H_0}^{H_1} \circ \ldots \circ \text{res}_{H_{n-2}}^{H_{n-1}})(\varphi_{n-1})
\]

\[\vdots\]

\[
= \sum_{\varphi_0 \in B(H_0)} \sum_{\varphi_1 \in B(H_1)} \ldots \sum_{\varphi_{n-1} \in B(H_{n-1})} m_{(H_0, \varphi_0)} \ldots m_{(H_{n-1}, \varphi_{n-1})} \cdot \varphi_0
\]

\[
= \sum_{\sigma = (H_0, \varphi_0) < \ldots < (H_n, \varphi_n)} m_{\sigma} \cdot \varphi_0,
\]

since \( m_{(H_i, \varphi_i)} \) \( m_{(H_{i-1}, \varphi_{i-1})} = 0 \) unless \( (H_{i-1}, \varphi_{i-1}) < (H_i, \varphi_i) \) for \( 1 \leq i \leq n \). If we substitute this expression for \( \text{res}_{H_0}^{H_n}(\varphi_0) \) in the above formula for \( a_H^{M,A,p}(\chi) \), the result follows. \( \square \)

In 4.4 and Theorem 4.5 we will use the following definitions to modify the factor \( |H_0| \) of the formula in the previous Lemma, and obtain integrality criteria for \( a^{M,A,p} \).

4.3 Let \( H \leq G \). For \( \sigma = (H_0, \varphi_0) < \ldots < (H_n, \varphi_n) \in \Gamma(M(H)) \) and a set \( \pi \) of primes we define

\[
N_H^\pi(\sigma) := \{ s \in N_H(\sigma) \mid s_{\sigma'} \in H_0 \},
\]

where \( N_H(\sigma) \) denotes the stabilizer in \( H \) of the chain \( \sigma \), i.e. the intersection of the stabilizers of the elements \( (H_i, \varphi_i) \), \( 0 \leq i \leq n \), in the \( H \)-set \( M(H) \). Note that,
2.4. INTEGRALITY

for \( s \in N_H(\sigma) \), the condition \( s_{\pi'} \in H_0 \) is equivalent to \( sH_0 \) being a \( \pi \)-element in \( N_H(\sigma)/H_0 \). Hence, we have

\[
|N_H^\pi(\sigma)| = |H_0| \cdot |(N_H(\sigma)/H_0)_\pi|
\]

(2.7)

For \( H \leq G \) and \( \pi \) as above we define

\[
\tilde{\Gamma}^\pi(M(H)) := \{(s, \sigma) \in H \times \Gamma(M(H)) \mid s \in N_G^\pi(\sigma)\}
\]

and give a partition

\[
\tilde{\Gamma}^\pi(M(H)) = \tilde{\Gamma}^\pi_0(M(H)) \cup \tilde{\Gamma}^\pi_1(M(H)) \cup \tilde{\Gamma}^\pi_2(M(H)) \cup \tilde{\Gamma}^\pi_3(M(H))
\]

of \( \tilde{\Gamma}^\pi(M(H)) \) into four subsets by defining for \( (s, \sigma) \in \tilde{\Gamma}^\pi(M(H)) \) with \( \sigma = ((H_0, \varphi_0) < \ldots < (H_n, \varphi_n)) \) that

\[
\begin{align*}
(s, \sigma) &\in \tilde{\Gamma}^\pi_0(M(H)) : \iff s \in H_0, \\
(s, \sigma) &\in \tilde{\Gamma}^\pi_1(M(H)) : \iff s \notin H_n, \\
(s, \sigma) &\in \tilde{\Gamma}^\pi_2(M(H)) : \iff s \in H_{i+1} \setminus H_i \text{ and } H_i(s) < H_{i+1} \text{ for some } i \in \{0, \ldots, n-1\}, \\
(s, \sigma) &\in \tilde{\Gamma}^\pi_3(M(H)) : \iff s \in H_{i+1} \setminus H_i \text{ and } H_i(s) = H_{i+1} \text{ for some } i \in \{0, \ldots, n-1\},
\end{align*}
\]

where \( H_i(s) \) denotes the subgroup of \( H \) generated by \( H_i \) and \( s \) (s normalizes \( H_i \)). Note that the index \( i \in \{0, \ldots, n-1\} \) in the definition of \( \tilde{\Gamma}^\pi_2(M(H)) \) and \( \tilde{\Gamma}^\pi_3(M(H)) \) is uniquely determined by the condition \( s \in H_{i+1} \setminus H_i \). Note also, that the definition of \( \tilde{\Gamma}^\pi_0(M(H)) \) does not depend on \( \pi \).

Next we consider the map

\[
f: \tilde{\Gamma}^\pi_3(M(H)) \to \tilde{\Gamma}^\pi_1(M(H)) \cup \tilde{\Gamma}^\pi_2(M(H)),
\]

\[
(s, (H_0, \varphi_0) < \ldots < (H_n, \varphi_n)) \mapsto (s, (H_0, \varphi_0) < \ldots < (H_{i+1}, \varphi_{i+1}) \ldots < (H_n, \varphi_n)),
\]

where \( i \in \{0, \ldots, n-1\} \) is determined by \( s \in H_{i+1} \setminus H_i \), and the pair \((H_{i+1}, \varphi_{i+1}) = (H_i(s), \varphi_{i+1})\) is omitted from \( \sigma \).

For \( (s, \sigma) \in \tilde{\Gamma}^\pi_1(M(H)) \) with \( \sigma = ((H_0, \varphi_0) < \ldots < (H_n, \varphi_n)) \) we have

\[
f^{-1}(\{(s, \sigma)\}) = \left\{ (s, (H_0, \varphi_0) < \ldots < (H_n, \varphi_n), (H_n(s), \varphi) \mid \varphi \in B(H_n(s)), m^{(H_n(s), \varphi)}_{(H_n(s), \varphi)} \neq 0 \right\},
\]

and for \( (s, \sigma) \in \tilde{\Gamma}^\pi_2(M(H)) \) with \( \sigma = ((H_0, \varphi_0) < \ldots < (H_n, \varphi_n)) \) and \( i \in \{0, \ldots, n-1\} \) with \( s \in H_{i+1} \setminus H_i \) we have

\[
f^{-1}(\{(s, \sigma)\}) = \left\{ (s, (H_0, \varphi_0) < \ldots < (H_i, \varphi_i) < (H_i(s), \varphi) < (H_{i+1}, \varphi_{i+1}) < \ldots < (H_n, \varphi_n) \mid \varphi \in B(H_i(s)), m^{(H_i(s), \varphi)}_{(H_i(s), \varphi)} \neq 0 \right\}.
\]
If we define for $H \leq G$, $\sigma = ((H_0, \varphi_0) < \ldots < (H_n, \varphi_n)) \in \Gamma(\mathcal{M}(H))$, and $\chi \in M(H)$ the abbreviation,

$$g(\chi, \sigma) := (-1)^{|\sigma|} m_\sigma m_{\varphi_n}(\text{res}_{H_n}^H(\chi))[H_0, \varphi_0]_H \in A_+(H),$$

(2.8)

we can rewrite Lemma 4.2 as

$$|H| \cdot a^M_{H,A,p}(\chi) = \sum_{(s,\sigma) \in \tilde{\Gamma}^\sigma_1} g(\chi, \sigma),$$

where the factor $|H_0|$ is replaced by summing over pairs $(s, \sigma)$ with $s \in H_0$. Hence, we may continue by

$$|H| \cdot a^M_{H,A,p}(\chi) = \sum_{(s,\sigma) \in \tilde{\Gamma}^\sigma_1(\mathcal{M}(H))} g(\chi, \sigma) - \sum_{(s,\sigma) \in \tilde{\Gamma}^\sigma_1(\mathcal{M}(H))} g(\chi, \sigma).$$

(2.9)

We examine the last sum in Equation (2.9) further using the function $f$:

$$\sum_{(s,\sigma) \in \tilde{\Gamma}^\sigma_1(\mathcal{M}(H))} g(\chi, \sigma) = \sum_{(s,\sigma) \in \tilde{\Gamma}^\sigma_1(\mathcal{M}(H))} \left( g(\chi, \sigma) + \sum_{(s',\sigma') \in f^{-1}((s,\sigma))} g(\chi, \sigma') \right) + \sum_{(s,\sigma) \in \tilde{\Gamma}^\sigma_1(\mathcal{M}(H))} \left( g(\chi, \sigma) + \sum_{(s',\sigma') \in f^{-1}((s,\sigma))} g(\chi, \sigma') \right).$$

(2.10)

For $(s,\sigma) \in \tilde{\Gamma}^\sigma_1(\mathcal{M}(H))$ with $\sigma = ((H_0, \varphi_0) < \ldots < (H_n, \varphi_n))$, i.e. $s \notin H_n$, we have

$$g(\chi, \sigma) + \sum_{(s',\sigma') \in f^{-1}((s,\sigma))} g(\chi, \sigma') = (-1)^{|\sigma|} m_\sigma \times$$

$$\times \left( m_{\varphi_n}(\text{res}_{H_n}^H(\vartheta)) - \sum_{\varphi \in \mathcal{B}(H_n(s))} m_{(H_n(s),\varphi)} m_{\varphi_n}(\vartheta) \right)[H_0, \varphi_0]_H$$

(2.11)

with $\vartheta := \text{res}_{H_n(s)}^H(\chi) \in M(H_n(s))$, and for $(s,\sigma) \in \tilde{\Gamma}^\sigma_2(\mathcal{M}(H))$ with $\sigma$ as above and $i \in \{0,\ldots, n-1\}$ with $s \in H_{i+1} \setminus H_i$ we have

$$g(\chi, \sigma) + \sum_{(s',\sigma') \in f^{-1}((s,\sigma))} g(\chi, \sigma')$$

$$= (-1)^{|\sigma|} \left( m_\sigma m_{\varphi_n}(\text{res}_{H_n}^H(\chi)) - \sum_{\varphi \in \mathcal{B}(H_i(s))} m_{(H_i(s),\varphi)} m_{(H_{i+1}(\varphi_{i+1}))} m_{\varphi_n}(\text{res}_{H_n}^H(\chi)) \right)[H_0, \varphi_0]_H$$

$$= (-1)^{|\sigma|} \frac{m_\sigma}{m_{(H_{i+1},\varphi_{i+1})}} m_{\varphi_n}(\text{res}_{H_n}^H(\chi)) \times$$

$$\times \left( m_{(H_{i+1},\varphi_{i+1})} - \sum_{\varphi \in \mathcal{B}(H_i(s))} m_{(H_i(s),\varphi)} m_{(H_{i+1}(\varphi_{i+1}))} \right)[H_0, \varphi_0]_H$$

$$= 0,$$
since
\[ m^{(H_{i+1}, \varphi_{i+1})}_{(H_i, \varphi_i)} = \sum_{\varphi \in B(H_i)} m^{(H_i(s), \varphi)}_{(H_{i+1}, \varphi_{i+1})} m^{(H_i(s), \varphi)}_{H_i(s), \varphi} \]
by transitivity of restriction. Combining this with Equations (2.11), (2.10) and (2.9) we obtain
\[ |H| \cdot a^{M,A,p}_{H}(\chi) = \sum_{(s,\sigma) \in \Gamma^s(M(H))} g(\chi, \sigma) - \sum_{(s,\sigma) \in \Gamma^s(M(H))} (-1)^{|\sigma|} m_{\sigma} \times \left( m_{\varphi_n}(\text{res}_{H_n}^{H}(\vartheta)) - \sum_{\varphi \in B(H_n)} m^{(H_n(s), \varphi)}_{(H_n, \varphi_n)} m_{\varphi}(\vartheta) \right) [H_0, \varphi_0]_{Hq} \]
for \( H \leq G \), \( \chi \in M(H) \) and \( \vartheta := \text{res}_{H_n}^{H}(\chi) \).

4.5 Theorem Let \( M \) and \( A \) satisfy Hypothesis (3.1) and let \( p: M \to A \) be a morphism of \( Z \)-conjugation functors on \( G \). Furthermore assume that for a set \( \pi \) of primes the following condition holds:

\[ \begin{align*}
& \text{Let } K \trianglelefteq H \leq G \text{ be such that } H/K \text{ is a cyclic } \pi\text{-group, let } \psi \in B(K) \text{ be fixed under } H \text{ (i.e. } \psi^H = \psi
\end{align*} \]

\[ \ (*)_{\pi} \]

for \( h \in H \), and let \( \chi \in M(H) \). Then the two elements \( p_K(\text{res}_K^H(\chi)), \text{res}_K^H(p_H(\chi)) \in A(K) \) have the same coefficient at the basis element \( \psi \).

Then we have
\[ a^{M,A,p}_{H}(\chi) = \frac{1}{|H|} \sum_{(s,\sigma) \in \Gamma^s(M(H))} (-1)^{|\sigma|} |N_H^s(\sigma)| m_{\sigma} \times \left( m_{\varphi_n}(\text{res}_{H_n}^{H}(\chi)) [H_0, \varphi_0]_{Hq} \right) = \sum_{(s,\sigma) \in \Gamma^s(M(H))} (-1)^{|\sigma|} \left| \frac{(N_H(\sigma)/H_0)_{\pi}}{|N_H(\sigma)/H_0|} \right| m_{\sigma} \times \left( m_{\varphi_n}(\text{res}_{H_n}^{H}(\chi)) [H_0, \varphi_0]_{Hq} \right) \]
for \( H \leq G \) and \( \chi \in M(H) \), where the second sum runs over a set of representatives for the \( H \)-orbits of \( \Gamma(M(H)) \).

Proof It is obvious that condition \( (*)_{\pi} \) implies
\[ m_{\varphi_n}(\text{res}_{H_n}^{H}(\vartheta)) = \sum_{\varphi \in B(H_n)} m^{(H_n(s), \varphi)}_{(H_n, \varphi_n)} m_{\varphi}(\vartheta) \]
in Equation (2.12), since the two sides of the above equation are equal to the coefficients of \( \varphi_n \) in \( p_{H_n}(\text{res}_{H_n}^{H}(\vartheta)) \) and \( \text{res}_{H_n}^{H}(p_{H_n}(\vartheta)) \) respectively. Therefore,
the last sum in Equation (2.12) vanishes, and we obtain for $H \leq G$ and $\chi \in M(H)$:

$$a_{H}^{M,A,p}(\chi) = \frac{1}{|H|} \sum_{(s,\sigma) \in \Gamma^m(M(H))} g(\chi, \sigma) = \frac{1}{|H|} \sum_{\sigma = (H_0, \varphi_0) < \ldots < (H_n, \varphi_n)} (-1)^{|\sigma| |N_H^\pi(\sigma)| m_\sigma} \times$$

$$\times m_{\varphi_n}(\text{res}^H_{H_n}(\chi)) |H_0, \varphi_0|_H.$$

Since the summands in the last sum are constant on $H$-orbits of $\Gamma(M(H))$, we may collect them and obtain further

$$a_{H}^{M,A,p}(\chi) = \frac{1}{|H|} \sum_{\sigma = (H_0, \varphi_0) < \ldots < (H_n, \varphi_n)} (-1)^{|\sigma| \frac{|H|}{|N_H(\sigma)|}} |N_H^\pi(\sigma)| m_\sigma \times$$

$$\times m_{\varphi_n}(\text{res}^H_{H_n}(\chi)) |H_0, \varphi_0|_H.$$

Now the second equation in the Theorem follows from Equation (2.7):

$$\frac{|N_H^\pi(\sigma)|}{|N_H(\sigma)|} = \frac{|H_0| \cdot |(N_H(\sigma)/H_0)_\pi|}{|H_0| \cdot |N_H(\sigma)/H_0|} = \frac{|(N_H(\sigma)/H_0)_\pi|}{|N_H(\sigma)/H_0|}.$$

\[\square\]

### 4.6 Remark

Note that in the first formula for $a_{H}^{M,A,p}(\chi)$ in Theorem 4.5 the factor $|N_H^\pi(\sigma)|$ replaces the factor $|H_0|$ in the original formula in Lemma 4.2 without changing the resulting sum. We obviously have $H_0 \subseteq N_H^\pi(\sigma)$, but in general not equality. Equation (2.7) even shows that $|H_0|$ divides $|N_H^\pi(\sigma)|$. In the case of the standard example, $M(H) := R(H)$, $A(H) := R^{ab}(H)$, $p_H$ the canonical projection, for $H \leq G$, the result of Theorem 4.5 was obtained in [Bo94, Theorem 3.2], where it was proved by comparing Euler characteristics of two chain complexes that the replacement of $|H_0|$ with $|N_H(\sigma)|$ does not change the alternating sum.

### 4.7 Corollary

Let $M, A, p$ be as in Theorem 4.5 and assume that $(*)_{\pi}$ holds for a set of primes $\pi$. Then

$$|H|_{\pi'} \cdot a_{H}^{M,A,p}(\chi) \in A_{+}(H)$$

for $H \leq G$, $\chi \in M(H)$, i.e. in $a_{H}^{M,A,p}(\chi)$ occur only $\pi'$-numbers as denominators.

**Proof**

It is well-known (cf. [Hu67, V.19.14]) that $|G_\pi|$ is a multiple of $|G|_\pi$ for any finite group $G$. Therefore we have in the second sum of Theorem 4.5:

$$\frac{|N_H(\sigma)/H_0|_{\pi'} |(N_H(\sigma)/H_0)_\pi|}{|N_H(\sigma)/H_0|} = \frac{|(N_H(\sigma)/H_0)_\pi|}{|N_H(\sigma)/H_0|_{\pi}} \in \mathbb{Z},$$

and a fortiori $|H|_{\pi'} \cdot a_{H}^{M,A,p}(\chi) \in A_{+}(H)$ for $H \leq G$, $\chi \in M(H)$. \[\square\]
4.8 Corollary Let $M, A, p$ be as in Theorem 4.5 and assume that $(\ast_{\pi})$ holds for the set $\pi$ of all primes. Then we have

$$a_{H}^{M, A, p}(\chi) = \sum_{\sigma = (H_0, \varphi_0) < \ldots < (H_n, \varphi_n) \in H \setminus \Gamma(M(H))} (-1)^n m_{\sigma} m_{\varphi_n}(\text{res}_{H_n}^{H}(\chi))[H_0, \varphi_0]_H$$

for $H \leq G$, $\chi \in \mathcal{M}(\chi)$, in particular $a_{M, A, p}$ is integral.

**Proof** This is a trivial consequence of Theorem 4.5.

4.9 Corollary Let $M, A, p$ be as in Theorem 4.5. Assume that $(\ast_{\pi})$ holds for the set $\pi$ of all primes, $A(H) = M(H)$ for all $H \in \mathcal{C}(M)$ (cf. I.2.6), and $p_H: M(H) \to A(H)$ is the identity map for all $H \in \mathcal{C}(M)$. Then $a_{M, A, p}$ is a canonical induction formula for $M$ from $A$.

**Proof** We have to show that condition (i) and (ii) in Proposition 3.5 are satisfied. Condition (i) holds by Corollary 4.8.

In order to show that condition (ii) holds, we first note that $a_{H}^{M, A, p}(\chi) = [H, \chi]_H$ for $H \in \mathcal{C}(M)$ and $\chi \in M(H) = A(H)$ by Lemma 3.7. This shows that $b_{H}^{M, A} \circ a_{H}^{M, A, p} = \text{id}_{M(H)}$ for $H \in \mathcal{C}(M)$.

Now we show that $(b_{H}^{M, A} \circ a_{H}^{M, A, p})(\chi) = \chi$ for arbitrary $H \leq G$ and $\chi \in M(H)$ by induction on the order of $H$. The trivial subgroup is always contained in $C(M)$. Let $H$ be non-trivial. Then either $H$ is contained in $C(M)$ (so there is nothing to show) or elements of $M(H)$ are uniquely determined by their restrictions to proper subgroups. But then we only have to show that $(\text{res}_{K}^{H} \circ b_{H}^{M, A} \circ a_{H}^{M, A, p})(\chi) = \chi$ for $K < H$ and $\chi \in M(H)$. However, this equation holds by our induction hypothesis and commutativity of $b_{i, A}$ and $a_{M, A, p}$ with restrictions.
Chapter 3

Examples of Canonical Induction Formulae

In this chapter we will give several examples of canonical induction formulae. They will all be of the form \( a_{M,A,p}^{M,A,p} : \mathbb{Q} \otimes M \rightarrow \mathbb{Q} \otimes A_+ \) (cf. Definition II.3.4) for a \( \mathbb{Z} \)-Mackey functor \( M \) on a finite group \( G \), a \( \mathbb{Z} \)-restriction subfunctor \( A \) of \( M \), a morphism \( p : M \rightarrow A \) of \( \mathbb{Z} \)-conjugation functors on \( G \), and a stable basis \( B \) of \( A \) which satisfy Hypothesis II.3.1 so that we are in the standard situation and can apply all results of Sections II.3 and II.4.

This chapter is divided into sections according to the different choices of the Mackey functor \( M \). More precisely, we will consider character rings (Section 1), Brauer character rings (Section 2), projective and \( \pi \)-projective character groups (Section 3), trivial source and linear source rings (Section 4), and Green rings (Section 5). Within each section we present for fixed \( M \) (or a family of related \( M \)'s) different choices of \( A \), \( p \), and \( B \) as examples for canonical induction formulae for \( M \) from \( A \) or for \( \mathbb{Q} \otimes M \) from \( \mathbb{Q} \otimes A \). These examples are followed by propositions which give information about the main properties of the resulting canonical induction formulae. We omit to mention in each example that Hypothesis II.3.1 is satisfied, since this is the case for all examples throughout this chapter.

For each choice of \( M \), \( A \), \( B \), and \( p \) we have associated a \( G \)-poset \( \mathcal{M}(G) \) of monomial pairs with \( H \)-subposets \( \mathcal{M}(H) \) for each \( H \leq G \), cf. Definition II.3.2, and multiplicities \( m_\varphi(\chi) \) and \( m_\sigma \) for \( H \leq G \), \( \varphi \in \mathcal{B}(H) \), \( m \in \mathbb{Q} \otimes M(H) \), \( \sigma \in \Gamma(\mathcal{M}(H)) \), cf. Definition II.4.1. Moreover, for any \( H \leq G \), \( \chi \in \mathbb{Q} \otimes M(H) \), \( (K, \psi) \in \mathcal{M}(H) \) we define a rational number \( \alpha^H_{(K,\psi)}(\chi) \) as the coefficient of the basis element \([K,\psi]_H \) of \( \mathbb{Q} \otimes A_+(H) \) in \( a_{H}^{M,A,p}(\chi) \) so that we can write

\[
a_{H}^{M,A,p}(\chi) = \sum_{(K,\psi) \in H\setminus \mathcal{M}(H)} \alpha^H_{(K,\psi)}(\chi)[K,\psi]_H,
\]

where \((K,\psi)\) runs through a set of representatives of the \( H \)-orbits of \( \mathcal{M}(H) \). Since \( a_{H}^{M,A,p} \) is a canonical induction formula, composition with \( b_{H}^{M,A} \) is the identity, and we obtain the formula

\[
\chi = \sum_{(K,\psi) \in H\setminus \mathcal{M}(H)} \alpha^H_{(K,\psi)}(\chi) \text{ind}^H_{K}(\psi).
\]
for $H \leq G$, $\chi \in \mathbb{Q} \otimes M(H)$.

In order to avoid overloaded notation we will abbreviate $a^{M,A,p}$ and $b^{M,A}$ by $a$ and $b$ respectively, and we also use the same symbols $M$, $A$, $B$, $p$, $M$, $m$, and $\alpha$ in the different sections, regardless of the example we are considering at a given moment.

In Section 6 we give an overview over the Mackey functors $M$ considered in the five previous sections together with the relations between them and the canonical induction formulae for them, and we conclude this chapter with an explicit example of the canonical induction formula for the character ring of an extraspecial $p$-group of exponent $p$ for odd $p$ in Section 7.

### 3.1 The character ring

Throughout this section let $K$ be a field of characteristic zero which is a splitting field for all finite groups occurring in this section. For a finite group $G$ we denote by $R_K(G)$ the Grothendieck ring of the category $KG-mod$ of finite dimensional left $KG$-modules. It is well-known that $R_K(G)$ can be canonically identified with the ring of virtual characters of $KG$-modules and that $R_K(G)$ is $\mathbb{Z}$-free on the set $\text{Irr}_K(G)$ of the characters of irreducible $KG$-modules. We denote by $\hat{G}(K) \subseteq \text{Irr}_K(G)$ the set of one-dimensional characters, i.e. $\hat{G}(K) = \text{Hom}(G, K^*)$, and by $R_K^{ab}(G) \subseteq R_K(G)$ the span of $\hat{G}(K)$.

It is well-known that $R_K(G)$, $R_K^{ab}(G)$, $\text{Irr}_K(G)$, and $\hat{G}(K)$ do not really depend on $K$. If $K'$ is another splitting field for $G$ of characteristic zero, then the corresponding rings and sets can be (non-canonically) identified. Even better, the identification can be chosen for all finite groups $G$ simultaneous in a way that conjugation, restriction and induction is respected. Therefore, we will omit the index $K$ in the sequel, which means that we work with the field of complex numbers, but we keep in mind that we could as well work with $K$.

For a finite group $G$ the rings $R(H)$, $H \leq G$, form a $\mathbb{Z}$-Green functor on $G$ with the usual conjugation, restriction, induction and multiplication. It is obvious that the set $C(R)$ of coprimordial subgroups for $R$ consists of the set of cyclic subgroups of $G$. The rings $R_K^{ab}(H)$, $H \leq G$, form a $\mathbb{Z}$-restriction subfunctor of $R$ on $G$ with stable basis $\mathcal{B}(H) := \hat{H}$ for $H \leq G$. By $\mathcal{M}(H)$, $H \leq G$, we denote the associated $H$-poset of monomial pairs, cf. Definition II.3.2.

For each finite group $G$ there is a Galois action on $R_K(G)$ and $R_K^{ab}(G)$ in the following sense: Let $\sigma$ be a field automorphism of $K$, and let $\chi \in R(G)$. Then $^\sigma \chi \in R(G)$ is defined by $(^\sigma \chi)(g) := \sigma(\chi(g))$ (note that $\chi(g) \in K$, since $K$ is a splitting field for $G$). If we consider $R_K$ as $\mathbb{Z}$-Green functor on $G$, then $\sigma : R_K \to R_K$ is an isomorphism of $\mathbb{Z}$-Green functors with $\sigma(R_K^{ab}) \subseteq R_K^{ab}$. This induces an isomorphism $\sigma : R_K^{ab} \to R_K^{ab}$ of $\mathbb{Z}$-Green functors given by $^\sigma[U, \mu](H) = [U, \sigma[\mu]]_H \in R_K^{ab}(H)$ for $H \leq G$, $(U, \mu) \in \mathcal{M}(H)$. Obviously this defines actions of the group of automorphism of $K$ on $R_K$, $R_K^{ab}$ and $R_K^{ab}$, which we all call Galois actions.

Now let $G'$ be another finite group and let $f : G' \to G$ be a group homomorphism. Then there is an induced ring homomorphism $\text{res}_f : R(G) \to R(G')$, $\chi \mapsto \chi \circ f$, which
maps $R^\text{ab}(G)$ to $R^\text{ab}(G')$. Moreover there is a ring homomorphism

$$\text{res}_{+f}: R^\text{ab}_+(G) \rightarrow R^\text{ab}_+(G'),$$

$$[H, \varphi]_G \mapsto \sum_{g \in f(G') \setminus G/H} \left[f^{-1}(gH), \text{res}_{f^{-1}(gH) \rightarrow gH}(\varphi)\right]_{G'},$$

with the property $\text{res}_{+f'} \circ \text{res}_{+f} = \text{res}_{+(f \circ f')}$ for another group homomorphism $f': G'' \rightarrow G'$.

In the case of an inclusion $f: H \rightarrow G$ of a subgroup $H$ of $G$, $\text{res}_{f}$ and $\text{res}_{+f}$ coincide with $\text{res}^G_H$ and $\text{res}_+^G_H$. In the case of a canonical surjection $f: G \rightarrow G/N$ for a normal subgroup $N$ of $G$, the ring homomorphisms $\text{res}_{f}$ and $\text{res}_{+f}$ are called inflation maps and will be denoted by $\text{inf}^G_{G/N}$ and $\text{inf}_+^G_{G/N}$ respectively. We have the explicit description

$$\text{inf}_+^G_{G/N}: R^\text{ab}_+(G/N) \rightarrow R^\text{ab}_+(G), \quad [H/N, \varphi]_{G/N} \mapsto [H, \varphi]_G,$$

for $N \leq H \leq G$ and $\varphi$ the inflation of a homomorphism $\varphi: H/N \rightarrow \mathbb{C}^\times$.

Note that with these definitions, $R$, $R^\text{ab}$, and $R^\text{ab}_+$ can be considered as contravariant functors from the category of finite groups to the category of abelian groups or to the category of commutative rings. The definition of $\text{res}_{f}$ is designed to make $b^{R, R^\text{ab}}$ a natural transformation between the functors $R^\text{ab}_+$ and $R$. Another motivation for the definition of $\text{res}_{+f}$ can be found in V.1.25.

Besides the well-known scalar product $(-, -)_G$ on $R(G)$ there is a bilinear form $[-, -]_G$ on $R^\text{ab}_+(G)$ given by

$$[[K, \psi], [H, \varphi]]_G := \# \{g \in K \setminus G/H \mid (K, \psi) \leq [H, \varphi]\}$$

for $(K, \psi), (H, \varphi) \in \mathcal{M}(G)$. Note that $[[K, \psi], [H, \varphi]]_G$ equals the coefficient of $[K, \psi]_G$ in $\text{res}_+^G_K([H, \varphi])$. For a motivation of this definition see V.1.27. This bilinear form is non-degenerate, since the describing matrix is an upper triangular matrix with the non-zero entries $|N_G(H, \varphi)/H|$ on the diagonal, if the basis elements $[H, \varphi]_G$ of $R^\text{ab}_+(G)$ are ordered in a way compatible with the partial order on $G \setminus \mathcal{M}(G)$. For $G \neq 1$ this bilinear form is not symmetric.

We define a morphism $\varepsilon: R \rightarrow R^\text{ab}$ of $\mathbb{Z}$-algebra restriction functors on $G$ by $\varepsilon_H(\chi) := \chi(1) \cdot 1$ for $H \leq G$, $\chi \in \mathcal{R}(H)$. The restriction of $\varepsilon$ to $R^\text{ab}$ induces a morphism $\varepsilon_+ \in \mathbb{Z} - \text{Mack}_{\text{alg}}(G)(R^\text{ab}_+, R^\text{ab}_+) \text{ with } \varepsilon_+^H([K, \psi]_H) = [K, 1]_H$ for $H \leq G$, $(K, \psi) \in \mathcal{M}(H)$.

For $H \leq G$, the rings $\mathcal{R}(H)$ and $R^\text{ab}_+(H)$ are $R^\text{ab}(H)$-algebras in a natural way, cf. the last paragraph in I.2.2. The maps $b_H$, $H \leq G$, are $R^\text{ab}(H)$-algebra homomorphisms.

For a $KG$-module $V$ and any monomial pair $(H, \varphi) \in \mathcal{M}(G)$ we define the $(H, \varphi)$-homogeneous part $V^{(H, \varphi)}$ of $V$ as

$$V^{(H, \varphi)} := \{v \in V \mid hv = \varphi(h)v \text{ for } h \in H\}.$$
The following Example is the standard example which was already considered in [Bo89], [Bo90], [BSS92], [BCS93] and [Bo94].

1.1 Example Let $G$ be a finite group, $M := R \in Z^{Mack_{\text{alg}}(G)}$, $A := R^{ab} \in Z^{\text{Res}_{\text{alg}}(G)}$, $B(H) := \hat{H}$ for $H \leq G$, and define $p \in Z^{\text{Con}(G)(M,A)}$ for $H \leq G$ and $\chi \in \text{Irr}(H)$ by

$$p_H(\chi) = \begin{cases} \chi, & \text{if } \chi \in \hat{H}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that for $H \leq G$, $\chi \in M(H)$, and $\varphi \in \hat{H}$ we have $m_{\varphi}(\chi) = (\chi, \varphi)_H$, cf. Definition II.4.1.

Most parts of the following proposition have already been proved in [Bo89] and [Bo90], but some of them with different methods from the ones we apply here. We include the full proof of the following proposition, since it illustrates how the very general results of Chapter II can be applied in a familiar case. The old methods in [Bo89] and [Bo90] do not apply to the various other examples in this chapter, but the new ones from Chapter II do. This is especially the case where integrality questions are concerned.

1.2 Proposition Let $G$ be a finite group, and let $M$, $A$, $B$, $M$, $p$, $m$, $\alpha$, $a$, and $b$ be defined according to Example 1.1.

(i) The morphism $a$ is an integral canonical induction formula, i.e. it can be considered as a morphism $a : M \to A_+$.

(ii) For $H \leq G$, the homomorphism $a_H$ is $A(H)$-linear, i.e. $a_H(\varphi \cdot \chi) = [H, \varphi]_H \cdot a_H(\chi)$ for $\varphi \in \hat{H}$, $\chi \in M(H)$.

(iii) For $H \leq G$, the map $a_H$ is the right adjoint of the map $b_H$ with respect to the scalar product $(-,-)_H$ on $M(H)$ and the bilinear form $[-,-]_H$ on $A_+(H)$ defined above.

(iv) The morphism $a$ respects the Galois action.

(v) For $H \leq G$ and $\varphi \in \hat{H}$ we have $a_H(\varphi) = [H, \varphi]_H$.

(vi) The morphism $a$ is given explicitly by

$$a_H(\chi) = \sum_{(H_0, \varphi_0) \prec \ldots \prec (H_n, \varphi_n) \in H \backslash \Gamma(M(H))} (-1)^n (\text{res}_{H_n}^H(\chi), \varphi)_{H_n} [H_0, \varphi_0]_H$$

for $H \leq G$ and $\chi \in M(H)$, and we have

$$V = \sum_{(H_0, \varphi_0) \prec \ldots \prec (H_n, \varphi_n) \in H \backslash \Gamma(M(H))} (-1)^n \text{ind}_{H_0}^H(V^{(H_n, \varphi_n)})$$

in $M(H)$ for any finite dimensional $\mathbb{C}H$-module $V$. 
(vii) For $H \leq G$, $\chi \in \text{Irr}(H)$ and $(K, \psi) \in \mathcal{M}(H)$ we have

$$\alpha_{(K, \psi)}^H(\chi) = \sum_{\left(\begin{array}{c}
(\mathcal{H}_0, \varphi_0) \ldots (\mathcal{H}_n, \varphi_n) \\
\in \mathcal{H} \setminus \mathcal{M}(H)
\end{array}\right) = (K, \psi)} (-1)^n(\text{res}_{H_n}^H(\chi), \varphi_n)_{H_n}$$

$$= \frac{|K|}{|N_G(K, \psi)|} \sum_{\left(\begin{array}{c}
(\mathcal{H}_0, \varphi_0) \ldots (\mathcal{H}_n, \varphi_n) \\
\in \mathcal{H} \setminus \mathcal{M}(H)
\end{array}\right) = (K, \psi)} (-1)^n(\text{res}_{H_n}^H(\chi), \varphi_n)_{H_n}.$$  

(viii) For $H \leq G$, $\chi \in \text{Irr}(H)$ and $(K, \psi) \in \mathcal{M}(H)$ we have

$$\text{res}_K^H(\chi), \psi)_K = 0 \implies \alpha_{(K, \psi)}^H(\chi) = 0.$$  

(ix) We have $b \circ \varepsilon_+ \circ a = \varepsilon$, or more explicitly

$$\sum_{(K, \psi) \in \mathcal{H} \setminus \mathcal{M}(H)} \alpha_{(K, \psi)}^H(\chi) \text{ind}_K^H(1) = \chi(1) \cdot 1$$

for $H \leq G$, $\chi \in \text{Irr}(H)$.

(x) For $H \leq G$ and $\chi \in \text{Irr}(H)$ we have

$$\sum_{(K, \psi) \in \mathcal{H} \setminus \mathcal{M}(H)} (H : K) \alpha_{(K, \psi)}^H(\chi) = \chi(1) = \sum_{(K, \psi) \in \mathcal{H} \setminus \mathcal{M}(H)} \alpha_{(K, \psi)}^H(\chi).$$  

(xi) Let $H \leq G$, $\chi \in \text{Irr}(H)$, and let $(Z(\chi), \lambda) \in \mathcal{M}(H)$ be the central pair of $\chi$ with $Z(\chi)$ denoting the centre of $\chi$, i.e. the maximal subgroup of $H$ such that the restriction of $\chi$ is a multiple of some one-dimensional character, and with $\lambda$ being exactly this one-dimensional character. Then we have for $(K, \psi) \in \mathcal{M}(H)$:

$$(Z(\chi), \lambda) \not\leq (K, \psi) \implies \alpha_{(K, \psi)}^H(\chi) = 0.$$  

(xii) For $H \leq G$, $\chi \in \text{Irr}(H)$, and $(K, \psi) \in \mathcal{M}(H)$ we have

$$Z(H) \not\leq K \implies \alpha_{(K, \psi)}^H(\chi) = 0.$$  

(xiii) Let $f : G' \to G$ be a group homomorphism. Then the diagram

$$\begin{array}{ccc}
\text{R}(G) & \xrightarrow{a_G} & \text{R}^{ab}(G) \\
\text{res}_f \downarrow & & \downarrow \text{res}_f \\
\text{R}(G') & \xrightarrow{a_{G'}} & \text{R}^{ab}(G')
\end{array}$$

commutes. In particular, the canonical induction formulae $a_G$, where $G$ runs over all finite groups, form a natural transformation between the contravariant functors $\mathcal{M}$ and $\mathcal{A}_+$ with values in abelian groups.

**Proof** (i), (vi) First we show that condition $(*)$ in Theorem II.4.5 is satisfied for the set $\pi$ of all primes. Let $K \leq H \leq G$ with $H/K$ cyclic and let $\psi \in \hat{K}$ be fixed under $H$. Let furthermore $\chi \in \text{Irr}(H)$. Then $\text{res}_K^H(\chi)$ is a sum of $H$-conjugates of some $\vartheta \in \text{Irr}(K)$. If $\vartheta \neq \psi$, then the multiplicities of $\psi$ in $\text{res}_K^H(p_H(\chi))$ and
$p_K(\res_K^H(\chi))$ are both zero. If $\vartheta = \psi$, then $\chi$ is a constituent of $\ind_K^H(\psi)$ by Frobenius reciprocity. But $\ind_K^H(\psi)$ is the sum of the $(H : K)$ different extensions of $\psi$, since $H/K$ is cyclic. Therefore, $\chi$ is an extension of $\psi$, and the multiplicities of $\psi$ in $\res_K^H(\rho_H(\chi))$ and $p_K(\res_K^H(\chi))$ are both equal to 1. Now part (i) and part (vi) follow from Corollaries II.4.8 and II.4.9.

(ii) This follows from Proposition II.2.9 (ii), since $p_H$ is $A(H)$-linear for $H \leq G$.

(iii) Let $H \leq G$, $\chi \in \Irr(H)$, and $(K, \psi) \in \mathcal{M}(H)$. Then $[[K, \psi], a_H(\chi)]_H$ is the coefficient of $[K, \psi]_H$ in $\res_K^H(a_H(\chi))$ as mentioned in the introductory part of this section. But $\res_K^H(a_H(\chi)) = a_K(\res_K^H(\chi))$ and the coefficient of $[K, \psi]_K$ in $a_K(\res_K^H(\chi))$ equals $(\psi, \res_K^H(\chi))_K$ by part (vi).

(iv) This follows from Diagram (2.2), since $\pi^A$, $p$, $\res$ and $\res_+$ respect the Galois action.

(v) This follows from Lemma II.3.7.

(vii) The first equation is immediate from part (vi) and the second equation follows from Lemma II.4.2 by collecting the summands with $\varphi$ action. a follows from Corollaries II.4.8 and II.4.9.

(vi) This follows from Lemma II.4.3.

(viii) This follows immediately from part (vi), since $(\res_K^H(\chi), \psi)_K = 0$ implies $(\res_K^H(\chi), \psi')_{K'} = 0$ for all $(K', \psi') \geq (K, \psi)$.

(ix) Since $b$, $\varepsilon_+$, $a$ and $\varepsilon$ commute with restrictions, it suffices to show that the equality $b_H \circ \varepsilon_+ \circ a_H = \varepsilon_H$ for subgroups $H \in \mathcal{C}(M)$, i.e. cyclic subgroups $H$ of $G$. By linearity it suffices then to prove equality, if both sides are applied to $\varphi \in \tilde{H}$, since $\tilde{H} = \Irr(H)$. Now part (v) yields the result.

(x) The first equation follows from evaluating formula (3.2) at the identity element of $H$, and the second equation follows from taking scalar products with the trivial character on both sides of the equation in part (ix).

(xi) Let $(K, \psi) \in \mathcal{M}(H)$ with $(Z(\chi), \lambda) \not\leq (K, \psi)$. By the second equation in part (vii) we have to show that

$$\sum_{(H_0, \varphi_0) < \ldots < (H_n, \varphi_n) \in \Gamma \text{ with } (H_n, \varphi_n) = (K, \psi)} (-1)^n(\res_{H_n}^H(\varphi_n))_{H_n} = 0.$$ 

Let $V$ be a $CH$-module affording the character $\chi$. Note that $Z(\chi)$ is normal in $H$. First we observe that if $(U, \mu) \in \mathcal{M}(H)$ with $(Z(\chi), \lambda) \not\leq (U, \mu)$, then there is a unique monomial pair $(Z(\chi)U, \nu)$ with $\res_{Z(\chi)U}^Z(\nu) = \lambda$ and $\res_{Z(\chi)U}^Z(\nu) = \mu$, since because both $Z(\chi)$ and $U$ act on $V(U, \mu) = \{v \in V \mid uv = \mu(u)v \text{ for } u \in U\}$ via scalar multiplication so does $Z(\chi)U$. We write $\lambda \cdot \mu$ instead of $\nu$. Moreover we have

$$\left(\mu, \res_{U}^H(\chi)\right)_U = \dim V(U, \mu) = \dim (V(U, \mu) \cap V(Z(\chi), \lambda)) = \dim V(Z(\chi)U, \lambda \cdot \mu) = (\lambda \cdot \mu, \res_{Z(\chi)U}^Z(\chi)Z(\chi)U).$$

For the moment we set

$$\Gamma := \{((H_0, \varphi_0) < \ldots < (H_n, \varphi_n)) \in \Gamma(\mathcal{M}(H)) \mid n \in \mathbb{N}_0, (H_0, \varphi_0) = (K, \psi), (\res_{H_n}^H(\chi), \varphi_n)_{H_n} \neq 0\}.$$
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and define a function \( f : \Gamma \to \Gamma \) in the following way: Let \( \sigma = ((H_0, \varphi_0) < \ldots < (H_n, \varphi_n)) \in \Gamma \) and let \( i \in \{0, \ldots, n\} \) be maximal such that \((Z(\chi), \lambda) \nsubseteq (H_i, \varphi_i)\). If \( i = n \) we define

\[
 f(\sigma) = ((H_0, \varphi_0) < \ldots < (H_n, \varphi_n) < (Z(\chi)H_n, \lambda \cdot \varphi_n)).
\]

If \( i < n \) and \( Z(\chi)H_i = H_{i+1} \) we define

\[
 f(\sigma) = ((H_0, \varphi_0) < \ldots < (H_i, \varphi_i) < (Z(\chi)H_i, \lambda \cdot \varphi_i) <
< (H_{i+1}, \varphi_{i+1}) < \ldots < (H_n, \varphi_n)).
\]

It is easy to verify that \( f^2 = \text{id}_\Gamma \), and that the parities of lengths of \( \sigma \) and \( f(\sigma) \)
are different so that the summands belonging to \( \sigma \) and \( f(\sigma) \) in the first sum cancel each other by the above equation.

(xii) This follows immediately from part (xi), since \( Z(H) \leq Z(\chi) \) for all \( \chi \in \text{Irr}(H) \).

(xiii) Since \( f \) can be written as a composition \( G' \xrightarrow{f} f(G') \leq G \) of an epimorphism and an inclusion, and since the result is already proved for subgroup inclusions, we may assume by functoriality that \( f \) is surjective. We proceed by induction on \(|G'|\). If \(|G'| = 1\), the result holds trivially. Now let \(|G'| > 1 \) and \( \chi \in \text{Irr}(G) \). If \( \chi \in \hat{G} \), then part (v) implies the result. So we can assume that \( \chi(1) > 1 \). By the injectivity of \( \rho_G \), it suffices to show that

\[
 \pi^{A}_{H'} \circ \text{res}_{+}^{G'}_{H'}((a_{G'} \circ \text{res}_{f})(\chi)) - (\text{res}_{+} \circ a_G)(\chi)) = 0
\]

for all \( H' \leq G' \).

For \( H' = G' \) we obtain

\[
 (\pi^{A}_{G'} \circ a_{G'} \circ \text{res}_{f})(\chi) = p_{G'}(\text{res}_{f}(\chi)) = 0,
\]
since \( \text{res}_{f}(\chi) \in \text{Irr}(G') \) is not one-dimensional, and on the other hand we obtain

\[
 (\pi^{A}_{G'} \circ \text{res}_{+} \circ a_G)(\chi) = (\pi^{A}_{G'} \circ \text{res}_{+})(\sum_{\varphi \in \hat{G}} \alpha_{(G, \varphi)}^{G}(\chi)|G, \varphi|_G),
\]

since \( (\pi^{A}_{G'} \circ \text{res}_{+})([H, \varphi]|_G) = 0 \) for \( H < G \). But \( \alpha_{(G, \varphi)}^{G}(\chi) = 0 \) for \( \varphi \in \hat{G} \) by part (viii).

Now let \( H' < G' \). Then the induction hypothesis for \( H' \) implies that

\[
 \text{res}_{+}^{G'}_{H'} \circ a_{G'} \circ \text{res}_{f} = a_{H'} \circ \text{res}_{H'}^{G'} \circ \text{res}_{f} = a_{H'} \circ \text{res}_{f} : H' \to f(H') \circ \text{res}_{f}(H')
\]

\[
 = \text{res}_{+} \circ f : H' \to f(H') \circ a_{f}(H') \circ \text{res}_{f}(H')
\]

\[
 = \text{res}_{+} \circ f : H' \to f(H') \circ \text{res}_{+} \circ f(H') \circ a_G
\]

\[
 = \text{res}_{+}^{G'}_{H'} \circ \text{res}_{+} \circ f \circ a_G,
\]

\[
 = \text{res}_{+}^{G'}_{H'} \circ \text{res}_{+} \circ f \circ a_G,
\]
and the result follows. 

1.3 Remark  
(i) In general, the maps \( a_H, H \leq G \), are not multiplicative (cf. Proposition II.2.9 (i)). Neither is \( a : M \to A_+ \) a morphism of Mackey functors, since condition (ii) in Lemma II.2.5 fails.

(ii) If we define \( p_H : M(H) \to A(H) \) for solvable subgroups \( H \leq G \) as in Example 1.1, and by \( p_H = 0 \) for non-solvable subgroups \( H \leq G \), we obtain another integral canonical induction formula for \( M \) from \( A \) which induces only from solvable subgroups. The statements of Proposition 1.2 still hold except for the parts (iii), (v), (xi), (xii), (xiii). Part (v) still holds for solvable \( H \). In parts (vi) and (vii) the sums have to run only over chains \((H_0, \varphi_0) < \ldots < (H_n, \varphi_n)\) with solvable \( H_n \).

More generally we may define a similar integral canonical induction formula by using instead of solvable subgroups a non-empty set \( T \) of subgroups of \( G \), which is closed under taking subgroups and conjugates, with the property that if \( K \leq H \leq G \), \( K \in T \), \( H/K \) cyclic, then \( H \in T \). In fact, then condition \((*_\pi)\) of Theorem II.4.5 holds for the set \( \pi \) of all primes. On the other hand \((*_\pi)\) is also a necessary condition. It is shown in [Bo90] that the those sets \( T \) with the above property lead to integral canonical induction formula, since otherwise already \( a_G(1) \) is not integral. This comes from an interpretation of \( a_G(1) \) as the idempotent in the Burnside ring associated to the set \( T \) of subgroups of \( G \). This is in contrast to Brauer’s induction theorem which needs only elementary subgroups. But there seems to be no integral canonical induction formula which induces only from elementary subgroups.

The following example can be found in [BCS93].

1.4 Example  
Let \( G \) be a finite group, \( M := R \in \mathbb{Z} - \text{Mack}_{\text{alg}}(G), A := R^{ab} \in \mathbb{Z} - \text{Res}_{\text{alg}}(G), B(H) := \hat{H} \) for \( H \leq G \), and define \( p \in \mathbb{Q} - \text{Con}(G)(\mathbb{Q} \otimes M, \mathbb{Q} \otimes A) \) by (cf. Proposition I.1.4 and Remark I.3.3)

\[
p_H(\chi) := e_H^{(H)} \cdot \chi = \frac{1}{|H|} \sum_{K \leq H} |K| \mu(K, H) \text{ind}_K^H(\text{res}_K^H(\chi))
\]

for \( H \leq G, \chi \in \mathbb{Q} \otimes M(H) \), and note that \( e_H^{(H)} \cdot \chi \in \mathbb{Q} \otimes A(H) \), since \( e_H^{(H)} \cdot \chi = 0 \) for non-cyclic \( H \leq G \), cf. Proposition I.6.2.

1.5 Proposition  
Let \( G \) be a finite group, and let \( M, A, B, \mathcal{M}, p, m, \alpha, a, \) and \( b \) be defined according to Example 1.4.

(i) The morphism \( a : \mathbb{Q} \otimes M \to \mathbb{Q} \otimes A_+ \) is a canonical induction formula.
(ii) The morphism \( a \) is a morphism of \( \mathbb{Q} \)-Green functors on \( G \).
(iii) For \( H \leq G, a_H \) is a morphism of \( \mathbb{Q} \otimes A(H) \)-modules.
(iv) For \( H \leq G \) and \( \chi \in \mathbb{Q} \otimes M(H) \) we have the explicit formulae

\[
a_H(\chi) = \frac{1}{|H|} \sum_{\substack{L \leq K \leq H \\ K \text{ cyclic}}} |L| \mu(|K/L|)[K, \text{ind}_L^K(\text{res}_L^K(\chi))]_H
\]

and

\[
\chi = \frac{1}{|H|} \sum_{\substack{L \leq K \leq H \\ K \text{ cyclic}}} |L| \mu(|K/L|) \text{ind}_L^K(\text{res}_L^K(\chi)),
\]
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where $\mu: \mathbb{N} \to \{-1, 0, 1\}$ denotes the number theoretic M"obius function. In particular, $\alpha^H_{(K,\psi)}(\chi) = 0$ for $(K,\psi) \in \mathcal{M}(H)$ with $K$ non-cyclic.

(v) For $H \leq G$ and $\chi \in M(H)$ we have

$$\alpha^H_{(1,1)}(\chi) = \frac{\chi(1)}{|H|}.$$ 

(vi) For $H \leq G$ and $\chi \in M(H)$ we have

$$\sum_{(K,\psi) \in H \setminus \mathcal{M}(H)} \alpha^H_{(K,\psi)}(\chi) \text{ind}_K^H(1) = \frac{\chi(1)}{|H|} \text{ind}_1^H(1).$$

(vii) For $H \leq G$ and $\chi \in M(H)$ we have

$$\sum_{(K,\psi) \in H \setminus \mathcal{M}(H)} \frac{1}{|K|} \alpha^H_{(K,\psi)}(\chi) = \frac{\chi(a)}{|H|} = \sum_{(K,\psi) \in H \setminus \mathcal{M}(H)} \alpha^H_{(K,\psi)}(\chi).$$

(viii) The morphism $a$ is Galois invariant.

(ix) For $H \leq G$ we have

$$a_H(\ker(\varepsilon_H: \mathbb{Q} \otimes M(H) \to \mathbb{Q} \otimes A(H))) \subseteq \ker(\varepsilon_+: \mathbb{Q} \otimes A_+(H) \to \mathbb{Q} \otimes A_+(H)),$$

i.e. virtual characters of degree zero are mapped under $a_H$ to the span of $[K,\psi]_H - [K,1]_H$, $(K,\psi) \in \mathcal{M}(H)$.

**Proof**

(i) This follows from Proposition II.2.2 (ii).

(ii) It follows from Proposition II.2.9 (i) that the maps $a_H$, $H \leq G$, are ring homomorphisms, and Lemma II.2.5 (iii) implies that $a$ is a morphism of Mackey functors on $G$.

(iii) This follows from Proposition II.2.9 (ii).

(iv) By Definition II.3.4 we have

$$a_H(\chi) = \frac{1}{|H|} \sum_{L \leq K \leq H \text{ cyclic}} |L| \mu(L,K) [L, \text{res}_L^K(e_K^K \cdot \text{res}_K^K(\chi))]_H.$$ 

Proposition I.6.2 implies that $\text{res}_L^K(e_K^K \cdot \text{res}_K^K(\chi)) = 0$ unless $L = K$ and $K$ is cyclic. Hence we obtain

$$a_H(\chi) = \frac{1}{|H|} \sum_{K \text{ cyclic}} |K| [K, e_K^K \cdot \text{res}_K^K(\chi)]_H$$

$$= \frac{1}{|H|} \sum_{K \text{ cyclic}} |L| \mu(|L/K|) [K, \text{ind}_L^K(\text{res}_L^K(\chi))]_H,$$

since $\mu(L,K) = \mu(|L/K|)$ for cyclic $K$ (the posets of intermediate subgroups of $L$ and $K$ is isomorphic to the poset of divisors of $|K/L|$). The second equation follows by applying $b_H$.

(v) This follows immediately from the first equation in part (iv) by evaluating the single summand with $K = 1$. 

(vi) Using the first equation in part (iv) we have
\[
\sum_{(K,\psi)\in H\setminus \mathcal{M}(H)} \alpha^H_{\{K,\psi\}}(\chi)\text{ind}_K^H(1) = \frac{\chi(1)}{|H|} \sum_{1 \leq L \leq K \leq H} |L|\mu(|K/L|) |K/L| \text{ind}_K^H(1)
\]
which equals $|H|^{-1}\chi(1)\text{ind}_H^H(1)$ by Möbius inversion of the second form in Corollary B.3 (ii) in the poset of cyclic subgroups of $H$.

(vii) The first equation follows immediately from Equation (3.2) by counting degrees. The second equation follows from taking scalar products with the trivial character on both sides of the equation in part (vi).

(viii) This follows from the commutativity of Diagramm (2.2), since $p_K$, $\pi_A$, $\text{res}_H^K$, and $\text{res}_K^H$, $K \leq H \leq G$, respect the Galois action.

(ix) This follows immediately from part (iv), since $\text{ind}_K^L(\text{res}_L^H(\chi))$ has degree zero, if $\chi$ has.

1.6 Remark  (i) Since $e^H_H \cdot \chi = 0$ for non-cyclic $H \leq G$ and $\chi \in M(H)$, the following definition leads to an equivalent canonical induction formula: $A(H)$ and $p_H$ are defined as in Example 1.4 for cyclic subgroups $H \leq G$, and $p_H = 0$ for non-cyclic subgroups $H \leq G$. This is exactly the canonical induction formula of Example II.2.8.

(ii) In [BCS93, Thm. 2.14] it is proved that the canonical induction formula $a: \mathbb{Q} \otimes M(H) \to \mathbb{Q} \otimes A(H)$ of Example 1.4 is the unique canonical induction formula for $\mathbb{Q} \otimes M$ from $\mathbb{Q} \otimes A$ which is a morphism of $\mathbb{Q}$-Mackey functors on $G$ such that $a_H$ is $\mathbb{Q} \otimes A(H)$-linear for all $H \leq G$. In [BCS93] the two canonical induction formulae of Example 1.1 and Example 1.4 where used together in order to obtain criteria for compatibility with induction of functions on the character ring which are defined as extensions of functions on $\mathbb{Z}^\ab$ using the canonical induction formulae. This was applied to define a conductor in the non-separable residue field case for arbitrary Galois representations.

(iii) The canonical induction formula of Example 1.4 does not commute with inflation or restriction along arbitrary group homomorphisms.

1.7 Example  Let $G$ be a finite group, $M := R \in \mathbb{Z}^\text{-Mack}_\text{alg}(G)$, $A := R^\text{ab} \in \mathbb{Z}^\text{-Res}_\text{alg}(G)$, $\mathcal{B}(H) := \hat{H}$ for $H \leq G$, and let $p \in \mathbb{Z}^\text{-Con}(G)(M,A)$ by defined for $H \leq G$ and $\chi \in \text{Irr}(H)$ by
\[
p_H(\chi) = \begin{cases}
\chi, & \text{if } H \text{ is cyclic,} \\
0, & \text{otherwise.}
\end{cases}
\]
Note that for $H \leq G$, $\varphi \in \hat{H}$, and $\chi \in \mathbb{Q} \otimes M(H)$ we have $m_\varphi(\chi) = 0$ unless $H$ is cyclic, and for cyclic $H$ we have $m_\varphi(\chi) = (\varphi,\chi)_H$.

1.8 Proposition  Let $G$ be a finite group, and let $M$, $A$, $\mathcal{B}$, $\mathcal{M}$, $p$, $m$, $\alpha$, $a$, and $b$ be defined according to Example 1.7.

(i) The morphism $a$ is a canonical induction formula.

(ii) The morphism $a$ is Galois invariant.

(iii) For $H \leq G$, $a_H$ is $\mathbb{Q} \otimes A(H)$-linear.
(iv) For $H \leq G$, $a_H$ is multiplicative, but does not preserve the unity, if $H$ is non-cyclic.

(v) For $H \leq G$ and $\chi \in M(H)$ we have

$$a_H(\chi) = \frac{1}{|H|} \sum_{\substack{L \leq K \leq H \atop K \text{ cyclic}}} |L| \mu(|K/L|) [L, \text{res}_L^H(\chi)]_H$$

and

$$\chi = \frac{1}{|H|} \sum_{\substack{L \leq K \leq H \atop K \text{ cyclic}}} |L| \mu(|K/L|) \text{ind}_L^H(\text{res}_L^H(\chi)).$$

(vi) For $H \leq G$ and $\chi \in M(H)$ we have

$$\sum_{(K, \psi) \in H \setminus M(H)} a_{(K, \psi)}^H(\chi) \text{ind}_K^H(1) = \chi(1) \cdot 1.$$

(vii) For $H \leq G$ and $\chi \in M(H)$ we have

$$\sum_{(K, \psi) \in H \setminus M(H)} (H : K) a_{(K, \psi)}^H(\chi) = \chi(1) = \sum_{(K, \psi) \in H \setminus M(H)} a_{(K, \psi)}^H(\chi).$$

(viii) If $H$ is cyclic, then $a_H(\varphi) = [H, \varphi]_H$ for $\varphi \in \hat{H}$. 

**Proof** (i) This follows from Corollary II.2.4.

(ii) This follows from Diagram (2.2) as in the previous proposition.

(iii) This follows from Proposition II.2.9 (ii).

(iv) This follows from Diagram (2.2) noting that $\pi^A$, $p$, res, and res$^+$ are multiplicative.

(v) The first equation follows from Definition II.3.4 and the same remark on the Möbius function on posets of cyclic subgroups as in the proof of Proposition 1.5 (vi). The second equation follows from the first one by applying $b_H$.

(vi) This is proved in the same way as Proposition 1.2 (ix).

(vii) The first equation follows from Equation (3.2) by counting degrees, and the second equation follows from part (vi) by taking scalar products with the trivial character.

(viii) This follows immediately from Lemma II.3.7.

1.9 Remark (i) The second equations in Propositions 1.5 (iv) and 1.8 (v) are exactly Brauer’s explicit version of Artin’s induction theorem from [Bra51], but the two canonical induction formulae of Examples 1.4 and 1.7 are not equivalent (cf. Lemma II.3.9).

(ii) If one chooses a set $\mathcal{T}$ of subgroups of $G$ which contains all cyclic subgroups, and which is closed under taking subgroups and conjugates, we may replace the set of cyclic subgroups in Example 1.7 with $\mathcal{T}$ and obtain again a canonical induction formula for which Proposition 1.8 holds with the appropriate changes. In [Bo90, Cor. 3.13] it is shown that such a canonical induction formula is integral if and only if, whenever $K \trianglelefteq H \leq G$ with $K \in \mathcal{T}$ and $H/K$ cyclic, then $H \in \mathcal{T}$ (cf. Remark 1.3 (ii)).
(iii) In general, the canonical induction formula $a$ of Example 1.7 is not integral, is not a morphism of Mackey functors (by Lemma II.2.5 (ii)), and does not commute with restrictions along arbitrary group homomorphisms.

The next example was suggested by H. Fottner.

1.10 Example  
Let $N$ be a normal subgroup of a finite group $G$, and let $\vartheta \in \Irr(N)$. For a subgroup $H \leq G$ with $N \leq H$ we define $R_{\vartheta}(H)$ as the span of

$$\Irr_{\vartheta}(H) := \{ \chi \in \Irr(H) \mid (\vartheta, \res^H_N(\chi))_N \neq 0 \text{ for some } g \in G \}$$

and we define $R_{\vartheta}^{ab}(H)$ as the span of

$$\Irr_{\vartheta}^{*}(H) := \{ \chi \in \Irr(H) \mid \res^H_N(\chi) = \vartheta \text{ for some } g \in G \}.$$

Then the definitions

$$M(H/N) := R_{\vartheta}(H), \quad A(H/N) := R_{\vartheta}^{ab}(H), \quad B(H/N) := \Irr_{\vartheta}^{*}(H)$$

for $N \leq H \leq G$ define a $\mathbb{Z}$-Mackey functor $M$ on $G/N$ and a $\mathbb{Z}$-restriction subfunctor $A$ of $M$ on $G/N$ which has a stable basis $B$. For $N \leq K \leq H \leq G$ and $g \in G$ the conjugation, restriction and induction maps for $M$ are defined by $c_{gN,H/N} := c_{g,H}$, $\res^{H/N}_{K/N} := \res^H_K$, and $\ind^{H/N}_{K/N} := \ind^K_H$. Furthermore, we define a morphism $p: M \to A$ of $\mathbb{Z}$-conjugation functors for $N \leq H \leq G$ and $\chi \in \Irr_{\vartheta}(H)$ by

$$p_{H/N}(\chi) = \begin{cases} \chi, & \text{if } \chi \in \Irr_{\vartheta}^{*}(H), \\ 0, & \text{otherwise.} \end{cases}$$

Note that $m_{\sigma} = 1$ for all chains $\sigma \in \Gamma(\mathcal{M}(G/N))$ and that $\mathcal{C}(M)$ is the set of all cyclic subgroups of $G/N$.

1.11 Proposition  
Let $G$ be a finite group, and let $M$, $A$, $B$, $\mathcal{M}$, $p$, $m$, $\alpha$, $a$, and $b$ be defined according to Example 1.10.

(i) The morphism $a$ is an integral canonical induction formula.

(ii) For $N \leq H \leq G$ and $\chi \in \Irr_{\vartheta}^{*}(H)$ we have $a_{H/N}(\chi) = [H/N, \chi]_{H/N}$.

(iii) The morphism $a$ is given explicitly by

$$a_{H/N}(\chi) = \sum_{(H_0/N, \varphi_0) \cdots < (H_n/N, \varphi_n)} (-1)^n (\res^H_{H_n}(\chi), \varphi_n)_{H_n} [H_0/N, \varphi_0]_{H/N}$$

for $N \leq H \leq G$ and $\chi \in R_{\vartheta}(H)$, and we have

$$V = \sum_{(H_0/N, \varphi_0) \cdots < (H_n/N, \varphi_n)} (-1)^n \ind^H_{H_0}(V^{(H_n, \varphi_n)})$$

in $R_{\vartheta}(H)$ for a $\mathbb{C}H$-module $V$ whose character is in $R_{\vartheta}(H)$, where $V^{(H_n, \varphi_n)}$ denotes the sum of all $\mathbb{C}H_n$-submodules of $\res^H_{H_n}(V)$ which have character $\varphi_n$. 


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Proof First we show that condition (∗π) in Theorem II.4.5 is satisfied for the set π of all primes. Let \( N \leq K \leq H \leq G \) with \( H/K \) cyclic and let \( \psi \in \text{Irr}_R^K(K) \) be fixed under \( H \). Let furthermore \( \chi \in \text{Irr}_R^\vartheta(H) \). Then \( \text{res}^H_K(\chi) \) is a sum of \( H \)-conjugates of some \( \xi \in \text{Irr}_R^\vartheta(K) \). If \( \xi \neq \psi \), then the multiplicities of \( \psi \) in \( \text{res}^H_K(p_{H/N}(\chi)) \) and \( p_{K/N}(\text{res}^H_K(\chi)) \) are both zero. If \( \xi = \psi \), then \( \chi \) is a constituent of \( \text{ind}^H_K(\psi) \), which is the sum of the \( (H : K) \) different extensions of \( \psi \), since \( H/K \) is cyclic and \( \psi \) is \( H \)-stable). Hence, \( \chi \) is an extension of \( \psi \), and therefore, \( \chi \in \text{Irr}_R^\vartheta(H) \). This implies that \( \text{res}^H_K(p_{H/n}(\chi)) = \psi = p_{K/N}(\text{res}^H_K(\chi)) \). Now the integrality of \( a \) and part (iii) follow from Corollary II.4.8.

Next we show that \( a \) is a canonical induction formula using Proposition II.2.2. We have to show that

\[
p_{H/N}(\chi) - \chi \in \sum_{N \leq K < H} \text{ind}^H_K(Q \otimes R_\vartheta(H))
\]

for all \( N \leq H \leq G \) with \( H/K \) cyclic (cf. Proposition I.6.2), and for all \( \chi \in \text{Irr}_R^\vartheta(H) \). If \( \chi \in \text{Irr}_R^\vartheta(H) \), then \( p_{H/N}(\chi) - \chi = 0 \). If \( \chi \notin \text{Irr}_R^\vartheta(H) \), then the stabilizer \( S \) of \( \vartheta \) in \( H \) is strictly contained in \( H \) and \( \chi \) is the induced character of an extension of a \( G \)-conjugate of \( \vartheta \) to \( S \) by Clifford theory. This completes the proof of part (i).

Part (ii) follows from Lemma II.3.7. □

1.12 Remark (i) If \( N \) is the trivial subgroup and \( \vartheta \) the trivial character, then Example 1.10 coincides with Example 1.1. Hence, Example 1.10 can be regarded as a generalization of Example 1.1.

(ii) Assume that \( \vartheta \) is stable under \( G \) and that \( \vartheta \) has an extension \( \hat{\vartheta} \in \text{Irr}(G) \). We define \( \vartheta_H := \text{res}^G_H(\hat{\vartheta}) \) for \( N \leq H \leq G \). Then it is well-known that

\[
\text{Irr}_R^\vartheta(H) = \{ \vartheta_H \cdot \text{inf}^H_{H/N}(\chi) \mid \chi \in \text{Irr}(H/N) \}
\]

and this defines a bijection between \( \text{Irr}_R^\vartheta(H) \) and \( \text{Irr}(H/N) \) which restricts to a bijection between \( \text{Irr}_R^\vartheta(H) \) and \( \text{Hom}(H/N, \mathbb{C}^\times) \). Moreover, this induces an isomorphism of Mackey functors \( R_\vartheta(H) \cong R(H/N), N \leq H \leq G \), which identifies \( R^{ab}_\vartheta(H) \) and \( R^{ab}(H/N) \) for \( N \leq H \leq G \). Moreover \( p_{H/N} \) is then identified with \( p_{H/N} \) of Example 1.1, and by Proposition II.3.11 we obtain a commutative diagram

\[
\begin{array}{ccc}
R_\vartheta(H) & \xrightarrow{a_{H/N}} & R^{ab}_\vartheta(H/N) \\
\downarrow \quad \downarrow & & \quad \downarrow \\
R(H/N) & \xrightarrow{a_{H/N}} & R^{ab}(H/N),
\end{array}
\]

where the lower map \( a_{H/N} \) is the canonical induction formula of Example 1.1.

3.2 The ring of Brauer characters

In this section we will derive canonical induction formulæ for the ring of Brauer characters. Throughout this section let \( F \) denote an algebraically closed field of characteristic \( p > 0 \). For a finite group \( G \) let \( R_F(G) \) denote the Grothendieck ring of the category \( FG-\text{mod} \) of finite dimensional \( FG \)-modules. For \( V \in FG-\text{mod} \),
the element in $R_F(G)$ associated to $V$ will be denoted by $[V]$. Arbitrary elements in $R_F(G)$ will often be denoted by $\chi$. The ring $R_F(G)$ is a free abelian group on the elements $[S]$, where $S$ runs through a set of representatives for the isomorphism classes of simple $FG$-modules. Whith the usual conjugation, restriction, induction and multiplication, the rings $R_F(H)$, $H \leq G$, form a $\mathbb{Z}$-Green functor on $G$. For $H \leq G$ let $R_{F,\mathrm{ab}}(H)$ denote the span of those elements $[S] \in R_F(H)$ with $S \in FH\mod$ of dimension one. Then the rings $R_{F,\mathrm{ab}}(H)$, $H \leq G$, form a $\mathbb{Z}$-algebra restriction subfunctor $R_{F,\mathrm{ab}}$ of $R_F$ on $G$. Obviously, for $H \leq G$, the basis $\{[S] \mid S \in FH\mod, \dim S = 1\}$ of $R_{F,\mathrm{ab}}(H)$ can be identified with $\hat{H}(F) := \hom(H, F^\times)$.

For $\varphi \in \hat{H}(F)$ we denote the corresponding $FH$-module by $F_\varphi$. If $\varphi = 1$, then we simply write $F$ instead of $F_1$.

Note that the set $C(R_F)$ of coprimordial subgroups for $R_F$ is the set of cyclic $p'$-subgroups of $G$. This is clear from the well-known interpretation of $R_F(H)$, $H \leq G$, as the ring of Brauer characters of $H$.

Similarly as in Section 1 the above construtions do not depend on the choice of $F$, but only on $p$.

For a finite group $G$, any automorphism of the field $F$ induces a ring automorphism of $R_F(G)$ by using the induced automorphism on FG. These ring automorphisms of $R_F(H)$, $H \leq G$, form an isomorphism of $\mathbb{Z}$-Green functors $R_F \to R_F$ on $G$. Similarly as in Section 1 we obtain actions of the group of automorphisms of $F$ on $R_F$, $R_{F,\mathrm{ab}}$, and $R_{F,\mathrm{ab}}^+$ by isomorphisms of $\mathbb{Z}$-Green functors, resp. $\mathbb{Z}$-algebra restriction functors on $G$ in the case of $R_{F,\mathrm{ab}}^+$, which will be called Galois actions.

If $G'$ is another finite group and $f : G' \to G$ is a group homomorphism, we obtain, as in Section 1, ring homomorphisms $\res_f : R_F(G) \to R_F(G')$, $\res_f : R_{F,\mathrm{ab}}(G) \to R_{F,\mathrm{ab}}(G')$, and $\res_{+f} : R_{F,\mathrm{ab}}^+(G) \to R_{F,\mathrm{ab}}^+(G')$. The latter is given again by the formula (3.3). The definition of $\res_{+f}$ can be motivated by V.1.25. We also define inflation maps on the level of $R_F$, $R_{F,\mathrm{ab}}$, and $R_{F,\mathrm{ab}}^+$, and note that formula (3.4) holds also in this situation. With these constructions, $R_F$, $R_{F,\mathrm{ab}}$, and $R_{F,\mathrm{ab}}^+$ can be considered as contravariant functors from the category of finite groups to the category of commutative rings or to the category of abelian groups.

As in Section 1 we define a morphism $\varepsilon : R_F \to R_{F,\mathrm{ab}}$ of $\mathbb{Z}$-algebra restriction functors by $\varepsilon_H([V]) := \dim_F(V) \cdot [F]$ for $H \leq G$ and $V \in FH\mod$. This induces a morphism $\varepsilon_+ : R_{F,\mathrm{ab}}^+ \to R_{F,\mathrm{ab}}^+$ of $\mathbb{Z}$-Green functors on $G$ which is given by $\varepsilon_{+H}([K, \psi]_H) = [K, 1]_H$ for $K \leq H \leq G$, $\psi \in \hat{K}(F)$.

Note that $m_\sigma = 1$ for $\sigma = ((H_0, \varphi_0) < \ldots < (H_n, \varphi_n)) \in \Gamma(\mathcal{M}(G))$ with $\mathcal{M}(G)$ based on $\mathcal{B}(H) := \hat{H}(F)$ for $H \leq G$.

2.1 Example Let $G$ be a finite group, $M := R_F \in \mathbb{Z}\text{-}\text{Mack}_{\text{alg}}(G)$, $A := R_{F,\mathrm{ab}} \in \mathbb{Z}\text{-}\text{Res}_{\text{alg}}(G)$, $\mathcal{B}(H) := \hat{H}(F)$ for $H \leq G$, and define $p \in \mathbb{Z}\text{-}\text{Con}(G)(M, A)$ for $H \leq G$ and a simple $FH$-module $S$ by

$$p_H([S]) := \begin{cases} [S], & \text{if } \dim_F S = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $m_\varphi([V])$ is the number of composition factors of $V$ isomorphic to $F_\varphi$ for $H \leq G$, $V \in FH\mod$, and $\varphi \in \hat{H}(F)$, cf. Definition II.4.1.
2.2 Proposition  Let $G$ be a finite group, and let $M$, $A$, $S$, $M$, $p$, $m$, $a$, and $b$ be defined according to Example 2.1.

(i) The morphism $a$ is an integral canonical induction formula, i.e. it can be considered as a morphism $a : M 	o A_+$.  
(ii) For $H \leq G$ the homomorphism $a_H$ is $A(H)$-linear, i.e. $a_H(\varphi[V]) = [H, \varphi]_H$. 
(iii) The morphism $a$ respects the Galois action.  
(iv) For $H \leq G$ and $\varphi \in \hat{H}(F)$ we have $a_H([\varphi]) = [H, \varphi]_H$.  
(v) The morphism $a$ is given explicitly by 

$$a_H([V]) = \sum_{(H_0, \varphi_0) \leq \cdots \leq (H_n, \varphi_n) \in H \setminus \Gamma(M(H))} (-1)^n m_{\varphi_n}(\text{res}_{H_0}^{H_n}([V]))[H_0, \varphi_0]_H$$

for $H \leq G$ and $V \in FH-\text{mod}$, and we have 

$$[V] = \sum_{(H_0, \varphi_0) \leq \cdots \leq (H_n, \varphi_n) \in H \setminus \Gamma(M(H))} (-1)^n m_{\varphi_n}(\text{res}_{H_0}^{H_n}([V])) \text{ind}_{H_0}^H([\varphi_0]).$$

(vi) For $H \leq G$, $V \in FH-\text{mod}$ and $(K, \psi) \in M(H)$ we have 

$$\alpha_{(K, \psi)}^H([V]) = \sum_{(H_0, \varphi_0) \leq \cdots \leq (H_n, \varphi_n) \in H \setminus \Gamma(M(H))} (-1)^n m_{\varphi_n}(\text{res}_{H_0}^{H_n}([V]))$$

$$= \frac{|K|}{|N_H(K, \psi)|} \sum_{(H_0, \varphi_0) \leq \cdots \leq (H_n, \varphi_n) \in \Gamma(M(H))} (-1)^n m_{\varphi_n}(\text{res}_{H_0}^{H_n}([V])).$$

(vii) For $H \leq G$, $S$ a simple $FH$-module, and $(K, \psi) \in M(H)$ we have 

$$m_{\psi}(\text{res}_K^H([S])) = 0 \implies \alpha_{(K, \psi)}^H([S]) = 0.$$ 

(viii) We have $b \circ \varepsilon_+ \circ a = \varepsilon$, or more explicitly 

$$\sum_{(K, \psi) \in H, M(H)} \alpha_{(K, \psi)}^H([V]) \text{ind}_K^H([F]) = \dim_F(V) \cdot [F]$$

for $H \leq G$, $V \in FH-\text{mod}$. 

(ix) Let $H \leq G$, $S$ a simple $FH$-module, and let $(Z(S), \lambda) \in M(H)$ be the central pair of $S$, where $Z(S)$ is defined as the maximal subgroup $K$ of $H$ which acts on $S$ by scalar multiplication via some element $\psi \in \hat{K}(F)$, i.e. $\text{res}_K^H(S) \cong \bigoplus_{i=1}^{\dim S} F_{\psi}$, and where $\lambda \in \text{Hom}(Z(S), F^\times)$ is such that $\text{res}_{Z(S)}^H(S) \cong \bigoplus_{i=1}^{\dim S} F_{\lambda}$. Then we have for $(K, \psi) \in M(H)$: 

$$(Z(S), \lambda) \not\preceq (K, \psi) \implies \alpha_{(K, \psi)}^H([S]) = 0.$$ 

(x) For $H \leq G$, $V \in FH-\text{mod}$ and $(K, \psi) \in M(H)$ we have 

$$Z(H)_V \not\subseteq K \implies \alpha_{(K, \psi)}^H([V]) = 0.$$
where $Z(H)_{p'}$ denotes the $p'$-part of the centre $Z(H)$ of $H$.

(xii) Let $f: G' \to G$ be a homomorphism of finite groups. Then the diagram

$$
\begin{array}{c}
R_F(G) \xrightarrow{\alpha_G} R_{F+}^{ab}(G) \\
\downarrow \text{res}_f \quad \quad \quad \downarrow \text{res}_f \\
R_F(G') \xrightarrow{\alpha_{G'}} R_{F+}^{ab}(G')
\end{array}
$$

commutes. In particular, the canonical induction formulae $\alpha_G$, where $G$ runs over all finite groups, form a natural transformation between the contravariant functors $R_F$ and $R_{F+}^{ab}$ with values in abelian groups.

**Proof** We only show that condition $(*)_f$ in Theorem II.4.5 is satisfied for the set $\pi$ of all primes. All other parts are proved in a similar way with the appropriate changes as the corresponding parts in Proposition 1.2.

So let $K \leq H \leq G$ with $H/K$ cyclic, and let $\psi \in \hat{K}(F)$ be fixed under $H$. Let furthermore $S$ be a simple $FH$-conjugate of some simple $FK$-module $T$. If $T$ is not isomorphic to $F_\psi$, then the multiplicity of $\psi$ in $p_K(\text{res}^H_F([S]))$ is zero as well as the multiplicity of $\psi$ in $\text{res}^H_F(p_H([S]))$. If $T$ is isomorphic to $F_\psi$, then

$$0 \neq \dim_F \text{Hom}_{FK}(F_\psi, \text{res}^H_K(S)) = \dim_F \text{Hom}_FH(\text{ind}^H_K(F_\psi), S),$$

and $S$ is isomorphic to a simple quotient of $\text{ind}^H_K(F_\psi)$. But $\text{ind}^H_K(F_\psi)$ has only one-dimensional composition factors, since $\text{ind}^H_K(F_\psi)$ comes via inflation from $H/\ker\psi$ which is an abelian group ($H$ fixes $\psi$). Therefore, $\dim S = 1$ and $\text{res}^H_K(S) \cong F_\psi$. This implies that $p_K(\text{res}^H_K([S])) = \psi = \text{res}^H_K(p_H([S]))$.

2.3 Remark (i) In general, the maps $\alpha_H$, $H \leq G$, are not multiplicative, since the maps $p_H$, $H \leq G$, aren’t (cf. Proposition II.2.9 (i)). Neither is $a: R_F \to R_{F+}^{ab}$ a morphism of Mackey functors, since condition (ii) in Lemma II.2.5 fails in general.

(ii) Let $\mathcal{T}$ be a non-empty set of subgroups of $G$ which is closed under taking subgroups and conjugates such that $\mathcal{T}$ contains with a subgroup $K \leq G$ all subgroups $H$ of $G$ with $K \leq H \leq N_G(K)$ and $H/K$ cyclic. For example, $\mathcal{T}$ may be the set of solvable subgroups of $G$. If we define $p_H: M(H) \to A(H)$ as in Example 2.1 for $H \in \mathcal{T}$, and $p_H := 0$ for $H \notin \mathcal{T}$, then we obtain again an integral canonical induction formula inducing only from the subgroups in $\mathcal{T}$. In this situation parts (i), (ii), (iii) of Proposition 2.2 hold without changes, part (iv) holds for $H \in \mathcal{T}$, parts (v) and (vi) hold with the summations taken over chains with $H_n \in \mathcal{T}$, and parts (vii) and (viii) hold without changes.

(iii) Let $\mathcal{O}$ be a complete discrete valuation ring with residue field $F$ and quotient field $K$ of characteristic zero containing all $|G|$-th roots of unity. Then, for $H \leq G$, there is the well-known decomposition map $d_H: R_K(H) \to R_F(H)$. The maps $d_H$, $H \leq G$, form a morphism of Mackey functors and induce morphisms $d: R_{K+}^{ab} \to R_{F+}^{ab}$ and $d+: R_{K+}^{ab} \to R_{F+}^{ab}$ of restriction functors and Mackey functors respectively. One might expect that the diagram

$$
\begin{array}{c}
R_K \xrightarrow{a} R_{K+}^{ab} \\
\downarrow d \quad \downarrow d+ \\
R_F \xrightarrow{a} R_{F+}^{ab}
\end{array}
$$
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commutes, where the upper canonical induction formula $a$ is the one of Example 1.1 and the lower one comes from our last Example 2.1. But this is not true in general by Proposition II.3.11, since $d_H(\chi)$ may have one-dimensional constituents for $\chi \in \text{Irr}_K(G)$ with $\chi(1) > 1$. Even worse, there are examples for groups $G$ (e.g. $G = S_4$) and virtual characters $\chi \in R_K(G)$ (e.g. $\chi = 1 + \varepsilon - \chi_2$, $\varepsilon$ the sign character, $\chi_2$ the unique irreducible character of degree 2) with $d_G(\chi) = 0$, but $d_G(a_G(\chi)) \neq 0$. This shows that there can’t exist a map $a: R_F \to R^\text{ab}_F$ rendering the above diagram commutative.

2.4 Example  Let $G$ be a finite group, $M := R_F \in \mathbb{Z} - \text{Mack}_{\text{alg}}(G)$, $A(H) := R^\text{ab}_F(H)$ and $B(H) := \hat{H}(F)$ for $H \leq G$ cyclic of $p'$-order, and $A(H) := 0$ and $B(H) := \emptyset$ otherwise. We define $p \in \mathbb{Q} - \text{Con}(G)(\mathbb{Q} \otimes M, \mathbb{Q} \otimes A)$ by (cf. Proposition I.1.4 and Remark I.3.3)

$$p_H(\chi) := e^{(H)}_H \cdot \chi = \frac{1}{|H|} \sum_{K \leq H} |K| \mu(K, H) \text{ind}_K^H(\text{res}_K^H(\chi))$$

for $H \leq G$, $\chi \in \mathbb{Q} \otimes M(H)$, and note that $e^{(H)}_H \cdot \chi \in \mathbb{Q} \otimes A(H)$, since $e^{(H)}_H \cdot \chi = 0$ for $H \notin \mathcal{C}(R_F)$, and since $M(H) = A(H)$ for $H \in \mathcal{C}(R_F)$. Note that automatically, $a^H_{(K, \psi)}(\chi) = 0$ unless $H$ is a cyclic subgroup of $p'$-order.

2.5 Proposition  Let $G$ be a finite group, and let $M, A, B, \mathcal{M}, p, m, \alpha, a, b$ be defined according to Example 2.4.

(i) The morphism $a: \mathbb{Q} \otimes M \to \mathbb{Q} \otimes A_+$ is a canonical induction formula.

(ii) The morphism $a: \mathbb{Q} \otimes M \to \mathbb{Q} \otimes A_+$ is a morphism of $\mathbb{Q}$-Green functors on $G$.

(iii) For $H \leq G$, $a_H$ is a morphism of $\mathbb{Q} \otimes A(H)$-modules, cf. the last paragraph in I.2.2.

(iv) For $H \leq G$ and $\chi \in \mathbb{Q} \otimes R_F(H)$ we have the explicit formulae

$$a_H(\chi) = \frac{1}{|H|} \sum_{L \leq K \leq H \atop K \text{ cyclic}, |K| = p = 1} |L| \mu(|K/L|) \left[ K, \text{ind}_K^L(\text{res}_K^H(\chi)) \right]_H$$

and

$$\chi = \frac{1}{|H|} \sum_{L \leq K \leq H \atop K \text{ cyclic}, |K| = p = 1} |L| \mu(|K/L|) \text{ind}_L^H(\text{res}_L^H(\chi)),$$

where $\mu: \mathbb{N} \to \{-1, 0, 1\}$ denotes the Möbius function from number theory.

(v) For $H \leq G$ and $V \in FH-\text{mod}$ we have

$$a^H_{(1,1)}([V]) = \frac{\dim_F V}{|H|}.$$

(vi) For $H \leq G$ and $V \in FH-\text{mod}$ we have

$$\sum_{(K, \psi) \in H \setminus \mathcal{M}(H)} a^H_{(K, \psi)}([V]) \text{ind}_K^H([F]) = \frac{\dim_F V}{|H|} \text{ind}_1^H([F]).$$
(vii) For $H \leq G$ and $V \in FH-\text{mod}$ we have
\[
\sum_{(K, \psi) \in H \setminus M(H)} \frac{1}{|K|} \alpha^H_{(K, \psi)}([V]) = \frac{\dim_F V}{|H|} = \sum_{(K, \psi) \in H \setminus M(H)} \alpha^H_{(K, \psi)}([V]).
\]

(viii) The morphism $a$ is Galois invariant.
(ix) For $H \leq G$ we have
\[
a_H(\ker(\varepsilon_H: \mathbb{Q} \otimes M(H) \to \mathbb{Q} \otimes A(H))) \subseteq \ker(\varepsilon_{+H}: \mathbb{Q} \otimes A_+(H) \to \mathbb{Q} \otimes A_+(H)),
\]
i.e. virtual Brauer characters of degree zero are mapped to the span of $[K, \psi]_H - [K, 1]_H$, $(K, \psi) \in M(H)$ under $a_H$.

Proof This follows by exactly the same arguments as in the proof of Proposition 1.5. \qed

2.6 Remark (i) The canonical induction formula of Example 2.4 does not commute with inflation or arbitrary group homomorphisms.
(ii) Let $\mathcal{O}$ be a complete discrete valuation ring with residue field $F$ and quotient field $K$ of characteristic zero containing the $|G|$-th roots of unity. Then we have a commutative diagram
\[
\begin{array}{ccc}
\mathbb{Q} \otimes R_K & \xrightarrow{a} & \mathbb{Q} \otimes R^\text{ab}_K \\
\downarrow{d} & & \downarrow{d_+} \\
\mathbb{Q} \otimes R_F & \xrightarrow{a} & \mathbb{Q} \otimes R^\text{ab}_F \\
\end{array}
\]
with the upper and lower $a$ being the canonical induction formulae of Example 1.4 and Example 2.4 respectively, and with $d$ being the decomposition map, cf. Remark 2.3 (iii). This follows from Proposition II.3.11, since
\[
d_H(e^H_H \cdot \chi) = \frac{1}{|H|} \sum_{K \leq H} |K| \mu(K, H) d_H(\text{ind}_K^H(\text{res}_K^H(\chi)))
\]
\[
= \frac{1}{|H|} \sum_{K \leq H} |K| \mu(K, H) \text{ind}_K^H(\text{res}_K^H(d_H(\chi)))
\]
\[
= e^H_H \cdot d_H(\chi),
\]
for $H \leq G$, $\chi \in \mathbb{Q} \otimes R_K(H)$, since $d$ is a morphism of $\mathbb{Q}$-Green functors on $G$.

2.7 Example Let $G$ be a finite group, $M := R_F \in \mathbb{Z}-\text{Mack}_{\text{alg}}(G)$, $A(H) := R^\text{ab}_F(H)$ and $B(H) := \hat{H}(F)$ for $H \leq G$ a cyclic $p'$-group, and $A(H) = 0$ and $B(H) = \emptyset$ otherwise. Define $p \in \mathbb{Z}-\text{Con}(G)(M, A)$ for $H \leq G$ an $\chi \in M(H)$ by
\[
p_H(\chi) := \begin{cases} 
\chi, & \text{if } H \text{ is a cyclic } p'\text{-group} \\
0, & \text{otherwise.}
\end{cases}
\]

2.8 Proposition Let $G$ be a finite group, and let $M, A, B, \mathcal{M}, p, m, \alpha, a,$ and $b$ be defined according to Example 2.7.
(i) The morphism $a$ is a canonical induction formula.
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(ii) The morphism \( a \) is Galois invariant.

(iii) For \( H \leq G \), \( a_H \) is \( \mathbb{Q} \otimes A(H) \)-linear.

(iv) For \( H \leq G \), \( a_H \) is multiplicative.

(v) For \( H \leq G \) and \( \chi \in \mathbb{Q} \otimes R_F(H) \) we have

\[
a_H(\chi) = \frac{1}{|H|} \sum_{L \leq K \leq H \atop K \text{ cyclic}, |K|_p = 1} |L| \mu(|K/L|) [L, \text{res}_L^H(\chi)]_H
\]

and

\[
\chi = \frac{1}{|H|} \sum_{L \leq K \leq H \atop K \text{ cyclic}, |K|_p = 1} |L| \mu(|K/L|) \text{ind}_L^H (\text{res}_L^H(\chi)).
\]

(vi) For \( H \leq G \) and \( V \in FH-\text{mod} \) we have

\[
\sum_{(K,\psi) \in H \setminus \mathcal{M}(H)} a^H_{(K,\psi)} \text{ind}_K^H([F]) = \dim_F V \cdot [F].
\]

(vii) For \( H \leq G \) and \( V \in FH-\text{mod} \) we have

\[
\sum_{(K,\psi) \in H \setminus \mathcal{M}(H)} (H : K) a^H_{(K,\psi)}([V]) = \dim_F V = \sum_{(K,\psi) \in H \setminus \mathcal{M}(H)} a^H_{(K,\psi)}([V]).
\]

(viii) If \( H \leq G \) is a cyclic \( p' \)-subgroup, then \( a_H(\varphi) = [H, \varphi]_H \) for \( \varphi \in \hat{H}(F) \).

**Proof**

(i) This follows from Corollary II.2.4.

(ii) This follows from Diagram (2.2).

(iii) This follows from Proposition II.2.9 (ii).

(iv) This follows from Diagram (2.2), since \( \pi^A, p, \text{res}, \text{res}_+ \) are multiplicative.

(v) The first equation follows from Definition II.3.4, and the second equation follows from applying \( b_H \) to the first one.

(vi) This is proved in the same way as Proposition 2.2 (viii).

(vii) This follows from Equation (3.2) by counting dimensions and counting multiplicities of \([F]\) on both sides of the equation of part (vi). Note that \([F]\) has multiplicity one in \( \text{ind}_K^H([F]) \), if \( H \) is a \( p' \)-group.

(viii) This follows from Lemma II.3.7. \( \square \)

2.9 Remark

(i) In general, the canonical induction formula of Example 2.7 is not a morphism of Mackey functors and does not commute with inflations or restrictions along arbitrary group homomorphisms.

(ii) If \( T \) is a set of subgroups of \( G \) containing the cyclic \( p' \)-subgroups of \( G \) which is closed under taking subgroups and conjugates, then there is a canonical induction formula for \( R_F \) from \( R_F^{ab} \) on subgroups in \( T \), which is defined in the same way as the one in Example 2.7 by replacing the set of cyclic \( p' \)-subgroups with \( T \). this canonical induction formula is integral if and only if \( T \) satisfies the condition of Remark 2.3 (ii), since for \( H \leq G \), \( a_H(1) \) is the sum of all primitive idempotents \( e^{(H)}_K \) of \( \mathbb{Q} \otimes \Omega(H) \subseteq \mathbb{Q} \otimes R_F^{ab} \) with \( K \in T \), and it is well-known that this sum is
integral if and only if $\mathcal{T}$ has the property mentioned in Remark 2.3 (ii), cf. [Yo83a, Thm. 3.1].

If we choose $\mathcal{T}$ to be the set of all cyclic subgroups, then the resulting canonical induction formula and the one of Example 1.7 commute with the decomposition map by Proposition II.3.11.

### 3.3 The Grothendieck group of ($\pi$-) projective modules

Throughout this section let $F$ be an algebraically closed field of characteristic $p > 0$. For a finite group $G$ let $P_F(G)$ denote the Grothendieck group of finite dimensional projective $FG$-modules. The abelian group $P_F(G)$ is free on the elements $[P]$ associated to the projective indecomposable $FG$-modules $P$. By $P_{F}^{\text{ab}}(G)$ we denote the span of the elements $[P]$, where $P$ is a one-dimensional projective $FG$-module. Thus, $P_{F}^{\text{ab}}(G) = 0$ unless $G$ is a $p'$-group, and if $G$ is a $p'$-group, then all $FG$-modules are projective, in particular $P_{F}(G) = R_{F}(G)$ and $P_{F}^{\text{ab}}(G) = R_{F}^{\text{ab}}(G)$. For a $p'$-group $G$ we can also identify the set of elements $[P]$, where $P$ is an $FG$-module of dimension one, with the multiplicative group $\hat{G}(F) := \text{Hom}(G, F^\times)$, and note that $P_{F}^{\text{ab}}(G)$ is free on $\hat{G}(F)$. For $\varphi \in \hat{G}(F)$ we denote the corresponding $F_{\varphi}$-module by $F_{\varphi}$, or just by $F$ in case that $\varphi = 1$.

For a finite group $G$, the groups $P_{F}(H)$, $H \leq G$, form a $\mathbb{Z}$-Mackey functor on $G$ with the usual conjugation, restriction and induction maps, and the groups $P_{F}^{\text{ab}}(H)$, $H \leq G$, form a $\mathbb{Z}$-restriction subfunctor of $P_{F}$. If we define for $H \leq G$, $\mathcal{B}(H) := \hat{H}(F)$, if $H$ is a $p'$-group, and $\mathcal{B}(H) := \emptyset$, if $H$ is not a $p'$-group, then $P_{F}$, $P_{F}^{\text{ab}}$, and $\mathcal{B}$ satisfy Hypothesis II.3.1. The set $\mathcal{C}(P_{F})$ of coprimordial subgroups for $P_{F}$ is the set of cyclic $p'$-subgroups of $G$. In fact, since $P_{F} \subseteq R_{F}$, we have $\mathcal{C}(P_{F}) \subseteq \mathcal{C}(R_{F})$, and one can verify easily that each cyclic $p'$-subgroup of $G$ is coprimordial.

As in the previous sections we have a Galois action of the automorphism group of $F$ on $P_{F}$, $P_{F}^{\text{ab}}$, and on $P_{F}^{\text{ab}}$, and the morphism $b_{P_{F}, P_{F}^{\text{ab}}}$ respects the Galois action. Moreover, we have a morphism $\varepsilon_{+} : P_{F}^{\text{ab}} \rightarrow P_{F}^{\text{ab}}$ of $\mathbb{Z}$-Mackey functors on $G$, given by $\varepsilon_{+}([[K, \psi]_{H}]) = [[K, 1]_{H}}$ for $(K, \psi) \in \mathcal{M}(H)$.

Note that $P_{F}$, $P_{F}^{\text{ab}}$, and $P_{F}^{\text{ab}}$ cannot be considered as contravariant functors on the categories of finite groups, since projectivity is not preserved under inflation.

For any chain $\sigma \in \Gamma(\mathcal{M}(G))$ the multiplicity $m_{\sigma}$ is 1, cf. Definition II.4.1.

#### 3.1 Example

Let $G$ be a finite group, $M := P_{F} \in \mathbb{Z} - \text{Mack}(G)$, $A := P_{F}^{\text{ab}} \in \mathbb{Z} - \text{Res}(G)$, $\mathcal{B}(H) := \hat{H}(F)$ for a cyclic $p'$-subgroup $H \leq G$, $\mathcal{B}(H) := \emptyset$ otherwise. We define $p \in \mathbb{Z} - \text{Con}(G)(M, A)$ by

$$p_H([P]) := \begin{cases} [P], & \text{if dim}_F P = 1, \\ 0, & \text{otherwise}, \end{cases}$$

for $H \leq G$ and a projective indecomposable $FH$-module $P$. Note that for a $p'$-subgroup $H \leq G$, $\varphi \in \hat{H}(F)$, and any (projective) $FH$-module $P$, the multiplicity $m_{\varphi}([P])$ is the number of summands of $P$ isomorphic to $F_{\varphi}$ in a direct decomposition of $P$. 
3.2 Proposition  Let $G$ be a finite group, and let $M, A, B, M, p, m, a, b$ be defined according to Example 3.1.

(i) The morphism $a$ is an integral canonical induction formula.

(ii) The morphism $a$ respects the Galois action.

(iii) For a $p'$-subgroup $H \leq G$ and $\varphi \in B(H)$ we have $a_H(\varphi) = [H, \varphi]_H$.

(iv) The morphism $a$ is given explicitly by

$$a_H([P]) = \sum_{\sigma = (\langle H_0, \varphi_0 \rangle, ..., \langle H_n, \varphi_n \rangle) \in H \backslash \Gamma(M)} (-1)^n \times$$

$$\times \frac{|N_H(\sigma)_{p'}| m_{\varphi_n}(\text{res}_{H_0}^H([P]))}{|N_H(\sigma)|} [H_0, \varphi_0]_H$$

for $H \leq G$ and each projective $P$-module, with

$$\frac{|N_H(\sigma)_{p'}| m_{\varphi_n}(\text{res}_{H_0}^H([P]))}{|N_H(\sigma)|} \in \mathbb{Z},$$

and we have

$$[P] = \sum_{\sigma = (\langle H_0, \varphi_0 \rangle, ..., \langle H_n, \varphi_n \rangle) \in H \backslash \Gamma(M)} (-1)^n \frac{|N_H(\sigma)_{p'}|}{|N_H(\sigma)|} \text{ind}_{H_0}^H(P^{(H_0, \varphi_0)}),$$

where

$$P^{(H_0, \varphi_0)} := \{ x \in P \mid h x = \varphi_n(h)x \text{ for } h \in H_n \}.$$ 

(v) For $H \leq G$, $P \in FH-\text{mod}$ projective indecomposable, and $(K, \psi) \in M(H)$ we have

$$m_{\psi}(\text{res}_{K}^H([P])) = 0 \Rightarrow a^H_{(K, \psi)}([P]) = 0.$$

(vi) For $H \leq G$ and $P \in FH-\text{mod}$ projective, we have

$$\sum_{(K, \psi) \in H \backslash \Gamma(M)} a^H_{(K, \psi)}([P]) = \frac{|H_{p'}|}{|H|} \dim_F P.$$ 

Proof  We first show that condition $(*)_p$ in Theorem II.4.5 is satisfied for the set $\pi$ of all primes different from $p$. Let $K \leq H \leq G$ with $H/K$ a cyclic $p'$-group, and let $\psi \in B(K)$ (in particular $K$, and hence also $H$, is a $p'$-group) be stable under $H$.

Let furthermore $P \in FH-\text{mod}$ be indecomposable (hence simple and projective). We have to show that the coefficients of $\psi$ in $\text{res}_K^H(p_H([P]))$ and $p_K(\text{res}_K^H([P]))$ coincide. Since $H$ is a $p'$-group, this can be proved exactly in the same way as in the proof of Proposition 1.2.

(iv) The first equation follows from Theorem II.4.5, and the second by applying $b_H$. Note that $|N_H(\sigma)/H_0)_{p'}| = |N_H(\sigma)_{p'}/|H_0|$, since $H_0$ is a $p'$-group. So it suffices to prove that

$$\frac{|N_H(\sigma)_{p'}|}{|N_H(\sigma)|_{p'}} \text{ and } \frac{m_{\varphi_n}(\text{res}_{H_0}^H([P]))}{|N_H(\sigma)|_p}$$
are integers. In fact $|N_H(\sigma)|'p'$ divides $|N_H(\sigma)|_p$ by [Hu67, V.19.14], and $|N_H(\sigma)|_p$ divides $m_{\phi}(res^{H_n}_{H_n}([P]))$ by the following argument:

Let $Q$ be a Sylow $p$-subgroup of $N_H(\sigma)$, $H' := QH_n$, and let $V$ be an indecomposable summand of $res^{H_n}_{H_n}(P)$. It follows from part (iv) that $\phi_n$ is stable under $H' \leq N_H(\sigma)$, we have $res^{H_n}_{H_n}(V) \cong \bigoplus_{i=1}^{l} res^{H_n}_{H_n}(F_{\phi_n})$, hence $m_{\phi_n}(res^{H_n}_{H_n}([V])) = |H'/H_n| = |Q|$. (i) It follows from Corollary II.2.4 that $a$ is a canonical induction formula. The integrality of $a$ follows from part (iv).

(ii) This follows from Diagram (2.2), since $\pi^A$, $p$, res, and $res_+$ respect the Galois action.

(iii) This follows from Lemma II.3.7.

(v) This follows immediately from part (iv).

(vi) This will be proved later in Remark IV.3.9.

3.3 Remark (i) The morphism $a$ of the last proposition is in general not a morphism of Mackey functors, since condition (ii) in Lemma II.2.5 does not hold.

(ii) There are more canonical induction formulae for $P_F$ inducing from cyclic $p'$-subgroups. They are defined similar to Example 1.4 and Example 1.7 in the case of the character ring.

(iii) Let $O$ be a complete discrete valuation ring with algebraically residue field $F$ and quotient field $K$ of characteristic zero which contains all $|G|$-th roots of unity. Let $H \leq G$. It is well-known that the functor $F \otimes_O -$ induces a bijection between the set of isomorphism classes of projective modules in $O \text{-mod}$ and in $FH \text{-mod}$. If we define $P_O(H)$ as the Grothendieck group of the category of projective $O$-modules, we obtain a $Z$-Mackey functor $P_O$ with the usual conjugation, restriction and induction maps. Tensoring with $F$ over $O$ induces an isomorphism $P_O \rightarrow P_F$ which identifies $P_F^{ab}(H)$ with

$$P_O^{ab}(H) := \begin{cases} Z\text{Hom}(H, O^\times), & \text{if } H \text{ is a } p' \text{-group}, \\ 0, & \text{otherwise.} \end{cases}$$

The morphism $p \in Z - \text{Con}(G)(P_F, P_F^{ab})$ of Example 3.1 is translated into a morphism $p \in Z - \text{Con}(G)(P_O, P_O^{ab})$, and we obtain a canonical induction formula $a \in Z - \text{Res}(G)(P_O, P_O^{ab})$.

Moreover, it is well-known that the functor $K \otimes_O -$ induces, for $H \leq G$, an isomorphism between $P_O(H)$ and the subgroup $R_{K,p}(H)$ of $R_K(H)$ consisting of all virtual characters vanishing on $p$-singular elements. These isomorphisms form an isomorphism $P_O \rightarrow R_{K,p}$ of $Z$-Mackey functors on $G$ which identifies, for $H \leq$
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Let $G$, $P_O^a(H)$ with the subgroup $R_{K,p}^a(H)$ of $R_K(H)$ spanned by one-dimensional characters vanishing on $p$-singular elements, i.e.

$$R_{K,p}^a(H) = \begin{cases} R_K^a(H), & \text{if } H \text{ is a } p'-\text{group}, \\ 0, & \text{otherwise}. \end{cases}$$

Now the morphism $p_H : P_O(H) \to P_O^a(H)$ translates into $p_H : R_{K,p}(H) \to R_{K,p}^a(H)$ with

$$p_H(\chi) = \begin{cases} \sum_{\varphi \in \hat{H}(K)} (\chi, \varphi) H \cdot \varphi, & \text{if } H \text{ is a } p'-\text{group}, \\ 0, & \text{otherwise}, \end{cases}$$

for $\chi \in R_{K,p}(H)$. Again we obtain a canonical induction formula $a : R_{K,p} \to R_{K,p}^a$, which corresponds to $a : P_O \to P_O^a$ under the canonical isomorphism. This leads to the following generalization, where the group $R_{K,p}(G)$ of $p$-projective virtual characters is replaced with the group $R_{K,\pi}(G)$ of $\pi$-projective virtual characters for a set $\pi$ of primes.

### 3.4 Example

Let $G$ be a finite group, $\pi$ a set of primes, and $K$ a field of characteristic zero which contains all $|G|$-th roots of unity. Let $M \in \mathbb{Z} \text{-} \text{Mack}(G)$ be defined by

$$M(H) := R_{K,\pi}(H) := \{ \chi \in R_K(H) \mid \chi(h) = 0 \text{ for } h \in H \text{ with } h_\pi \neq 1 \}$$

for $H \leq G$, and let

$$\mathcal{B}(H) := \{ \varphi \in \text{Hom}(H, K^\times) \mid \varphi \in R_{K,\pi}(H) \},$$

hence $\mathcal{B}(H) = \hat{H}(K)$ for $\pi'$-subgroups $H \leq G$ and $\mathcal{B}(H) = \emptyset$ for $H \leq G$ with $|H|_\pi \neq 1$. Furthermore, for $H \leq G$, let $A(H) := P_{K,\pi}^a(H)$ denote the span of $\mathcal{B}(H)$. Then $R_{K,\pi}^a$ is a restriction funtor of the Mackey functor $R_{K,\pi}$ on $G$. We define $p \in \mathbb{Z} \text{-} \text{Con}(G)(M, A)$ by

$$p_H(\chi) := \sum_{\varphi \in \mathcal{B}(H)} (\chi, \varphi) H \cdot \varphi$$

for $H \leq G$ and $\chi \in R_{K,\pi}(H)$. In particular, $p_H = 0$ for $H \leq G$ with $|H|_\pi \neq 1$.

### 3.5 Proposition

Let $G$ be a finite group, and let $M$, $A$, $\mathcal{B}$, $\mathcal{M}$, $p$, $m$, $\alpha$, $a$, and $b$ be defined according to Example 3.4.

(i) The morphism $a$ is an integral canonical induction formula.

(ii) The morphism $a$ respects the Galois action.

(iii) For $H \leq G$ of $\pi'$-order and $\varphi \in \mathcal{B}(H)$ we have $a_H(\varphi) = [H, \varphi]_H$.

(iv) The morphism $a$ is given explicitly by

$$a_H(\chi) = \sum_{\sigma = (H_0, \varphi_0) < \ldots < (H_n, \varphi_n) \in H \setminus \Gamma(M(H))} (-1)^n x \frac{|N_H(\sigma)| \cdot (\varphi_n, \res_H^{H_n}(\chi))_{H_n} H_n}{|N_H(\sigma)|} [H_0, \varphi_0]_H$$
for \( H \leq G \) and \( \chi \in R_{K,\pi}(H) \), with

\[
\frac{|N_H(\sigma)|_{\pi^r} \cdot (\varphi_n, \text{res}_{H_n}^H(\chi))_{H_n}}{|N_H(\sigma)|} \in \mathbb{Z},
\]

and we have

\[
\chi = \sum_{\sigma = ((H_0, \varphi_0) < \ldots < (H_n, \varphi_n)) \in H \setminus \Gamma(M(H))} (-1)^n \times 
\frac{|N_H(\sigma)|_{\pi^r} \cdot (\varphi_n, \text{res}_{H_n}^H(\chi))_{H_n}}{|N_H(\sigma)|} \cdot \text{ind}_{H_0}^H(\varphi_0).
\]

(v) For \( H \leq G \) and \( \chi \in R_{K,\pi} \) we have

\[
\sum_{(U, \mu) \in H \setminus M(H)} \alpha_{(U, \mu)}^H(\chi) = \frac{|H_{\pi^r}|}{|H|} \chi(1).
\]

**Proof** We first show that \((\ast_{\pi^r})\) in Theorem II.4.5 is satisfied. Let \( U \leq H \leq G \) with \( H/U \) a cyclic \( \pi^r \)-group and let \( \psi \in B(U) \) (in particular \( U \), and hence \( H \), is a \( \pi^r \)-group) be stable under \( H \). Let furthermore \( \chi \in R_{K,\pi}(H) \). If \( \pi \) is empty, then we are in the situation of Example 1.1, and everything is proved there. If \( \pi \) is non-empty we choose a prime \( p \in \pi \). We may replace \( K \) without loss of generality by a complete discrete valuation field with algebraically closed residue field of characteristic \( p \). Then we may interpret \( \psi \) and \( \chi \) as an \( FU \)-module and an \( FH \)-module respectively, and we may proceed as in the proof of Proposition 1.2.

(iv) The first equation follows now from Theorem II.4.5, and the second by applying \( b_H \). Note that \(|(N_H(\sigma)/H_0)_{\pi^r}| = |N_H(\sigma)_{\pi^r}|/|H_0|\), since \( H_0 \) is a \( \pi^r \)-group. We will show that

\[
\frac{|N_H(\sigma)|_{\pi^r}}{|N_H(\sigma)|_{\pi^r}} \quad \text{and} \quad \frac{(\varphi_n, \text{res}_{H_n}^H(\chi))_{H_n}}{|N_H(\sigma)|_{\pi^r}}
\]

are integers. In fact, \(|N_H(\sigma)|_{\pi^r}\) divides \(|N_H(\sigma)|_{\pi^r}|\) by [Hu67, V.19.14], and it follows from the proof of Proposition 3.2 (iv) that for each \( p \in \pi \) the \( p \)-part \(|N_H(\sigma)|_p\) divides \((\varphi_n, \text{res}_{H_n}^H(\chi))_{H_n}\), since we may translate everything into Example 3.1 for the prime \( p \) as already done above.

(i) This follows from part (iv).

(ii) This follows from Diagram (2.2):

(iii) This follows from Lemma II.3.7.

(v) This will be proved in Remark IV.3.9.

**3.6 Remark** Remark 3.3 (i) and (ii) still hold in this situation with \( p \) replaced by \( \pi \). Note that in \( R_{K,\pi}(H) \) there is no distinguished basis having a module theoretic interpretation as this was the case for \( \pi = \{p\} \).

**3.4 The trivial source ring and the linear source ring**

Throughout this section \( G \) denotes a finite group and \( O \) is a complete discrete valuation ring with algebraically closed residue field \( F \) and with a quotient field of characteristic zero which is a splitting field for all groups occurring in this section.
By $\mathcal{O}G-\text{lin}$ we denote the category of those $\mathcal{O}$-free $\mathcal{O}$-modules of finite $\mathcal{O}$-rank whose indecomposable direct summands have vertices of $\mathcal{O}$-rank one. The category $\mathcal{O}G-\text{triv}$ of direct sums of indecomposable $\mathcal{O}$-free $\mathcal{O}$-modules of finite $\mathcal{O}$-rank with trivial source is a full subcategory of $\mathcal{O}G-\text{lin}$. Considering linear source modules, i.e. objects of $\mathcal{O}G-\text{lin}$, does not add serious difficulties to the proofs in this section compared with considering trivial source modules only, i.e. objects of $\mathcal{O}G-\text{triv}$.

The isomorphism classes of objects $W \in \mathcal{O}G-\text{lin}$ with $\text{rk}_\mathcal{O}W = 1$ are obviously parametrized by $\hat{\mathcal{G}}(\mathcal{O}):= \text{Hom}(G, \mathcal{O}^\times)$, and those of objects $W \in \mathcal{O}G-\text{triv}$ with $\text{rk}_\mathcal{O}W = 1$ by the $p'$-part $\hat{\mathcal{G}}(\mathcal{O})_{p'} := \text{Hom}(G, \mathcal{O}^\times)_{p'}$. In fact, if for $\varphi \in \hat{\mathcal{G}}(\mathcal{O})$ the corresponding $\mathcal{O}G$-module is denoted by $\mathcal{O}_\varphi$, the Sylow $p$-subgroups of $G$ are vertices of $\mathcal{O}_\varphi$ and a source of $\mathcal{O}_\varphi$ is given by $\mathcal{O}_\psi$, where $\psi$ is the restriction of $\varphi$ to a Sylow $p$-subgroup. Obviously, $\psi$ is trivial, if and only if $\varphi$ is of $p'$-order.

An $\mathcal{O}$-free $\mathcal{O}G$-module of finite $\mathcal{O}$-rank is called monomial, if it is isomorphic to a direct sum of modules of the type $\text{ind}_{H}^{G}(\mathcal{O}_\varphi)$ with $H \leq G$, $\varphi \in \hat{\mathcal{G}}(\mathcal{O})$. The following proposition, which is well-known for trivial source modules if ‘monomial module’ is replaced with ‘permutation module’, (cf. [Bro85, Props. (0.2), (0.4)] for example), collects some basic properties of linear source modules. We will omit the proofs which are just modifications of the proofs in the trivial source case.

4.1 Proposition Let $G$ be a finite group and let $P$ denote a Sylow $p$-subgroup of $G$.

(a) For $V \in \mathcal{O}G-\text{mod}$ the following are equivalent:

(i) $V \in \mathcal{O}G-\text{lin}$.

(ii) The module $\text{res}_{P}^{G}(V)$ is a monomial $\mathcal{O}P$-module.

(iii) $V$ is a direct summand of a monomial module.

(b) For $H \leq G$, $M, M' \in \mathcal{O}G-\text{lin}$, and $N \in \mathcal{O}H-\text{lin}$ we have:

(i) The modules $M \oplus M'$ and $M \otimes M'$ are again linear source modules.

(ii) The modules $\text{res}_{H}^{G}(M)$ and $\text{ind}_{H}^{G}(N)$ are again linear source modules.

(iii) Any direct summand of $M$ is a linear source module. □

By $L_{\mathcal{O}}(G)$ (resp. $T_{\mathcal{O}}(G)$) we denote the Grothendieck ring with respect to direct sums of the category $\mathcal{O}G-\text{lin}$ (resp. $\mathcal{O}G-\text{triv}$) and call it the linear source ring (resp. trivial source ring). For $V \in \mathcal{O}G-\text{lin}$ (resp. $V \in \mathcal{O}G-\text{triv}$) we denote by $[V]$ the associated element in $L_{\mathcal{O}}(G)$ (resp. $T_{\mathcal{O}}(G)$). Since the Krull-Schmidt-Azumaya theorem holds for $\mathcal{O}G-\text{lin}$ and $\mathcal{O}G-\text{triv}$ (cf. [CR81, 6.12]), $L_{\mathcal{O}}(G)$ and $T_{\mathcal{O}}(G)$ are free abelian groups on the finite set of isomorphism classes $[V]$ of indecomposable objects $V$ in $\mathcal{O}G-\text{lin}$ and $\mathcal{O}G-\text{triv}$ respectively. We view $T_{\mathcal{O}}(G)$ in the natural way as a subring of $L_{\mathcal{O}}(G)$. We denote by $L_{\mathcal{O}}^{\text{ab}}(G)$ the subring of $L_{\mathcal{O}}(G)$ spanned by the linear independent elements $[\mathcal{O}_\varphi]$, $\varphi \in \hat{\mathcal{G}}(\mathcal{O})$. Similarly, $T_{\mathcal{O}}^{\text{ab}}(G)$ is defined as the subring of $T_{\mathcal{O}}(G)$ spanned by the elements $[\mathcal{O}_\varphi]$, $\varphi \in \hat{\mathcal{G}}(\mathcal{O})_{p'}$, and we have $T_{\mathcal{O}}^{\text{ab}}(G) = T_{\mathcal{O}}(G) \cap L_{\mathcal{O}}^{\text{ab}}(G)$. With the usual conjugation, restriction, induction, and multiplication we obtain $\mathbb{Z}$-Green functors $T_{\mathcal{O}} \subseteq L_{\mathcal{O}}$ on $G$ and $\mathbb{Z}$-algebra restriction.
subfunctors $T^{ab}_O \subseteq L^{ab}_O$ of $T_O \subseteq L_O$ on $G$. For $\varphi \in \hat{G}(O)$ we will abbreviate $[O,\varphi]$ again by $\varphi$, and consider $\hat{G}(O)$ as a $\mathbb{Z}$-basis of $L^{ab}_O(G)$.

By Conlon’s induction theorem [CR87, Thm. 80.51] and Proposition I.6.2, the sets $C(T_O)$ and $C(L_O)$ of coprimordial subgroups for $T_O$ and $L_O$ are contained in the set of $p$-\textit{hypo-elementary} subgroups of $G$, i.e. subgroups $H$ such that $H/O_p(H)$ is a cyclic $p'$-group. Note that subgroups and factor groups of $p$-hypo-elementary groups are again $p$-hypo-elementary. We will see in Proposition 4.15 that a $p$-hypo-elementary subgroup $H \leq G$ is in fact coprimordial for $L_O$ and $T_O$.

Note that we may define Galois actions on $T_O, T_O^{ab}, T_O^{ab}+, L_O, L_O^{ab}, L_O^{ab}+$ by automorphisms of $O$.

If $f: G' \to G$ is a homomorphism of finite groups, we obtain an induced ring homomorphism $\text{res}_f: L_O(G) \to L_O(G')$ which restricts to ring homomorphisms $L^{ab}_O(G) \to L^{ab}_O(G'), T_O(G) \to T_O(G'), T^{ab}_O(G) \to T^{ab}_O(G')$, and induces ring homomorphisms $\text{res}_{+f}: L^{ab}_O+(G) \to L^{ab}_O+(G')$ and $\text{res}_{+f}: T^{ab}_O+(G) \to T^{ab}_O+(G')$ such that the diagram

\[
\begin{array}{ccc}
L^{ab}_O+(G) & \xrightarrow{\text{res}_{+f}} & L^{ab}_O+(G') \\
\uparrow & & \uparrow \\
T^{ab}_O+(G) & \xrightarrow{\text{res}_{+f}} & T^{ab}_O+(G')
\end{array}
\]

commutes, where the vertical maps are induced by the inclusion $T^{ab}_O \subseteq L^{ab}_O$. Note that these maps form a morphism $T^{ab}_O+ \to L^{ab}_O+$ of $\mathbb{Z}$-Green functors on $G$ which is injective, since, for $H \leq G$, the basis $[K,\psi|_H], K \leq H, \psi \in \hat{K}(O)_H$, of $T^{ab}_O+(H)$ is mapped injectively into the basis $[K,\psi|_H], K \leq H, \psi \in \hat{K}(O)$ of $L^{ab}_O+(H)$. The ring homomorphisms $\text{res}_{+f}$ are again described by formula (3.3). There are also inflation maps $\inf^G_{G/N}$ and $\inf^+_G$ described by formula (3.4).

This allows to consider $T_O, T_O^{ab}, T_O^{ab}+, L_O, L_O^{ab}, L_O^{ab}+$ as contravariant functors from the category of finite groups to the category of commutative rings.

For two modules $M$ and $N$ over some ring we write $M \mid N$, and say that $M$ is a summand of $N$, if $M$ is isomorphic to a direct summand of $N$.

**4.2 Example** Let $G$ be a finite group, $M := L_O \in \mathbb{Z}-\text{Mack}_\text{alg}(G), A := L^{ab}_O \in \mathbb{Z}-\text{Res}_\text{alg}(G), B(H) := \hat{H}(O)$ for $H \leq G$, and let $p \in \mathbb{Z}-\text{Con}(G)(M,A)$ be defined for $H \leq G, V \in \mathcal{O}H-\mathfrak{lin}$ indecomposable, by

\[
p_H([V]) := \begin{cases} 
[V], & \text{if } \rk_O V = 1, \\
0, & \text{otherwise.}
\end{cases}
\]

Note that $m_\sigma = 1$ for all $\sigma \in \Gamma(M(G))$, and that $m_\varphi([V])$ is the number of summands isomorphic to $O_\varphi$ in a direct sum decomposition of $V \in \mathcal{O}H-\mathfrak{lin}$ for $H \leq G, \varphi \in \mathcal{B}(H)$.

**4.3 Proposition** Let $G$ be a finite group, and let $M, A, B, \mathcal{M}, p, m, \alpha, a,$ and $b$ be defined according to Example 4.2.

(i) The morphism $a$ is an integral canonical induction formula.

(ii) For $H \leq G$ the homomorphism $a_H$ is $A(H)$-linear.

(iii) The morphism $a$ respects the Galois action.
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(iv) For $H \leq G$ and $\varphi \in \hat{H}(\mathcal{O})$ we have $a_H(\varphi) = [H, \varphi]_H$.

(v) The morphism $a$ is given explicitly by

$$a_H([V]) = \sum_{\sigma = (H_0, \varphi_0) < \ldots < (H_n, \varphi_n)} (-1)^n \times \frac{|N_H(\sigma)/H_0|}{|N_H(\sigma)/H_0|} m_{\varphi_n}(\text{res}_{H_n}^H([V])) [H_0, \varphi_0]_H$$

for $H \leq G$ and $V \in \mathcal{O}H - \text{lin}$, and we have

$$[V] = \sum_{\sigma = (H_0, \varphi_0) < \ldots < (H_n, \varphi_n)} (-1)^n \times \frac{|N_H(\sigma)/H_0|}{|N_H(\sigma)/H_0|} m_{\varphi_n}(\text{res}_{H_n}^H([V])) \text{ind}_{H_0}^H([\varphi_0]).$$

(vi) For $H \leq G$, $V \in \mathcal{O}H - \text{lin}$, and $(K, \psi) \in \mathcal{M}(H)$ we have

$$m_\psi(\text{res}_K^H([V])) = 0 \implies \alpha_{(K, \psi)}^H([V]) = 0.$$

(vii) For $H \leq G$ and $(K, \psi) \in \mathcal{M}(H)$ we have $\alpha_{(K, \psi)}^H([V]) = 0$ unless a Sylow $p$-subgroup of $K$ is contained in some vertex of $V$.

(viii) Let $f : G' \to G$ be a homomorphism of finite groups. Then the diagram

$$\begin{array}{ccc}
L_{\mathcal{O}}(G) & \xrightarrow{a_G} & L_{\mathcal{O}}^{ab}(G) \\
\text{res}_f \downarrow & & \downarrow \text{res}_f \\
L_{\mathcal{O}}(G') & \xrightarrow{a_{G'}} & L_{\mathcal{O}}^{ab}(G')
\end{array}$$

is commutative. In particular, the maps $a_G$, where $G$ runs over all finite groups, form a natural transformation between the contravariant functors $L_{\mathcal{O}}$ and $L_{\mathcal{O}}^{ab}$ with values in abelian groups.

**Proof** First we show that condition $(\ast_{p'})$ in Theorem II.4.5 is satisfied for $L_{\mathcal{O}}$, $L_{\mathcal{O}}^{ab}$, $\mathcal{B}$, and $p \in \mathbb{Z} - \text{Con}(G)(L_{\mathcal{O}}, L_{\mathcal{O}}^{ab})$. Let $K \leq H \leq G$ with $H/K$ a cyclic $p'$-group, and let $\psi \in \text{Hom}(K, \mathcal{O}^\times)$ be stable under $H$. Let furthermore $V \in \mathcal{O}H - \text{lin}$ be indecomposable. If $\text{rk}_{\mathcal{O}}V = 1$, then

$$p_K(\text{res}_K^H([V])) = \text{res}_K^H([V]) = \text{res}_K^H(p_H([V]));$$

and condition $(\ast_{p'})$ holds for $V$. If $\text{rk}_{\mathcal{O}}V > 1$, then $p_H([V]) = 0$, and we have to show that the multiplicity of $\mathcal{O}_\psi$ in $\text{res}_K^H(V)$ is zero. We assume the $\mathcal{O}_\psi | \text{res}_K^H(V)$ and will obtain a contradiction.

Since $(H : K)$ is a $p'$-number, $V$ is $K$-projective, and there is some indecomposable $W \in OK - \text{lin}$ with $V | \text{ind}_K^H(W)$. This implies

$$\mathcal{O}_\psi | \text{res}_K^H(V) | \text{res}_K^H(\text{ind}_K^H(W)) \cong \bigoplus_{h \in H/K} \text{ind}_K^{K \cap K}(\text{res}_K^{K \cap K}(hW)) \cong \bigoplus_{h \in H/K} hW.$$
Therefore, \( \mathcal{O}_\psi \) is isomorphic to \( ^hW \) for some \( h \in H \), and, since \( \psi \) is \( H \)-stable, this implies \( \mathcal{O}_\psi \cong W \). Hence,

\[
V \mid \text{ind}^H_K(\mathcal{O}_\psi) \cong \bigoplus_{\varphi \in \hat{H}(\mathcal{O}), \text{res}^K_K(\varphi) = \psi} \mathcal{O}_\varphi,
\]

and \( \text{rk}_\mathcal{O} V = 1 \) which is a contradiction.

(i) Since condition \((*')\) in Theorem II.4.5 holds for \( L_\mathcal{O}, L^{ab}_\mathcal{O}, \mathcal{B} \), and \( p \in \mathbb{Z} - \text{Con}(G)(L_\mathcal{O}, L^{ab}_\mathcal{O}) \). Corollary II.4.7 implies that \( |H|_p \cdot a_H \) is integral for \( H \leq G \).

(ii) This follows from Proposition II.2.9 (ii).

(iii) This follows from Diagram (2.2).

(iv) This follows from Lemma II.3.7.

(v) This follows from part (v).

(vi) This follows from part (v). In fact, for \((K, \psi) \in \mathcal{M}(H), m_\psi(\text{res}^H_K([V])) \neq 0\), i.e. \( \mathcal{O}_\psi | \text{res}^H_K(V) \), implies that a vertex of \( \mathcal{O}_\psi \) is contained in a vertex of \( V \).

(vii) This is proved exactly in the same way as Proposition 1.2 (viii). Here we first prove that the \( \mathbb{Q} \)-tensored version of the diagram commutes, since at this state we do not know that \( a \) is integral at this state. The original version will then follow from the integrality of \( a \).

4.4 Lemma Let \( G \) be a \( p \)-hypo-elementary group with Sylow \( p \)-subgroup \( P \).

(i) Let \( V \in \mathcal{O}G-\text{lin} \) be indecomposable with vertex \( P \) and source \( \mathcal{O}_\psi \) with \( \psi \in \text{Hom}(P, \mathcal{O}^\times) \), and let \( H := N_G(P, \psi) \). Then we have \( V \cong \text{ind}^H_H(\mathcal{O}_\varphi) \) for some extension \( \varphi \in \text{Hom}(H, \mathcal{O}^\times) \) of \( \psi \). Moreover, we have for \( H \leq U \leq G \):

\[
p_U(\text{res}^G_U([V])) = \begin{cases} \sum_{g \in G/H} g \varphi, & \text{if } U = H, \\ 0, & \text{if } U > H. \end{cases}
\]

(ii) The map

\[
r_G := \rho_G \circ a_G = (p_H \circ \text{res}^G_H)_{H \leq G} : L_\mathcal{O}(G) \to L^{ab+}_\mathcal{O}(G)
\]

is injective.

(iii) We have \( p \circ b \circ a = p \) as morphisms in \( \mathbb{Q} - \text{Con}(G)(\mathbb{Q} \otimes L_\mathcal{O}, \mathbb{Q} \otimes L^{ab}_\mathcal{O}) \).

Proof (i) We have \( V \mid \text{ind}^G_G(\mathcal{O}_\psi) \), and

\[
\text{ind}^H_H(\mathcal{O}_\psi) \cong \bigoplus_{\varphi \in \hat{H}(\mathcal{O}), \text{res}^H_H(\varphi) = \psi} \mathcal{O}_\varphi,
\]

hence \( V \mid \text{ind}^G_G(\mathcal{O}_\varphi) \) for some extension \( \varphi \in \hat{H}(\mathcal{O}) \) of \( \psi \). If \( K \) denotes the quotient field of \( \mathcal{O} \), then \( K \otimes \mathcal{O} \text{ind}^H_H(\mathcal{O}_\varphi) \) is a simple \( KG \)-module, since \( H = \text{stab}_G(\varphi) \).
Therefore \( \text{ind}_H^G(\mathcal{O}_\varphi) \) is indecomposable. The last assertion follows immediately from Mackey’s decomposition theorem

\[
\text{res}_H^G(\text{ind}_H^G(\mathcal{O}_\varphi)) \cong \bigoplus_{g \in G/U} \text{ind}_H^G(\mathcal{O}_{(g\varphi)}),
\]

and the fact, that \( \text{ind}_H^G(\mathcal{O}_{(g\varphi)}) \) is indecomposable, since \( \text{ind}_H^G(\mathcal{O}_{(g\varphi)}) \cong \text{ind}_H^G(\mathcal{O}_\varphi) \cong V \) is.

(ii) We proceed by induction on \( |G| \). If \( |G| = 1 \), the assertion is trivial. Let \( |G| > 1 \). We show that for \( V, W \in \mathcal{O}_G - \text{lin} \) we have

\[
p_H(\text{res}_H^G([V])) = p_H(\text{res}_H^G([W])) \quad \text{for all } H \leq G \implies V \cong W.
\]

We may assume that \( V \) and \( W \) have no indecomposable summand in common. Under this condition we will show that \( V = W = 0 \).

First assume that \( V \) or \( W \) has an indecomposable summand with vertex \( P \). Choose \( X \) among the indecomposable summands of \( V \) and \( W \) with vertex \( P \) with maximal \( \mathcal{O} \)-rank, and assume \( X \mid V \). Let \( \mathcal{O}_\psi \) be a source of \( X \) with \( \psi \in \hat{P}(\mathcal{O}) \), let \( H := N_G(P, \psi) \), and let \( \varphi \in \hat{H}(\mathcal{O}) \) be an extension of \( \psi \) with \( X \cong \text{ind}_H^G(\mathcal{O}_\varphi) \) as granted by part (i). From \( p_H(\text{res}_H^G([V])) = p_H(\text{res}_H^G([W])) \) and part (i) we can see that \( \mathcal{O}_\varphi \mid \text{res}_H^G(W) \). Hence \( W \) has an indecomposable summand \( Y \) with vertex \( P \) and \( \mathcal{O}_\varphi \mid \text{res}_H^G(Y) \). Since \( \mathcal{O}_\psi \mid \text{res}_H^G(\mathcal{O}_\varphi) \mid \text{res}_H^G(Y) \), also \( Y \) has source \( \mathcal{O}_\psi \), and by part (i) we have \( Y \cong \text{ind}_H^G(\mathcal{O}_{(g\varphi)}) \) for some extension \( \varphi' \in \hat{H}(\mathcal{O}) \) of \( \psi \). But

\[
\mathcal{O}_\varphi \mid \text{res}_H^G(Y) \cong \text{res}_H^G(\text{ind}_H^G(\mathcal{O}_{(g\varphi)})) \cong \bigoplus_{g \in G/H} \mathcal{O}_{(g\varphi')}
\]

implies \( \varphi = g\varphi' \) for some \( g \in G \), and

\[
X \cong \text{ind}_H^G(\mathcal{O}_\varphi) \cong \text{ind}_H^G(\mathcal{O}_{(g\varphi)}) \cong Y,
\]

which contradicts our assumption that \( V \) and \( W \) have no indecomposable summand in common. Hence, all indecomposable summands of \( V \) and \( W \) have vertices strictly smaller than \( P \).

If \( V \) and \( W \) are not both trivial, we choose among the indecomposable direct summands of \( V \) and \( W \) one, say \( X \), with maximal vertex \( Q \). Assume that \( X \mid V \).

If \( N := N_G(Q) < G \), then we have by our induction hypothesis that \( \text{res}_N^G(V) \cong \text{res}_N^G(W) \). But then the Green correspondent \( X' \in \mathcal{O}_N - \text{lin} \) of \( X \) is also a summand of \( \text{res}_N^G(W) \). Hence \( X \) is a summand of \( V \) by the Green correspondence. This contradicts our assumption.

Hence, we are left with the case that our maximal vertex \( Q \) is normal in \( G \). Let \( C \leq G \) be a cyclic \( p' \)-subgroup with \( G = PC \). Then by our induction hypothesis we have \( \text{res}_Q^G(V) \cong \text{res}_Q^G(W) \). Let \( \mathcal{O}_\psi \) be a source of \( X \) for some \( \psi \in \hat{Q}(\mathcal{O}) \).

Since \( X \mid \text{ind}_Q^G(\mathcal{O}_\psi) \), there is an indecomposable module \( X' \in \mathcal{O}_Q - \text{lin} \) with \( X' \mid \text{ind}_Q^G(\mathcal{O}_\psi) \) and \( X \mid \text{ind}_Q^G(X') \), hence \( X = \text{ind}_Q^G(X') \) by Green’s indecomposability theorem. By Mackey’s decomposition theorem we have

\[
X' \mid \text{res}_Q^G(X) \mid \text{res}_Q^G(V) \cong \text{res}_Q^G(W).
\]
Therefore, \( W \) has an indecomposable summand \( Y \) with \( X' \mid \res_{O_C}^G(Y) \). Since \( X' \) has vertex \( Q \), and since \( Q \) is maximal among the vertices of summands of \( V \) and \( W \), also \( Y \) has vertex \( Q \), and, similar as for \( X, Y \cong \ind_{QC}^G(Y') \) for some indecomposable module \( Y' \in \mathcal{O}[QC]\). Now we have
\[
X' \mid \res_{QC}^G(Y) \cong \res_{QC}^G(\ind_{QC}^G(Y')) \cong \bigoplus_{g \in G/QC} gY',
\]
and we obtain the contradiction
\[
X \cong \ind_{QC}^G(X') \cong \ind_{QC}^G(Y') \cong Y \mid W.
\]
Hence, \( V = W = 0 \), and the proof of part (ii) is complete.

(iii) It suffices to show \( p_G \circ b_G \circ a_G = p_G \), since all subgroups of \( G \) are again \( p \)-hypo-elementary. Let \( V \in \mathcal{O} G-\lin \) be indecomposable. If \( \rk_O V = 1 \), then
\[
(p_G \circ b_G \circ a_G)([V]) = [V] = p_G([V])
\]
by Proposition 4.3 (iv). Now assume that \( \rk_O V > 1 \) and write
\[
a_G([V]) = \sum_{(H, \varphi) \in G \backslash \mathcal{M}(G)} \alpha_{(H, \varphi)}^G([V])[H, \varphi]_G.
\]
If the vertices of \( V \) are strictly contained in \( P \), then \( \alpha_{(H, \varphi)}^G([V]) = 0 \) for all \((H, \varphi) \in \mathcal{M}(G) \) with \( P \leq H \) by Proposition 4.3 (vii). For \((H, \varphi) \in \mathcal{M}(G) \) with \( P \nleq H \) we have always \( p_G(b_G([H, \varphi]_G)) = p_G(\ind_{H}^G([O_\varphi])) = 0 \). Hence, \( (p_G \circ b_G \circ a_G)([V]) = 0 \).

If \( V \) has vertex \( P \) and source \( O_\psi \) for some \( \psi \in \hat{P}(O) \), then by the same argument it suffices to show that
\[
p_G(\alpha_{(H, \varphi)}^G([V])\ind_{H}^G([O_\varphi])) = 0
\]
for \((H, \varphi) \in \mathcal{M}(G) \) with \( P \leq H \). By Proposition 4.3 (vi), we may assume that \( O_\varphi \mid \res_{H}^G(V) \), since otherwise \( \alpha_{(H, \varphi)}^G([V]) = 0 \). In this case we have
\[
\ind_{H}^G(O_\varphi) \mid \ind_{H}^G(\res_{H}^G(V)) \mid \ind_{H}^G(\res_{H}^G(\ind_{H}^G(O_\psi)))
\]
\[
\cong \ind_{H}^G\left( \bigoplus_{g \in G/H} \ind_{H}^G(O_{(\varphi)}^g) \right)
\]
\[
\cong \bigoplus_{g \in G/H} \ind_{H}^G(O_{(\varphi)}^g).
\]
But since \( \psi \) is not \( G \)-stable (by part (i)), the module \( \ind_{H}^G(O_{(\varphi)}^g) \cong \ind_{\hat{P}}(O_\psi) \) has no summands of \( O \)-rank one.

Now we are able to prove that \( b \circ a = \id_{Q \otimes L_O} \). It suffices to prove \( b_H \circ a_H = \id_{Q \otimes L_O}(H) \) for all \( p \)-hypo-elementary subgroups \( H \leq G \), since \( a \) and \( b \) commute with restrictions, and since elements of \( Q \otimes L_O(K), K \leq G \), are uniquely determined by restrictions to all \( p \)-hypo-elementary subgroups of \( K \). So let \( H \leq G \) be \( p \)-hypo-elementary. Since \( \rho_H^A \circ a_H = (p_K \circ \res_{K}^H)_{K \leq H} \) is injective by Lemma 4.4 (ii), it suffices
to show that $p_K \circ \text{res}_K^H \circ b_H \circ a_H = p_K \circ \text{res}_K^H$ for $K \leq H$. Since $b$ and $a$ commute with restrictions, it suffices to prove $p_K \circ b_K \circ a_K = p_K$ for all $p$-hypo-elementary subgroups $K$ of $G$. But this is granted by Lemma 4.4 (iii).

Next we turn to the integrality of $a$ and introduce another canonical induction formula $a'$ for $Q \otimes L_O$ from $Q \otimes L_O^\text{ab}$.

4.5 Example Let $G$ be a finite group, $M := L_O \in \mathbb{Z} - \text{Mack}_\text{alg}(G)$, and let for $H \leq G$, $A'(H) \subseteq M(H)$ be defined as the $\mathbb{Z}$-span of the set $B'(H)$ of elements $[V]$ with $V \in \mathcal{O}H \cdot \mathbf{lin}$ indecomposable such that $V/\ker(V)$ is $p$-elementary, i.e. a direct product of a $p$-group and a cyclic $p'$-group. Then $A' \subseteq L_O$ is a $\mathbb{Z}$-algebra restriction subfunctor of $M$. For $H \leq G$ and indecomposable $V \in \mathcal{O}H \cdot \mathbf{lin}$ we define

$$p'_H([V]) := \begin{cases} [V], & \text{if } [V] \in B'(H), \\ 0, & \text{otherwise.} \end{cases}$$

This defines a morphism $p' \in \mathbb{Z} - \text{Con}(G)(L_O, A')$. We denote by $M'$, $m'$, $\alpha'$, $a'$, and $b'$ the features resulting from $M$, $A'$, $p'$, and $B'$.

4.6 Lemma (i) Let $G$ be a $p$-elementary group, and let $V \in \mathcal{O}G \cdot \mathbf{lin}$ be indecomposable. Then there is a subgroup $H \leq G$ of $p$-power index and some $\varphi \in \hat{H}(\mathcal{O})$ with $V \cong \text{ind}_H^G(\mathcal{O}_\varphi)$. Moreover, the pair $(H, \varphi) \in \mathcal{M}(G)$ is uniquely determined up to $G$-conjugacy by the condition $V \cong \text{ind}_H^G(\mathcal{O}_\varphi)$.

(ii) Let $G$ be an arbitrary finite group, and let $V \in \mathcal{O}G \cdot \mathbf{lin}$ be indecomposable such that $[V] \in B'(G)$. Then there is a unique pair $(H, \varphi) \in \mathcal{M}(G)$ up to $G$-conjugacy such that $V \cong \text{ind}_H^G(\mathcal{O}_\varphi)$. Moreover $\ker(V) \leq H$ and $(G : H)$ is a $p$-power. In particular $\text{rk}_G V$ is a $p$-power.

Proof (i) Let $G = P \times C$ with $P = O_p(G)$ and $C = O_p'(G)$. Let $Q \leq P$ be a vertex of $V$, and let $\mathcal{O}_\psi, \psi \in \hat{Q}(\mathcal{O})$, be a source of $V$. Then

$$V \mid \text{ind}_Q^G(\mathcal{O}_\psi) \cong \text{ind}_Q^C(\text{ind}_Q^C(\mathcal{O}_\psi)) \cong \bigoplus_{\varphi \in \text{Hom}(Q,C, \mathcal{O}^\times)} \text{ind}_Q^C(\mathcal{O}_\varphi).$$

Since $\text{ind}_Q^C(\mathcal{O}_\varphi)$ is indecomposable by Green’s indecomposability theorem ($C \leq G$ of $p$-power index), we have $V \cong \text{ind}_H^G(\mathcal{O}_\varphi)$ with $H = QC$ of $p$-power index in $G$. Now let $V \cong \text{ind}_H^G(\mathcal{O}_\varphi)$ for an arbitrary subgroup $H' \leq G$ and some $\varphi' \in \text{Hom}(H', \mathcal{O}^\times)$. From the first part of the proof we know that the $\mathcal{O}$-rank of $V$ is a $p$-power, hence $C \leq H'$. Now we have

$$\mathcal{O}_\varphi \mid \text{res}_H^G(\text{ind}_H^G(\mathcal{O}_\varphi)) \cong \text{res}_H^G(V) \cong \text{res}_H^G(\text{ind}_H^G(\mathcal{O}_\varphi')) \cong \bigoplus_{g \in H \cap gH'} \text{ind}_H^G(\text{res}_H^G(\mathcal{O}_{(g\varphi')})).$$

Since $C \leq H \cap gH'$ for all $g \in G$, Green’s indecomposability theorem implies that each of the summands in the last sum is indecomposable. Hence, there is some $g \in G$ with $(H, \varphi) \leq (gH', g\varphi')$. Now $(G : H) = \text{rk}_G V = (G : H')$ implies $(H, \varphi) = (gH, g\varphi)$. 

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(ii) By part (i) there is a unique \((H, \varphi) \in \mathcal{M}(G)\) up to \(G\)-conjugacy with \(\ker(V) \leq H\) and \(\text{ind}_H^G(\mathcal{O}_\varphi) \cong V\). Moreover \((G : H)\) is a \(p\)-power. Let \((H', \varphi')\) be another element in \(\mathcal{M}(G)\) with \(V \cong \text{ind}_H^G(\mathcal{O}_{\varphi'})\). We will show that \(\ker(V) \leq H'\) and \(\text{res}_{\ker(V)}(\varphi') = 1\). Then part (i) implies that \((H, \varphi)\) and \((H', \varphi')\) are \(G\)-conjugate.

With \(K := \ker(V)\) we have

\[
\mathcal{O} \mid \text{res}_K^G(V) \cong \text{res}_K^G(\text{ind}_{H'}^H(\mathcal{O}_{\varphi'})) \cong \bigoplus_{g \in K \cap G/H'} \text{ind}_K^G(\text{res}_{K \cap gH'}^K(\mathcal{O}_{(g\varphi')})).
\]

Hence, there is some \(g \in G\) with \(\mathcal{O} \mid \text{ind}_K^G(\text{res}_{K \cap gH'}^K(\mathcal{O}_{(g\varphi')}))\). Since the vertices of \(\mathcal{O}\) are the Sylow \(p\)-subgroups of \(K\), the index \((K : K \cap gH')\) is a \(p\)-index. On the other hand \((K : K \cap gH')\) is a \(p\)-power, since \((G : H') = \text{rk}_Q V\) is a \(p\)-power and \((K : K \cap gH') = (K : gH')\). Hence, \((K : K \cap gH') = K\) and \(\mathcal{O} \mid \text{res}_K^G(\mathcal{O}_{(g\varphi')})\), i.e. \((K, 1) \leq (gH', g\varphi')\), and then also \((K, 1) \leq (H', \varphi')\), since \(K\) is normal in \(G\).

\[\square\]

**4.7 Lemma** Let \(G\) be a finite group and \(H\) a normal subgroup of \(G\) of \(p\)-power index. Let \(V \in \mathcal{O}G-\text{lin}\) and \(W \in \mathcal{O}H-\text{lin}\) be indecomposable modules such that \(W\) is stable under \(G\) and \(W \mid \text{res}_H^G(V)\). Then, if \(H/\ker(W)\) is \(p\)-elementary so is \(G/\ker(V)\).

**Proof** Let \(K := \ker(W)\), then \(K\) is normal in \(G\), since \(W\) is \(G\)-stable. We assume that \(H/K\) is \(p\)-elementary. First note that \(G/K\) is again \(p\)-elementary. In fact, let \(L \leq P, C \leq H\) be intermediate groups with \(C/K = \mathcal{O}_p(H/K)\) and \(P/K = \mathcal{O}_p(H/K)\), hence \(H/K = C/K \times P/K\). Then \(W \cong \text{ind}_U^G(\mathcal{O}_\mu)\) for some \(C \leq U \leq H\) and \(\mu \in \mathcal{U}(\mathcal{O})\) by Lemma 4.6 (i). Since \(K = \ker(W)\), \(K\) is also the kernel of \(\text{res}_C^G(\mu)\). By the uniqueness part of Lemma 4.6 (ii), for each element \(g \in G\) there is some \(h \in H\) with \(g(U, \mu) = h(U, \mu)\). Therefore, \(\text{res}_C^G(\mu) = h\text{res}_C^G(\mu) = \text{res}_C^G(\mu)\), since \(H\) centralizes \(C/K\). Since \(\text{res}_C^G(\mu)\) has kernel \(K\), also \(G\) centralizes \(C/K\). Hence, \(G/K\) is again \(p\)-elementary, and it suffices to prove that \(K \leq \ker(V)\).

Let \(Q\) be a vertex of \(V\) and let \(\varphi \in \mathcal{Q}(\mathcal{O})\) such that \(\mathcal{O}_\varphi\) is a source of \(V\). Then \(V \mid \text{ind}_K^Q(\mathcal{O}_\varphi)\). Hence there exists an indecomposable module \(V' \in \mathcal{O}(KQ)-\text{lin}\) with \(V' \mid \text{ind}_Q^K(\mathcal{O}_\varphi)\) and \(V \mid \text{ind}_K^Q(V')\). We will show that \(V' \cong \mathcal{O}_\varphi\) for an extension \(\varphi' \in \text{Hom}(KQ, \mathcal{O}^\times)\) of \(\varphi\) with \(\text{res}_K^Q(\varphi') = 1\).

We have \(\mathcal{O} \mid \text{res}_K^Q(V')\), since

\[
\mathcal{O} \mid \text{res}_K^Q(W) \mid \text{res}_K^Q(\text{res}_H^G(V)) \cong \text{res}_K^Q(V) \mid \text{res}_K^Q(\text{ind}_K^Q(\mathcal{O}_\varphi)) \cong \bigoplus_{g \in K \cap G/KQ} \text{ind}_K^Q(g(KQ))(\text{res}_{K \cap gQ}^Q(g(V'))) \cong \bigoplus_{g \in G/KQ} g\text{res}_K^Q(V').
\]
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From

\[
\mathcal{O} | \res_K^Q(V') \mid \res_K^Q(\ind_K^Q(\mathcal{O}_\varphi)) = \bigoplus_{g \in K \cap Q} \ind_{K \cap Q}^Q(\res_{K \cap Q}^Q(\mathcal{O}_\varphi))
\]

we see that \( K \cap Q \) is a Sylow \( p \)-subgroup of \( K \), since the vertices of \( \mathcal{O} \in \mathcal{O}K-lin \) are the Sylow \( p \)-subgroups of \( K \). Therefore, \( Q \) is a Sylow \( p \)-subgroup of \( KQ \). Moreover, we obtain from \( \mathcal{O} | \ind_K^Q(\res_{K \cap Q}^Q(\mathcal{O}_\varphi)) \) that \( \res_{K \cap Q}^Q(\mathcal{O}) = 1 \), since \( \mathcal{O} \in \mathcal{O}K-lin \) has trivial source. Hence we can extend \( \varphi \in \Hom(Q, \mathcal{O}^\times) \) to \( \varphi' \in \Hom(KQ, \mathcal{O}^\times) \) by setting \( \varphi'(xy) := \varphi(y) \) for \( x \in K, y \in Q \). Since the multiplicity of \( \mathcal{O} \) in

\[
\res_K^Q(\ind_K^Q(\mathcal{O}_\varphi)) \cong \ind_{K \cap Q}^Q(\mathcal{O})
\]
equals 1, there is exactly one indecomposable summand \( X \) of \( \ind_K^Q(\mathcal{O}_\varphi) \) with \( \mathcal{O} \mid \res_{K}^Q(X) \). But both \( V' \) and \( \mathcal{O}_\varphi' \) have this property. Hence, \( V' \cong \mathcal{O}_\varphi' \), and

\[
\res_K^G(V) \mid \res_K^G(\ind_K^G(\mathcal{O}_\varphi')) \cong \bigoplus_{g \in G/KQ} g \res_K^G(\mathcal{O}_\varphi') \cong \bigoplus_{g \in G/KQ} \mathcal{O}.
\]

Therefore, \( K \leq \ker(V) \), and the result follows. \( \square \)

4.8 Proposition Let \( G \) be a finite group, let \( M, A', B', M', p', m', \alpha', a' \), and \( b' \) be defined according to Example 4.5, and let \( \Gamma, A, B, M, p, m, \alpha, a, \) and \( b \) be defined according to Example 4.2.

(i) Condition \((*_p)\) of Theorem II.4.5 holds for \( M, A', B', \) and \( p' \).

(ii) For \( H \leq G \) we define \( \eta_H : A'_+(H) \to A_-+H \) for \( K \leq H \) and indecomposable \( V \in \mathcal{O}K-lin \) with \( [V] \in B'(K) \) by

\[
\eta_H([K, [V]]_H) := [L, \varphi]_H,
\]
where \( [L, \varphi]_H \) is uniquely determined by \( V \cong \text{ind}^K_L(\mathcal{O}_\varphi) \), cf. Lemma 4.7 (ii). Then \( \eta \in \mathbb{Z} - \text{Res}(G)(A'_+, A_+) \) and the diagram

\[
\begin{array}{ccc}
\mathbb{Q} \otimes L_\mathcal{O} & \xrightarrow{a'} & \mathbb{Q} \otimes A'_+ & \xrightarrow{b'} & \mathbb{Q} \otimes L_\mathcal{O} \\
\downarrow \quad \eta & & \downarrow & & \\
\mathbb{Q} \otimes A_+ & = & \mathbb{Q} \otimes A_+ & = & \mathbb{Q} \otimes A_+
\end{array}
\]

is commutative.

(iii) The morphism \( a' \) is a canonical induction formula for \( \mathbb{Q} \otimes L_\mathcal{O} \) from \( \mathbb{Q} \otimes A' \).

**Proof**  

(i) Let \( K \leq H \leq G \) with \( H/K \) a cyclic \( p \)-group, and let \( W \in \mathcal{O}K-\text{lin} \) be indecomposable and \( H \)-stable such that \( W/\ker(W) \) is \( p \)-elementary (i.e. \( [W] \in \mathcal{B}'(K) \)). Let furthermore \( V \in \mathcal{O}H-\text{lin} \) be indecomposable. If \( [V] \in \mathcal{B}'(H) \), then

\[
p_K(\text{res}^H_V([V])) = \text{res}^H_K([V]) = \text{res}^H_K(p_K([V])).
\]

If \( V \notin \mathcal{B}'(H) \), then \( p_H([V]) = 0 \), and we have to prove that the multiplicity of \( W \) as a summand in \( \text{res}^H_K(V) \) is zero. But this is exactly the statement of Lemma 4.7.

In fact, if \( W \mid \text{res}^H_K(V) \), then we obtain \( [V] \in \mathcal{B}'(H) \).

(ii) First we show that \( \eta \) is a morphism of \( \mathbb{Z} \)-restriction functors on \( G \). Obviously \( \eta \) commutes with conjugations. Now let \( U \leq H \leq G \), and let \( (K, [V]) \in \mathcal{M}'(H) \), i.e. \( K \leq H, V \in \mathcal{O}K-\text{lin} \) indecomposable with \( [V] \in \mathcal{B}'(K) \). Let \( (L, \mu) \in \mathcal{M}(K) \) with \( V \cong \text{ind}^L_K(\mathcal{O}_\mu) \). Then we have

\[
\text{res}^H_U(\eta_H([K, [V]]_H)) = \text{res}^H_U(\mu) = \sum_{h \in U \cap H/L} [U \cap hL, \text{res}^h_L(\mu)]_U,
\]

and

\[
\eta_U(\text{res}^H_U([K, [V]]_H)) = \sum_{x \in U \cap H/K} \eta_U([U \cap xK, \text{res}^x_U(\mu)])_U.
\]

For \( x \in H \) we have

\[
\text{res}^x_{U \cap xK}(\varphi) \cong \text{res}^x_{U \cap xK}(\text{ind}^x_U(\mathcal{O}(\varphi)))
\]

\[
\cong \bigoplus_{y \in U \cap xK/\varphi} \text{ind}^x_{U \cap xK/\varphi}(\text{res}^y_U(\mathcal{O}(\varphi)))
\]

\[
\cong \bigoplus_{y \in U \cap xK/\varphi} \text{ind}^x_{U \cap xK/\varphi}(\mathcal{O}(\varphi)).
\]

Each of the summands in the last sum is indecomposable by Green’s indecomposability theorem. In fact, let \( C \leq L \leq K \) be such that \( C/\ker(V) = \mathcal{O}_\mu(K/\ker(V)) \). Then \( C \) is normal in \( K \) and \( (K : C) \) is a power of \( p \). Hence, \( U \cap \varphi C \) is normal in \( U \cap \mathcal{O}_\mu \) of \( p \)-power index, and \( U \cap \mathcal{O}_\mu \leq U \cap \mathcal{O}_\mu L \leq U \cap \mathcal{O}_\mu K \), so that Green’s indecomposability theorem can be applied. Therefore, by the definition of \( \eta \), we obtain

\[
\eta_U([U \cap xK, \text{res}^x_{U \cap xK}(\varphi)_U]) = \sum_{y \in U \cap \mathcal{O}_\mu} [U \cap \mathcal{O}_\mu, \text{res}^y_{U \cap \mathcal{O}_\mu}(\mathcal{O}(\varphi))]_U.
\]
and summing over \( x \in U \setminus H/K \) we obtain exactly the result of the above transformation of \( \text{res}^H_U (\eta_H([K,[V]]_H)) \), since the elements \( yx \) run through a set of representatives for \( U \setminus H/L \) if \( x \) runs through a set of representatives for \( U \setminus H/K \), and for each \( x, y \) runs through a set of representatives for \( U \cap L\setminus K \setminus K'/L \). Hence, we have \( \eta \in \mathbb{Z} - \text{Res}(G)(A_+^\prime, A_+) \).

Next we show the commutativity of the diagram in the assertion. Let \( H \leq G \) and let \( K, V, L, \) and \( \mu \) be as in the last paragraph, then we have

\[
b_H(\eta_H([K,[V]]_H)) = b_H([L,\mu]_H) = \text{ind}^H_L ([O,\mu]) = \text{ind}^H_K (\text{ind}^L_K (O,\mu)) = \text{ind}^H_K ([V]) = b_H([K,[V]]_H).
\]

This shows the commutativity of the right square of the diagram. For the commutativity of left square of the diagram it suffices to show that \( \rho^A_H \circ \eta_H \circ a'_H = \rho^A_H \circ a_H \) for \( H \leq G \), since \( \rho^A_H \) is injective. By the definition of \( \rho^A_H \) and the commutativity of \( a, a' \), and \( \eta \) with restrictions it suffices to show that \( \pi^A_H \circ \eta_H \circ a'_H = \pi^A_H \circ a_H \) for \( H \leq G \). Since \( \pi^A_H \circ a_H = p_H \), it suffices to show that

\[
\pi^A_H \circ \eta_H \circ a'_H = p_H
\]

for \( H \leq G \). Let \( V \in \mathcal{O}H - \text{lin} \) be indecomposable. If \( [V] \in \mathcal{B}(H) \), i.e. \( V \cong \mathcal{O}_\varphi \) for some \( \varphi \in \mathcal{H}(O) \), then

\[
(\pi^A_H \circ \eta_H \circ a'_H)([V]) = (\pi^A_H \circ \eta_H)([H,[V]]_H) = \pi^A_H([H,\varphi]_H) = \varphi = p_H(\varphi)
\]

by Lemma II.3.7. If \( [V] \in \mathcal{B}'(H) \setminus \mathcal{B}(H) \), and \( V \cong \text{ind}^H_K (O,\varphi) \) with \( K < H \), \( \varphi \in \mathcal{K}(O) \), then

\[
(\pi^A_H \circ \eta_H \circ a'_H)([V]) = (\pi^A_H \circ \eta_H)([H,[V]]_H) = \pi^A_H([K,\varphi]_H) = 0 = p_H([V])
\]

again by Lemma II.3.7. If \( [V] \notin \mathcal{B}'(H) \), then \( \pi^A_H(a'_H([V])) = p_H([V]) = 0 \). Hence \( a'_H([V]) \) is a linear combination of basis elements \([K,[W]]_H\), with \( K < H \), \([W] \in \mathcal{B}'(K) \). This implies that \( \eta_H(a'_H([V])) \) is a linear combination of basis elements \([K,\psi]_H \) with \( K < H \), \( \psi \in \mathcal{K}(O) \), and therefore we have \( (\pi^A_H \circ \eta_H \circ a'_H)([V]) = 0 \). On the other hand, also \( p_H([V]) = 0 \). This completes the proof of part (ii).

(iii) This follows from the commutative diagram in part (ii) and the fact that \( b \circ a = \text{id}_{\mathcal{O}(O)} \).

Now we can prove that \( a \) is integral. We already know that \( [H]_{\mathcal{O}} \cdot a_H(L_O(H)) \subseteq L^{ab}_O(H) \) for \( H \leq G \). From Proposition 4.8 (i) and Corollary II.4.7 we obtain that \( [H]_{\mathcal{O}} \cdot a'_H(L_O(H)) \subseteq A_+^\prime(H) \) for \( H \leq G \). Hence, Proposition 4.8 (ii) implies that

\[
[H]_{\mathcal{O}} \cdot a_H(L_O(H)) = [H]_{\mathcal{O}} \cdot (\eta_H \circ a'_H)(L_O(H)) \subseteq \eta_H([H]_{\mathcal{O}} \cdot a'_H(L_O(H))) \subseteq \eta_H(A_+^\prime(H)) \subseteq A_+(H).
\]

This implies \( a_H(L_O(H)) \subseteq A_+(H) \) for \( H \leq G \), and completes the proof of (part (i) of) Proposition 4.3.

4.9 Example Let \( G \) be a finite group, \( \tilde{M} := T_{\mathcal{O}} \in \mathbb{Z} - \text{Mack}_{\text{alg}}(G) \), \( \tilde{A} := T^{ab}_{\mathcal{O}} \in \mathbb{Z} - \text{Res}_{\text{alg}}(G) \), \( \tilde{B}(H) := \tilde{H}(O)_{\mathcal{O}} \) for \( H \leq G \), and let \( \tilde{p} \in \mathbb{Z} - \text{Con}(G)(\tilde{M}, \tilde{A}) \).
be defined for $H \leq G$ and for indecomposable $V \in \mathcal{O}H - \text{triv}$ by

$$\tilde{p}_H([V]) := \begin{cases} [V], & \text{if } \text{rk}_\mathcal{O}V = 1, \\ 0, & \text{otherwise.} \end{cases}$$

4.10 Proposition  Let $G$ be a finite group, let $M = L_O$, $A = L^\text{ab}_O$, $B$, $\mathcal{M}$, $p$, $m$, $\alpha$, $a$, and $b$ be defined according to Example 4.2, and let $\tilde{M} = T_O$, $\tilde{A} = T^\text{ab}_O$, $\tilde{B}$, $\tilde{\mathcal{M}}$, $\tilde{p}$, $\tilde{m}$, $\tilde{\alpha}$, $\tilde{a}$, and $\tilde{b}$ be defined according to Example 4.9. Then the diagram

$$
\begin{array}{c}
T_O & \xrightarrow{\tilde{a}} & T^\text{ab}_O + & \xrightarrow{\tilde{b}} & T_O \\
i & & \downarrow{i} & & \downarrow{i} \\
L_O & \xrightarrow{a} & L^\text{ab}_O + & \xrightarrow{b} & L_O
\end{array}
$$

commutes, where $i: T_O \to L_O$ and $i: T^\text{ab}_O \to L^\text{ab}_O$ denote the inclusion maps. The morphism $i_+$ is injective and $\alpha^H_{(K,\psi)}([V]) = \tilde{\alpha}^H_{(K,\psi)}([V])$ for $H \leq G$, $V \in \mathcal{O}H - \text{triv}$, and $(K,\psi) \in \tilde{\mathcal{M}}(H)$. In particular, $\tilde{a}$ is an integral canonical induction formula, and all the other parts of Proposition 4.3 hold with the appropriate changes in the situation of Example 4.9.

Proof  The morphism $i_+$ is injective, as has already been observed at the beginning of this section. The left part of the diagram commutes by Proposition II.3.11. All the other assertions are easy consequences or can be reproved with similar arguments as used in the proof of Proposition 4.3.

In the next chapter we will need a stronger version of Lemma 4.4 for trivial source modules.

4.11 Lemma  Let $G$ be a $p$-hypo-elementary group and let $P$ denote the Sylow $p$-subgroup of $G$. Furthermore, let $\tilde{M} = T_O$, $\tilde{A} = T^\text{ab}_O$, $\tilde{B}$, $\tilde{\mathcal{M}}$, $\tilde{p}$, $\tilde{m}$, $\tilde{\alpha}$, $\tilde{a}$, and $\tilde{b}$ be defined according to Example 4.9.

(i) For indecomposable $V \in \mathcal{O}G - \text{triv}$ we have:

$$\text{rk}_\mathcal{O}V = 1 \iff V \text{ has vertex } P$$

(ii) The map

$$\tilde{r}_G := \tilde{p}_G \circ \tilde{a}_G = (\tilde{p}_H \circ \text{res}^G_H)_{H \leq G}: T_O(G) \to T^\text{ab}_O(G)$$

is injective.

(iii) For $f \in \mathbb{Z} - \text{Res}(G)(T^\text{ab}_O, T^\text{ab}_O)$ we have

$$\tilde{p} \circ \tilde{b} \circ f_+ \circ \tilde{a} = f \circ \tilde{p}, \quad \text{in particular,} \quad \tilde{p} \circ \tilde{b} \circ \tilde{a} = \tilde{p}.$$

(iv) $\tilde{p}_G$ is a ring homomorphism.

Proof  (i) Let $\text{rk}_\mathcal{O}V = 1$. Then $V$ has vertex $P$. Conversely, if $V$ has vertex $P$, then

$$V \mid \text{ind}_P^G(\mathcal{O}) = \bigoplus_{\varphi \in \mathcal{B}(G)} \mathcal{O}_{\varphi},$$

where $\varphi \in \mathcal{B}(G)$.\]
and \( \text{rk}_O V = 1 \).

(ii) This follows from Lemma 4.4 (ii).

(iii) Let \( H \leq G \) and let \( V \in \mathcal{O}G - \text{triv} \) be indecomposable. If \( \text{rk}_O V = 1 \), then

\[
(\tilde{p}_H \circ \tilde{b}_H \circ f_{+H} \circ \tilde{a}_H)([V]) = f_H([V]) = (f_H \circ \tilde{p}_H)([V])
\]

for \( H \leq G \) by the analogue of Proposition 4.3 (iv). If \( \text{rk}_O V > 1 \), then we have \( \tilde{p}_H([V]) = 0 \), and \( \tilde{a}_H([V]) = 0 \) for all \( (K, \psi) \in \hat{M}(H) \) with \( O_p(H) \leq K \) by part (i) and the analogue Proposition 4.3 (vii). Hence, \( (\tilde{b}_H \circ f_{+H} \circ \tilde{a}_H)([V]) \) is a linear combination of elements \( \text{ind}^G_K(\psi) \) with \( (K, \psi) \in \hat{M}(H) \) such that the Sylow \( p \)-subgroup of \( K \) is strictly contained in \( O_p(H) \). Therefore, \( (\tilde{b}_H \circ f_{+H} \circ \tilde{a}_H)([V]) \) is a linear combination of elements \([W] \), where \( W \in \mathcal{O}H - \text{triv} \) is indecomposable with vertex strictly smaller than \( O_p(H) \). Now part (i) implies that \( (\tilde{p}_H \circ \tilde{b}_H \circ f_{+H} \circ \tilde{a}_H)([V]) = 0 \).

(iv) Let \( H \leq G \) and let \( V, W \in \mathcal{O}H - \text{triv} \) be indecomposable. If \( \text{rk}_O V = \text{rk}_O W = 1 \), then obviously \( \tilde{p}_H([V] \cdot [W]) = [V \otimes W] = [V] \cdot [W] = \tilde{p}_H([V]) \cdot \tilde{p}_H([W]) \).

Hence, we may assume that \( \text{rk}_O V > 1 \). Then \( \tilde{p}_H([V]) = 0 \), and by part (i), \( V \) has a vertex \( Q \) strictly contained in \( O_p(H) \). Since \( V \) is \( Q \)-projective, so is \( V \otimes W \) by the Frobenius axiom (cf. Definition I.1.1 (vi)). Part (i) again shows that \( V \otimes W \) has no summand of \( \mathcal{O} \)-rank one. Hence \( \tilde{p}_H([V \otimes W]) = 0 \). \( \square \)

4.12 Remark  
(i) In Example 4.9 we may replace \( O \) by its residue field \( F \). It is well-known (see [Bro85, 3.5] for example) that reduction yields an isomorphism \( T_O \to T_F \) of \( Z \)-Green functors identifying also \( T_O^{ab} \) with \( T_F^{ab} \). Here \( T_F \) is the Mackey functor arising from the categories \( FH - \text{triv} \) defined in the same way as \( \mathcal{O}H - \text{triv} \) for \( H \leq G \), and \( T_F^{ab}(H) \) is \( Z \)-free on the set \( \hat{H}(F) := \text{Hom}(H, F^\times) \) for \( H \leq G \). The morphism \( \tilde{p} \) translates to a morphism \( T_F \to T_F^{ab} \) annihilating the indecomposable trivial source modules over \( F \) of dimension greater than 1, and being the identity on one-dimensional modules. The resulting canonical induction formula \( T_F \to T_F^{ab} \) shares all properties of Proposition 4.10, or better of Proposition 4.3 with the appropriate changes.

Moreover, this canonical induction formula commutes with the one of Example 3.1 with respect to the embedding \( P_F \to T_F \). In contrast to the situation in Example 3.1, the coefficient

\[
\frac{|(N_H(\sigma)/H_0)p'| \cdot m_\sigma(\text{res}^H_{H_0}([V]))}{|N_H(\sigma)/H_0|}
\]

in the explicit formula for \( a \) in Proposition 4.3 (v) is in general not an integer, as one can see easily for \( V = \mathcal{O} \) and \( \sigma = ((1,1)) \), where it reduces to \( |H_0'|/|H| \).

(ii) There are other canonical induction formulae for \( T_O \) or \( L_O \) in the spirit of Examples 1.4, 1.7 and Remark 1.9 (ii).

(iii) In general, the canonical induction formulae \( a \) and \( \tilde{a} \) are not morphisms of Mackey functors, since condition (ii) in Lemma II.2.5 does not hold, and they are not multiplicative, since \( p \) and \( \tilde{p} \) aren’t, cf. Proposition II.2.9 (i).

In the next chapter we will need the notion of a species (cf. [Be84b, 2.2 and 2.13]) on \( T_O(G) \), and some of its properties. For the rest of this section let \( K \)
denote the quotient field of \( \mathcal{O} \) which is of characteristic zero and which is assumed to be a splitting field for all finite groups occurring in this section.

### 4.13 Definition
Assume that notation of Example 4.9. (i) Let \( G \) be a finite group. A **species** of \( T_{\mathcal{O}}(G) \) is a ring homomorphism \( s: L_{\mathcal{O}}(G) \rightarrow K \).

(ii) For a \( p \)-hypo-elementary group \( G, g \in G \), and \( V \in \mathcal{O}G - \text{triv} \) with \([V] \in T_{\mathcal{O}}^{ab}\) we define

\[
s_g: T_{\mathcal{O}}^{ab}(G) \rightarrow K, \quad [V] \mapsto \sum_{\varphi \in \hat{G}(\mathcal{O})} m_{\varphi}([V]) \cdot \varphi(g).
\]

In other words, \( s_g([V]) \) is the trace of the \( \mathcal{O} \)-linear action of \( g \) on \( V \). Note that \( s_g \) is a ring homomorphism.

(iii) For a finite group \( G \), a \( p \)-hypo-elementary subgroup \( H \leq G \), and \( h \in H \) we define a species (cf. Lemma 4.11 (iv))

\[
s_{(H,h)} := s_h \circ \overline{p}_H \circ \text{res}_H: T_{\mathcal{O}}(G) \rightarrow K
\]

of \( T_{\mathcal{O}}(G) \).

### 4.14 Proposition
For a finite group \( G \), the ring homomorphism

\[
s_G := \prod_{(H,h)} s_{(H,h)}: L_{\mathcal{O}}(G) \rightarrow \prod_{(H,h)} K,
\]

where \( (H,h) \) runs through all pairs with \( H \leq G \) \( p \)-hypo-elementary and \( h \in H \), is injective. In particular, the \( K \)-algebra \( K \otimes T_{\mathcal{O}}(G) \), is semisimple, i.e. isomorphic to a direct product of copies of \( K \).

**Proof** This follows from Lemma 4.11 (ii) together with the fact that the set of \( p \)-hypo-elementary subgroups of \( G \) contains the set of coprimordial subgroups for \( T_{\mathcal{O}} \).

### 4.15 Proposition
Let \( G \) be a \( p \)-hypo-elementary group. Then \( G \) is coprimordial for \( T_{\mathcal{O}} \) and \( L_{\mathcal{O}} \).

**Proof** Assume that \( G \notin \mathcal{C}(T_{\mathcal{O}}) = \mathcal{C}(\mathbb{Q} \otimes T_{\mathcal{O}}) \). Then by Proposition I.6.2, we have \( e_{T_{\mathcal{O}}}(G) \cdot 1_{T_{\mathcal{O}}(G)} \neq 0 \), i.e.

\[
[\mathcal{O}] = \sum_{H < G} \alpha_H \text{ind}_H^G([\mathcal{O}])
\]

in \( \mathbb{Q} \otimes T_{\mathcal{O}}(G) \) for certain rational coefficients \( \alpha_H \in \mathbb{Q}, \ H < G \), by the explicit formula for \( e_{T_{\mathcal{O}}}(G) \), cf. Remark I.3.3. We apply the species \( s_{(G,g)} \) to both sides of this equation, where \( g \in G \) is chosen in such a way that \( gO_p(G) \) generates \( G/O_p(G) \). Obviously,

\[
s_{(G,g)}([\mathcal{O}]) = 1,
\]

and we will show that

\[
s_{(G,g)}(\text{ind}_H^G([\mathcal{O}])) = 0
\]

for \( H < G \), which yields the desired contradiction. In fact, if \( O_p(G) \leq H < G \), then \( s_{(G,g)}(\text{ind}_H^G([\mathcal{O}])) \) equals the \( K \)-character of \( \text{ind}_H^G(1) \) evaluated at \( g \), and this is zero.
If \( O_p(G) \not\leq H \), then each indecomposable summand in \( \text{ind}_H^G(O) \) has a vertex which is strictly smaller than \( O_p(G) \), and Lemma 4.11 (i) implies that \( \tilde{p}_G(\text{ind}_H^G([O])) = 0 \). This shows that \( G \) is coprimordial for \( T_O \). By Proposition I.6.2 the same is true for \( L_O \).

4.16 Remark It is surprising that the ring \( L_O(G) \) is not treated in the literature to the same extent as \( T_O(G) \), since they are comparably easy or difficult to investigate. However, \( L_O(G) \) has the advantage that it can serve as a link between \( R_F(G) \) and \( R_K(G) \), since we have surjections of Mackey functors

\[
L_O \xrightarrow{\text{res}} R_F \xrightarrow{\text{res}} K
\]

Moreover, one can show that \( K \otimes L_O(G) \) is semisimple, so that we have for all three rings a character theory. In fact, we may extend the species \( s_{(H,h)} : T_O(G) \rightarrow K \), for \( H \leq G \) \( p \)-hypo-elementary and \( h \in H \), to a ring homomorphism

\[
s_{(H,h)} : L_O(G) \rightarrow K
\]

by the following construction:

For a \( p \)-hypo-elementary subgroup \( H \) of \( G \) with Sylow \( p \)-subgroup \( P \) and \( V \in OH-\text{lin} \) we define

\[
p_H^* : L_O(G) \rightarrow L_O(G), \quad [V] \mapsto [V'],
\]

where we use a decomposition \( V = V' \oplus V'' \) with \( V' \) having only indecomposable summands with vertex \( P \), and \( V'' \) having only indecomposable summands with vertex strictly smaller than \( P \). Note that \( p_H^* \) is a ring homomorphism. Moreover, for \( h \in H \) and \( V \in OH-\text{lin} \) we denote by \( s_h([V]) \) the trace of the multiplication by \( h \) on \( V \). This defines a ring homomorphism

\[
s_h : L_O(G) \rightarrow K.
\]

Now, \( s_{(H,h)} \) is defined as

\[
s_{(H,h)} := s_h \circ p_H^* \circ \text{res}_H^G.
\]

Using arguments of Lemma 4.4 (ii) it is not difficult to see that the collection

\[
s_G := \prod_{(H,h)} s_{(H,h)} : L_O(G) \rightarrow \prod_{(H,h)} K
\]

is injective.

### 3.5 The Green ring

Throughout this section let \( O \) and \( F \) be defined as in the previous section.

For a finite group \( G \) we denote by \( OG-\text{lat} \) the category of \( OG \)-lattices, i.e. finitely generated \( O \)-free \( OG \)-modules. The Grothendieck ring of \( OG-\text{lat} \) with respect to direct sums will be called the Green ring of \( G \) over \( O \), and will be
denoted by $Gr_{\mathcal{O}}(G)$. The element in $Gr_{\mathcal{O}}(G)$ associated to $V \in \mathcal{O}G-\mathbf{lat}$ will be again denoted by $[V]$. By the Krull-Schmidt-Azumaya theorem (cf. [CR81, 6.12]) the ring $Gr_{\mathcal{O}}(G)$ is a free abelian group on the set of elements $[V]$, where $V \in \mathcal{O}G-\mathbf{lat}$ runs through a set of representatives of the isomorphism classes of indecomposable $\mathcal{O}G$-lattices.

With the usual conjugation, restriction and induction maps, the rings $Gr_{\mathcal{O}}(H)$, $H \leq G$, form a $\mathbb{Z}$-Green functor on $G$. Since $\mathbb{Q}\otimes T_{\mathcal{O}} \subseteq \mathbb{Q}\otimes Gr_{\mathcal{O}}$ is an inclusion of quasi-Green functors on $G$, Proposition I.6.2 and Proposition 4.15 imply that $\mathcal{C}(Gr_{\mathcal{O}}) = \mathcal{C}(\mathbb{Q} \otimes Gr_{\mathcal{O}})$ is the set of $p$-hypo-elementary subgroups of $G$.

For a finite group $G$ we define $\mathcal{B}$ as the set of elements $[V] \in Gr_{\mathcal{O}}(G)$, where $V \in \mathcal{O}G-\mathbf{lat}$ is indecomposable and $G/\ker(V)$ is solvable with a normal Sylow $p$-subgroup. Further we define the subring $Gr'_{\mathcal{O}}(G)$ of $Gr_{\mathcal{O}}(G)$ as the span of $\mathcal{B}(G)$. Obviously, $Gr'_{\mathcal{O}}(G)$ is free on $\mathcal{B}(G)$ and the rings $Gr'_{\mathcal{O}}(H)$, $H \leq G$, form a $\mathbb{Z}$-algebra restriction subfunctor of $Gr_{\mathcal{O}}$ on $G$.

If $f: G' \to G$ is a homomorphism of finite groups, there is an induced ring homomorphism $\text{res}_f: Gr_{\mathcal{O}}(G) \to Gr_{\mathcal{O}}(G')$ restricting along the obvious ring homomorphism $\mathcal{O}G' \to \mathcal{O}G$. This ring homomorphism maps $Gr'_{\mathcal{O}}(G)$ to $Gr'_{\mathcal{O}}(G')$, and we can define a ring homomorphism

$$\text{res}_{+f}: Gr'_{\mathcal{O}+}(G) \to Gr'_{\mathcal{O}+}(G'),$$

$$[H, [V]]_G \mapsto \sum_{g \in f(G') \setminus G/H} [f^{-1}(gH), \text{res}_f: f^{-1}(gH) \to gH([V])]_{G'}$$

as in Formula (3.3) which allows to consider $Gr_{\mathcal{O}}$, $Gr'_{\mathcal{O}}$, and $Gr'_{\mathcal{O}+}$ as contravariant functors from the category of finite groups to the category of commutative rings.

It is easily verified that $b^{Gr_{\mathcal{O}}, Gr'_{\mathcal{O}}}: Gr'_{\mathcal{O}+} \to Gr_{\mathcal{O}}$ is a natural transformation. If $f$ is the inclusion of a subgroup $H \leq G$, we write $\text{res}_{+f}^G_H$ instead of $\text{res}_{+f}$, which is consistent with the definition of $\text{res}_{+}$ in I.2.2. And if $f$ is the natural epimorphism $G \to G/N$ for $N \trianglelefteq G$ we write $\text{inf}_{+f}^G_H$ instead of $\text{res}_{+f}$.

As in the previous sections we have Galois actions of the automorphism group of $\mathcal{O}$ on $Gr_{\mathcal{O}}$, $Gr'_{\mathcal{O}}$, and $Gr'_{\mathcal{O}+}$.

5.1 Example Let $G$ be a finite group, $M := Gr_{\mathcal{O}} \in \mathbb{Z}-\text{Mack}_{\mathcal{O}}(G)$, $A := Gr'_{\mathcal{O}} \in \mathbb{Z}-\text{Res}_{\mathcal{O}}(G)$, $\mathcal{B}(H)$ as defined at the beginning of this section, and let $p \in \mathbb{Z}-\text{Con}(G)(M, A)$ be defined for $H \leq G$ and indecomposable $V \in \mathcal{O}H-\mathbf{lat}$ by

$$p_H([V]) := \begin{cases} [V], & \text{if } [V] \in \mathcal{B}(H) \\ 0, & \text{otherwise.} \end{cases}$$

Note that, for $H \leq G$ and $V, W \in \mathcal{O}H-\mathbf{lat}$ with $[W] \in \mathcal{B}(H)$, the number $m_{[W]}([V])$ is the multiplicity of $W$ in a direct sum decomposition of $V$.

5.2 Proposition Let $G$ be a finite group, and let $M, A, \mathcal{B}, \mathcal{M}, p, m, a, \alpha, a$ and $b$ be defined according to Example 5.1.

(i) The morphism $a$ is a canonical induction formula and $|H|^p \cdot a_H$ is integral for $H \leq G$.

(ii) For $H \leq G$ the homomorphism $a_H$ is $A(H)$-linear.

(iii) The morphism $a$ respects the Galois action.
G \otimes Q \text{ form a natural transformation between the contravariant functors}

\text{is commutative. In particular, the maps } a_H, a_H[V] = [H, [V]]_H.

(v) The morphism \( a \) is given explicitly by

\[ a_H([V]) = \sum_{\sigma \in \Gamma(M(H))} (-1)^n \times \frac{\left( N_H(\sigma)/H_0 \right)_{p^i}}{|N_H(\sigma)/H_0|} m_{\sigma[H_n]}(\text{res}^H_{H_n}([V])) [H_0, [V]]_H \]

for \( H \leq G \) and \( V \in \mathcal{O}H-\text{lat} \), and we have

\[ [V] = \sum_{\sigma \in \Gamma(M(H))} (-1)^n \times \frac{\left( N_H(\sigma)/H_0 \right)_{p^i}}{|N_H(\sigma)/H_0|} m_{\sigma[H_n]}(\text{res}^H_{H_n}([V])) \text{ind}^H_{H_0}([V_0]) \]

in \( \text{Gr}_{\mathcal{O}}(G) \).

(vi) For \( H \leq G \), \( V \in \mathcal{O}H-\text{lat} \) and \( (K, [W]) \in M(H) \) we have

\[ m_{[W]}(\text{res}^H_K([V])) = 0 \implies a^H_{(K, [W])}([V]) = 0. \]

(vii) Let \( f: G' \to G \) be a homomorphism of finite groups. Then the diagram

\[
\begin{array}{ccc}
\mathbb{Q} \otimes \text{Gr}_{\mathcal{O}}(G) & \xrightarrow{a_G} & \mathbb{Q} \otimes \text{Gr}_{\mathcal{O}'}^{G'}(G) \\
\text{res}_f & & \text{res}_{f'} \\
\mathbb{Q} \otimes \text{Gr}_{\mathcal{O}}(G') & \xrightarrow{a_{G'}} & \mathbb{Q} \otimes \text{Gr}_{\mathcal{O}'}^{G'}(G')
\end{array}
\]

is commutative. In particular, the maps \( a_G \), where \( G \) runs over all finite groups, form a natural transformation between the contravariant functors \( \mathbb{Q} \otimes \text{Gr}_{\mathcal{O}} \) and \( \mathbb{Q} \otimes \text{Gr}_{\mathcal{O}'}^{G} \) with values in \( \mathbb{Q} \)-vector spaces.

**Proof** First we show that condition \( \ast_{p'} \) in Theorem II.4.5 is satisfied for \( M, A, B, \) and \( p \in \mathbb{Z} - \text{Con}(G)(M, A) \) for the set of primes different from \( p \). Let \( K \leq H \leq G \) with \( H/K \) acyclic \( p' \)-group, and let \( W \in \mathcal{O}K-\text{lat} \) be indecomposable and stable under \( H \) with \( [W] \in B(H) \). Let furthermore \( V \in \mathcal{O}H-\text{lat} \) be indecomposable. If \([V] \in B(H)\), then

\[ p_K(\text{res}^H_K([V])) = \text{res}^H_K([V]) = \text{res}^H_K(p_H([V])) \]

and the condition \( \ast_{p'} \) holds for \( V \). Now assume that \([V] \notin B(H)\). Then \( p_H([V]) = 0 \), and we have to show that \( W \) is not a constituent of \( \text{res}^H_K(V) \). We assume that \( W \mid \text{res}^H_K(V) \) and will deduce a contradiction. Since \( (H : K) \) is a \( p' \)-number, \( V \) is \( K \)-projective and there is some indecomposable \( V' \in \mathcal{O}K-\text{lat} \) with \( V' \mid \text{res}^H_K(V) \) and \( V \mid \text{ind}^H_K(V') \). This implies

\[ W \mid \text{res}^H_K(V) \mid \text{res}^H_K(\text{ind}^H_K(V')) \cong \bigoplus_{h \in H/K} hV' \]
by the Mackey decomposition formula. Hence $W \cong V'$, since $W$ is $H$-stable. This implies that $V$ is a summand of $\text{ind}^H_K(W)$. But $\ker(W)$ is normal in $H$ and $\ker(\text{ind}^H_K(W)) \geq \ker(W)$ by the Mackey decomposition formula. Hence $H/\ker(V)$ is again solvable with normal $p$-Sylow subgroup, and therefore $[V] \in \mathcal{B}(H)$ which is a contradiction.

(v) This follows from Theorem II.4.5.

(i) This follows from Corollary II.2.4, since $Gr'_O(H) = Gr_O(H)$ for $p$-hypo-elementary subgroups $H \leq G$, and from Corollary II.4.7.

(ii) This follows from Proposition II.2.9 (ii).

(iii) This follows from Diagram (2.2).

(iv) This follows from Lemma II.3.7.

(v) This follows from part (v).

(vi) This is proved by the appropriate modifications of the proof of Proposition 1.2 (xiii).

5.3 Remark  (i) We do not know whether $a$ is integral or not. In particular we do not know how to imitate the trick of the previous section, where $a$ was proved to be a homomorphic image of a formula with $p'$-denominators.

(ii) It is not clear whether $Gr'_O$ is the right candidate for a canonical induction formula for $Gr_O$. There are suggestions by Puig, to replace the condition ‘solvable with normal Sylow $p$-subgroup’ with ‘nilpotent modulo $p$’ or ‘elementary modulo $p$’. But then we are not even able to prove that $|H|_p \cdot a_H$ is integral. If we replace our condition by ‘abelian modulo $p$’ the resulting formula is definitely not integral as examples show.

(iii) By a suggestion of Dipper we may replace the condition ‘solvable with normal Sylow $p$-subgroup’ with ‘abelian modulo $p$’ and replace $p \in \mathbb{Z} - \text{Con}(G)(M, A)$ by taking fixed points under the smallest normal subgroup with ‘abelian modulo $p$’ factor group. But again, it is definitely not integral as examples show. One might as well try a combination of the suggestions of Puig in (ii) and the modified definition of $p \in \mathbb{Z} - \text{Con}(G)(M, A)$ using a kind of Harish-Chandra restriction by taking fixed points under suitable distinguished subgroups. We just have no answer to the question of integrality in all these examples.

(iv) We set $M := Gr_O \in \mathbb{Z} - \text{Mack}_{\text{alg}}(G)$. Let $A(H) := M(H) = Gr_O(H)$ for all solvable subgroups $H \leq G$, and $A(H) = 0$, if $H \not\leq G$ is not solvable. Let $\mathcal{B}(H)$ consist of the isomorphism classes of indecomposable $OH$-lattices for solvable $H \leq G$, and let $\mathcal{B}(H)$ be the empty set for all other subgroups. Moreover, let $p_H: M(H) \rightarrow A(H)$ be the identity for solvable $H \leq G$, and the trivial map otherwise. Then it is very easy to see that condition $(\ast_{\pi})$ of Theorem II.4.5 is satisfied for the set $\pi$ of all primes, and since each $p$-hypo-elementary subgroup is solvable, Corollary II.4.9 implies that the corresponding morphism $a^{M,A,p}$ is an integral canonical induction formula for $Gr_O$ from $Gr'_O$, but it seems that we allow too many modules to be induced. We would rather like to induce modules of restricted type from arbitrary subgroups than inducing all modules from subgroups of restricted type. At least this was the theme in all the previous sections.

(v) We may replace the ring $\mathcal{O}$ with its residue field $F$ everywhere in this section, and obtain the corresponding results of Proposition 5.2 also in this situation. The question of integrality is not easier over $F$ than over $\mathcal{O}$.

(vi) There are other non-integral canonical induction formulae for $Gr_O$ inducing
3.6. SUMMARY

from \(p\)-hypo-elementary subgroups only. They are the precise analogues of Examples 1.4 and 1.7.

3.6 Summary

Let \(G\) be a finite group and let \(\mathcal{O}\) be a complete discrete valuation ring with algebraically closed residue field \(F\) of characteristic \(p > 0\), and with quotient field \(K\) of characteristic zero containing all \(|G|\)-th roots of unity. Furthermore, let \(\pi\) be a set of primes containing \(p\).

Recalling the definitions of the previous sections we have a commutative diagram

\[
\begin{array}{ccc}
R_{K,\pi} & \supset & P_{\mathcal{O}} \\
\cap & \supset & \cap \\
R_{K,p} & \leftarrow & P_{\mathcal{O}} \rightarrow P_F \\
\cap & \supset & \cap \\
R_K & \leftarrow & T_{\mathcal{O}} \rightarrow T_F \\
\Vert & \supset & \Vert \\
R_K & \leftarrow & L_{\mathcal{O}} \rightarrow T_F \\
\Vert & \supset & \Vert \\
R_K & \leftarrow & G_{\mathcal{T}\mathcal{O}} \rightarrow G_{\mathcal{T}F} \\
\Vert & \supset & \Vert \\
R_K & \leftarrow & \rightarrow \\
\Vert & \supset & \Vert \\
R_K & \rightarrow & d \rightarrow R_F
\end{array}
\]

(3.5)

in the category \(Z-\text{Mack}(G)\), involving the different choices for \(M\) in the Examples of the previous sections, where the horizontal left bound arrows are induced by the functor \(K \otimes_{\mathcal{O}} -\), and the horizontal right bound arrows are induced by the functor \(F \otimes_{\mathcal{O}} -\). The vertical morphism \(G_{\mathcal{T}F}(H) \rightarrow R_F(H)\) maps \([V]\) to \([V]\) for \(V \in \mathcal{O}H-\text{lat}\) and \(H \leq G\). The morphism \(d: R_K \rightarrow R_F\) is uniquely determined by \(G_{\mathcal{T}O} \rightarrow R_F\), since \(G_{\mathcal{T}O} \rightarrow R_K\) is surjective. It is the well-known decomposition map. The \(cde\)-triangle (cf. [Se78, §15]) is part of the diagram, with the Cartan map \(c: P_F \rightarrow R_F\) given by the vertical arrows, and \(e: P_F \cong P_{\mathcal{O}} \rightarrow R_K\) given by any path in the diagram connecting the two Mackey functors. Moreover, we obtain a similar commutative diagram

\[
\begin{array}{ccc}
R_{ab,K,\pi} & \supset & P_{ab} \\
\cap & \supset & \cap \\
R_{ab,K,p} & \leftarrow & P_{ab} \rightarrow P_{ab} \\
\cap & \supset & \cap \\
R_{ab,K} & \leftarrow & T_{ab} \rightarrow T_{ab} \\
\Vert & \supset & \Vert \\
R_{ab,K} & \leftarrow & L_{ab} \rightarrow T_{ab} \\
\Vert & \supset & \Vert \\
R_{ab,K} & \leftarrow & \rightarrow \\
\Vert & \supset & \Vert \\
R_{ab,K} & \rightarrow & d \rightarrow R_{ab,F}
\end{array}
\]

(3.6)

in the category \(Z-\text{Res}(G)\), involving the different choices of \(A\), and by functoriality.
a commutative diagram

\[
\begin{align*}
R_{ab}^{\kappa,\pi} \\
\cap \\
R_{ab}^{\kappa,p} & \xleftarrow{\sim} P_{ab}^{\sigma} \xrightarrow{\sim} P_{F}^{\sigma} \\
\cap \\
T_{ab}^{\kappa} & \xleftarrow{\sim} T_{F}^{ab} \\
L_{ab}^{\kappa} & \xrightarrow{d} R_{F}^{ab}
\end{align*}
\]

in the category $\mathbb{Z} \text{-Mack}(G)$. The induction morphisms $b^{M,A} : A_+ \to M$ from Diagram (3.7) to Diagram (3.5) yield a three-dimensional commutative diagram (which we do not write down). Since the maps $p \in \mathbb{Z} \text{-Con}(G)(M,A)$ from the appropriate part of Diagram (3.5) to Diagram (3.6) only commute on the skeleton

\[
\begin{align*}
R_{K,p} & \xleftarrow{\sim} P_{ab}^{\sigma} \xrightarrow{\sim} P_{F}^{\sigma} \\
\cap \\
T_{ab}^{\sigma} & \xleftarrow{\sim} T_{F}^{ab} \\
L_{ab}^{\sigma} & \xrightarrow{d} R_{F}^{ab}
\end{align*}
\]

of Diagram (3.5), the canonical induction formulae commute only on Diagram (3.8), cf. Proposition II.3.11. In particular, we obtain a commutative diagram (cf. Proposition 4.10 and Remark 4.11 (i))

\[
\begin{align*}
P_{ab}^{\sigma} & \xrightarrow{a} P_{ab}^{\sigma} \\
\cap \\
T_{ab}^{\sigma} & \xrightarrow{a} T_{ab}^{\sigma} \\
L_{ab}^{\sigma} & \xrightarrow{a} L_{ab}^{\sigma}
\end{align*}
\]

Note moreover, that none of the three morphisms in the $cde$-triangle commutes with the respective canonical induction formulae.

**6.1 Remark** If we consider the $\mathbb{Q}$-tensored version of the Diagrams (3.5)–(3.7) together with the morphisms $p \in \mathbb{Q} \text{-Con}(G)(\mathbb{Q} \otimes M, \mathbb{Q} \otimes A)$ which are defined by multiplication with the primitive idempotent $e_H^{(H)} \in \mathbb{Q} \otimes \Omega(H)$ on $M(H)$, for $H \leq G$, then all corresponding induction formulae commute with each other by Proposition II.3.11, since for any morphism $f \in \mathbb{Q} \text{-Mack}(G)(M,N)$ between $\mathbb{Q}$-Mackey functors on $G$ we have

\[f_H(e_H^{(H)} \cdot x) = e_H^{(H)} \cdot x\]

for $H \leq G$, $x \in M(H)$. See also Remark 2.6 (ii) for an example of this statement.
3.7 Extraspecial $p$-groups: an explicit example

In this section we will compute the map $a_G: R(G) \to R^ab_\ast(G)$ in the case of Example 1.1 for the extraspecial $p$-group $G$ of exponent $p$ and order $p^{2n+1}$, where $n \geq 0$ and $p$ is an odd prime. For details about extraspecial $p$-groups see for example [Hu67, III.13] and [Su86, IV.4, Exerc. 4], for symplectic forms see [Hu67, II.9], and for the representation theory of $G$ see [Hu67, V.16.14].

7.1 The centre $Z$ of $G$ has order $p$ and $G/Z$ is a symplectic $\mathbb{F}_p$-vector space of dimension $2n$, where the symplectic form is given by the commutator, if one identifies $Z$ with $\mathbb{F}_p$ once and for all. Note that $Z$ equals the Frattini subgroup of $G$ and also the commutator subgroup of $G$. A subgroup $H \geq Z$ of $G$ is abelian if and only if $H/Z$ is a totally isotropic subspace of $G/Z$. The automorphism group $\text{Aut}(G)$ has the normal subgroup $\text{Inn}(G)$ of inner automorphisms which consists precisely of those automorphisms which act trivially on $Z$ and $G/Z$. Each automorphism of $G$ induces an automorphism of $Z$. This yields a surjective group homomorphism $\beta: \text{Aut}(G) \to \text{Aut}(Z) \cong \mathbb{F}_p^\times$, and we obtain a series

$$1 < \text{Inn}(G) < \ker(\beta) < \text{Aut}(G)$$

of normal subgroups of $\text{Aut}(G)$ with $\text{Aut}(G)/\ker(\beta) \cong \mathbb{F}_p^\times$ and $\ker(\beta)/\text{Inn}(G) \cong \text{Sp}_{2n}(\mathbb{F}_p)$, the group of symplectic $2n$ by $2n$ matrices over $\mathbb{F}_p$, since each element of $\ker(\beta)$ respects the symplectic structure of $G/Z$, and since each element in $\text{Sp}_{2n}(\mathbb{F}_p)$ can be realized as an automorphism of $G$. Therefore we will denote $\ker(\beta)$ also by $\tilde{\text{Sp}}_{2n} \leq \text{Aut}(G)$. Note that, by a theorem of Witt, $\text{Sp}_{2n}(\mathbb{F}_p)$ acts transitively on the set of totally isotropic subspaces of $G/Z$ of fixed dimension. Hence, $\tilde{\text{Sp}}_{2n}$ acts transitively on the set of abelian subgroups $H \geq Z$ of $G$ of fixed order. The maximal abelian subgroups of $G$ are all conjugate under $\tilde{\text{Sp}}_{2n}$, since they all contain $Z$. Moreover, a maximal abelian subgroup of $G$ has order $p^{n+1}$.

The set $\text{Irr}(G)$ consists of $p^{2n}$ one-dimensional characters and $p - 1$ characters of degree $p^n$. Let $\lambda$ be one of the $p - 1$ non-trivial irreducible characters of $Z$, let $\tilde{\lambda}$ be an extension of $\lambda$ to a maximal abelian subgroup $H \leq G$, then $\text{ind}^{G}_{H}(\tilde{\lambda})$ is irreducible and depends only on $\lambda$ and not on $H$ and $\tilde{\lambda}$. Since $\text{ind}^{G}_{H}(\tilde{\lambda})$ is irreducible, the stabilizer of $\tilde{\lambda}$ in $G$ is $H$, and therefore, $\tilde{\lambda}$ has exactly $p^n$ $G$-conjugates which have to be precisely the $p^n$ different extensions of $\lambda$ to $H$, since $\lambda$ is stable under $G$-conjugation.

7.2 Proposition Let $\chi \in \text{Irr}(G)$ be of degree $p^n$ and let $\lambda \in \text{Irr}(Z)$ be such that $\text{res}^{G}_{H}(\chi) = p^n \cdot \lambda$. Then

$$a_G(\chi) = \sum_{d=0}^{n} (-1)^d p^{d(d-1)} \sum_{Z \leq H \text{ abelian} \atop |H/Z|=p^{n-d}} [H, \varphi]_G,$$

where $\varphi \in \hat{H}$ is an arbitrary extension of $\lambda$ to $H$.

Proof We write

$$a_G(\chi) = \sum_{(H, \varphi) \in G \cdot \mathcal{M}(G)} a^{G}_{(H, \varphi)}(\chi)[H, \varphi]_G,$$
with $\alpha_G^H(H, \varphi)(\chi) \in \mathbb{Z}$ as in Equation (3.1). By Proposition 1.2 (xi) we obtain $\alpha_G^H(H, \varphi)(\chi) = 0$ unless $(Z, \lambda) \leq (H, \varphi)$. But in this case $H$ is abelian. In fact, if the commutator group $H'$ of $H$ were non-trivial, then $H' \leq G' = Z$, hence $H' = Z$, and $\text{res}^H_H(\varphi) = 1$, which is a contradiction to $\text{res}^H_Z(\varphi) = \lambda$. For given abelian subgroup $H$ with $Z \leq H$, the extensions $\varphi$ of $\lambda$ form a full $G$-orbit, since they do in the case of a maximal abelian subgroup of $G$. Therefore, we obtain

$$a_G(\chi) = \sum_{Z \leq H \text{ abelian}} \alpha_H[H, \varphi]_G,$$

where $\alpha_H \in \mathbb{Z}$ and $\varphi$ is an extension of $\lambda$ to $H$. Let $X$ denote the poset of abelian subgroups of $G$ containing $Z$. For each $d \in \{0, \ldots, n\}$, the subgroup $\mathbb{S}_{p^{2n}} \in \text{Aut}(G)$ (which fixes $\lambda$ and hence $\chi$) acts transitively on the set of elements $H \in X$ with $|H/Z| = p^{n-d}$. Hence, by Proposition 1.2 (xiii) we obtain $\alpha_H = \alpha_H'$ for $H, H' \in X$ with $|H| = |H'|$. Therefore we can write

$$a_G(\chi) = \sum_{d=0}^n \alpha_d \sum_{|H| = p^{n-d} \in X} [H, \varphi]_G,$$

where $\alpha_d \in \mathbb{Z}$ and $\varphi$ is an extension of $\lambda$ to $H$. Now it suffices to show

$$\alpha_d = (-1)^d p^{d(d-1)}$$

for $d \in \{0, \ldots, n\}$. Let $H_0 \in X$ be a maximal abelian subgroup of $G$, and for $d \in \{0, \ldots, n\}$ let $H_d \in X$ be a subgroup of $H_0$ of index $p^d$. We fix $e \in \{0, \ldots, n\}$. Then Diagram (2.2) implies

$$p_{H_e}(\text{res}^G_{H_e}(\chi)) = \sum_{d=0}^n \alpha_d \sum_{|H| = p^{n-d} \in X} \pi_{H_e}(\text{res}^G_{H_e}([H, \varphi]_G))$$

(3.10)

in $R^{ab}(H_e)$. Now we fix an extension $\tilde{\lambda}_e \in \text{Irr}(H_e)$ of $\lambda$ and compare the coefficients of $\tilde{\lambda}_e$ in both sides of the above equation. Since $\text{res}^G_{H_0}(\chi)$ is the sum of all different extensions of $\lambda$ to $H_0$, we have $\text{res}^G_{H_e}(\chi) = p^e \sum \lambda$ where the sum runs over all extensions $\tilde{\lambda}$ of $\lambda$ to $H_e$. Hence the coefficient of $\tilde{\lambda}_e$ in the left hand side of the above equation equals $p^e$. Recall that

$$\text{res}^G_{H_e}([H, \varphi]_G) = \sum_{g \in H_e \setminus G/H} [H_e \cap gH, \text{res}^g_{H_e} g \text{res}^H_{H_e \cap gH} (g \varphi)]_{H_e},$$

and that the coefficient of $\tilde{\lambda}_e$ in $(\pi_{H_e} \circ \text{res}^G_{H_e})([H, \varphi]_G)$ is the coefficient of $[H_e, \tilde{\lambda}_e]_{H_e}$ in $\text{res}^G_{H_e}([H, \varphi]_G)$. If $H_e \not\leq H$, this coefficient is zero. If $H_e \leq H$, then each of the $|H/Z|$ different $G$-conjugates (extensions of $\lambda$) of $\varphi$ occur $(G : H)/(H : Z)$ times in the above sum. Among these $|H/Z|$ different $G$-conjugates of $\varphi$, precisely $(H : H_e)$ i.e above $\tilde{\lambda}_e$. Thus, the coefficient of $\tilde{\lambda}_e$ in $(\pi_{H_e} \circ \text{res}^G_{H_e})([H, \varphi]_G)$ equals

$$\frac{|H|}{|H_e|} \cdot \frac{|G|}{|H|} \cdot \frac{Z}{|H|} = p^{e+d},$$
if $|H/Z| = p^{n-d}$. Hence by comparing the $\tilde{\lambda}_e$-coefficients in both sides of Equation (3.10) we obtain (after dividing by $p^e$) for every $e \in \{0, \ldots, n\}$,

$$1 = \sum_{d=0}^e \alpha_d \cdot p^d \cdot \#\{H \in X \mid H_e \leq H, |H/Z| = p^{n-d}\}.$$

Since these equations determine $\alpha_d$ inductively, it suffices to prove that for given $U \in X$ we have

$$1 = \sum_{U \leq H \in X} (-1)^{d(H)} p^{d(H)}^2,$$

where $d(H)$ is defined by $|H/Z| = p^{n-d(H)}$. Proposition B.2 (ii) in Appendix B shows that the above equations for $U \in X$ are equivalent to the equations

$$(-1)^{d(U)} p^{d(U)}^2 = \sum_{U \leq H \in X} \mu_X(U, H),$$

for $U \in X$. Now we fix $U \in X$. Note that each $H \in X$ with $U \leq H$ is contained in the centralizer $C_G(U)$, and $C_G(U)/Z = (U/Z)^\perp$ with respect to the symplectic form. The original symplectic form on $G/Z$ induces a non-degenerate symplectic form on $(C_G(U)/Z)/(U/Z) \cong C_G(U)/U$, and the poset $\{H \in X \mid U \leq H\}$ is isomorphic to the poset of totally isotropic subspaces in $(C_G(U)/Z)/(U/Z)$. Now the above equation follows from the following proposition, noting that $\mu_X(U, H)$ is the number of chains $\sigma$ connecting $U$ and $H$ counted with multiplicity $(-1)^{d(H)}$, and that the above sum equals $-\mu_{X \cup \hat{1}}(U, \hat{1})$, if we add an element $\hat{1}$ to $X$ which is bigger than every element in $X$.

**7.3 Lemma** Let $V$ be a non-degenerate symplectic $\mathbb{F}_p$-vector space of dimension $2d$, and let $X_V$ denote the poset of all totally isotropic subspaces of $V$ together with a greatest element $\hat{1}$. Then we have

$$\mu_{X_V}(0, \hat{1}) = (-1)^{d+1} p^d.$$  

**Proof** We prove the above equation by induction on $d$. The cases $d = 0$ and $d = 1$ can be verified easily. Let $d \geq 2$ and consider a decomposition

$$V = W \perp H$$

of $V$ into a non-degenerate symplectic subspace $W \neq 0$ of dimension $2d - 2$ and a hyperbolic plane $H$. We consider the two maps

$$f: X_W \to X_V^{\text{op}}, \quad U \mapsto U, \quad \hat{1} \mapsto \hat{1},$$

$$g: X_V^{\text{op}} \to X_W, \quad U \mapsto U \cap W, \quad \hat{1} \mapsto \hat{1},$$

where $X_V^{\text{op}}$ denotes the opposite poset of $X_V$, i.e. the elements are the same and the relations are reversed. Obviously $f$ and $g$ are order reversing, and they form a Galois connection, i.e. $f(g(v)) \geq v$ and $g(f(w)) \geq w$ for $v \in X_V^{\text{op}}$ and $w \in X_V$. A
fundamental property of Galois connections (cf. [Walk81, Thm. 4.1] for example) is
the fact that, for each \( v \in X_V^\text{op} \) and \( w \in X_W \), we have
\[
\sum_{t \in f^{-1}(\{v\})} \mu_{X_W}(w,t) = \sum_{s \in g^{-1}(\{w\})} \mu_{X_V^\text{op}}(v,s).
\]
We apply this to \( v = 1 \in X_V^\text{op} \) and \( w = 0 \in X_W \), and obtain
\[
\mu_{X_W}(0,1) = \sum_{U \in X_V \setminus \{1\}} \mu_{X_V}(U,1).
\]

For arbitrary \( U, U' \in X_V \setminus \{1\} \), the relation \( U \leq U' \) implies \( U' \in U^\perp \). The factor
space \( U^\perp/U \) is again a non-degenerate symplectic \( \mathbb{F}_p \)-vector space of dimension
\( 2(\dim V - \dim U) \), and we have for given \( U \in X_V \setminus \{1\} \) an isomorphism of posets
\[
\{U' \in X_V \mid U \leq U'\} \rightarrow X_{U^\perp/U}, \quad U' \mapsto U'/U, \quad 1 \mapsto 1.
\]
If \( U \cap W = 0 \), we have \( \dim U \leq 2 \), and the last equation can be written as
\[
a_d - 1 = n_0 a_d + n_1 a_{d-1} + n_2 a_{d-2}, \tag{3.11}
\]
where \( a_i := \mu_{X_V^\text{op}}(0,1) \) for a non-degenerate symplectic \( \mathbb{F}_p \)-vector space \( V' \) of
dimension \( 2i, i \in \mathbb{N} \), and where
\[
n_i := \{U \in X_V \setminus \{1\} \mid \dim U = i, U \cap W = 0\}
\]
for \( i \in \{0, 1, 2\} \). Obviously, \( n_0 = 1 \), and
\[
n_1 = \frac{p^{2d} - p^{2d-2}}{p - 1} = p^{2d-1} + p^{2d-2},
\]
since we may generate \( U \) by a vector in \( V \setminus W \) and have \( p - 1 \) choices.

The computation of \( n_2 \) is more difficult. First note that there are
\[
\frac{(p^{2d-1} - 1)(p^{2d-2} - 1)}{(p^2 - 1)(p - 1)}
\]
elements \( U \in X_V \setminus \{1\} \) with \( \dim U = 2 \). In fact, there are \( (p^{2d} - 1)(p^{2d-1} - p) \)
ordered pairs \( (v, v') \) of linearly independent vectors \( v, v' \in V \) with \( v' \perp v \). On the
other hand each such \( U \) has \( (p^2 - 1)(p^2 - p) \) different ordered bases.

Among these \( U \in X_V \setminus \{1\} \) with \( \dim U = 2 \) there are by the same argument
\[
\frac{(p^{2(d-1)} - 1)(p^{2(d-1)-2} - 1)}{(p^2 - 1)(p - 1)}
\]
elements with \( \dim(U \cap W) = 2 \), and there are
\[
\frac{(p^{2d-2} - 1)(p^{2d-2} - p^{2d-4})}{(p - 1)^2}
\]
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elements with $\dim(U \cap W) = 1$, since there are $(p^{2d-2} - 1)(p^{2d-1} - p^{2d-3})$ pairs $(w, v)$ of linearly independent vectors $w \in W$ and $v \in W^\perp \setminus (W \cap W^\perp)$, and each such $U$ has exactly $(p-1)(p^2 - p)$ ordered bases $(w, v)$ with $w \in W$.

Hence we have

\[
    n_2 = \frac{(p^{2d} - 1)(p^{2d-2} - 1) - (p^{2d-2} - 1)(p^{2d-4} - 1)}{(p^2 - 1)(p - 1)} - \frac{(p + 1)(p^{2d-2} - 1)(p^{2d-2} - p^{2d-4})}{(p^2 - 1)(p - 1)}
\]

\[
    = \frac{p^{2d-2} - 1}{(p^2 - 1)(p - 1)}[p^{2d} - 1 - p^{2d-4} + 1 - p^{2d-1} + p^{2d-3} - p^{2d-2} + p^{2d-4}]
\]

\[
    = \frac{p^{2d-2} - 1}{(p^2 - 1)(p - 1)}[p^{2d} - p^{2d-2} - p^{2d-1} + p^{2d-3}]
\]

\[
    = \frac{p^{2d-2} - 1}{(p^2 - 1)(p - 1)}(p^2 - 1)(p^{2d-2} - p^{2d-3})
\]

\[
    = p^{2d-3}(p^{2d-2} - 1).
\]

Using the induction hypothesis for $a_{d-1}$ and $a_{d-2}$, we obtain from Equation (3.11)

\[
a_d = (1 - n_1)a_{d-1} - n_2a_{d-2}
\]

\[
    = (1 - (p + 1)p^{2d-2})(-1)^d p^{(d-1)^2} - p^{2d-3}(p^{2d-2} - 1)(-1)^{d-1}p^{(d-2)^2}
\]

\[
    = (-1)^{d+1}( -p^{2d-2d+1} + p^{d^2} + p^{d^2-1} - p^{4d-5+(d-2)^2} + p^{2d-3+(d-2)^2})
\]

\[
    = (-1)^{d+1}( -p^{d-2d+1} + p^{d^2} + p^{d^2-1} - p^{d^2-1} + p^{d^2-2d+1})
\]

\[
    = (-1)^{d+1}p^{d^2},
\]

which completes the proof of the lemma. \qed
Chapter 4

Applications and Further Properties of Canonical Induction Formulae

In this chapter we give applications of the existence of the canonical induction formulae constructed in Chapter III. In Section 1 we consider the question whether a morphism of restriction functors \( f: A \rightarrow N \) between a restriction subfunctor \( A \) of a Mackey functor \( M \) and a Mackey functor \( N \) can be lifted to a morphism \( F: M \rightarrow N \) of restriction functors. The answer is positive, if there is a canonical induction formula for \( M \) from \( A \) whose residue is the identity on \( A \). Moreover, the extension is unique, if \( A \) and \( M \) coincide on the set of coprimordial subgroups of \( N \). This extension theorem is applied in Sections 2, 3, and 4 to the determinant, the Adams operations, and the Chern classes. By this procedure we are able to give explicit algebraic definitions for Chern classes of Brauer characters, trivial source modules and linear source modules with all the natural properties that are known for the classical Chern classes on the character ring. In Sections 5 and 6 we consider the semidirect product \( G S \) of two finite groups \( G \) and \( S \) of coprime order (\( S \) acts on \( G \)). Using the canonical induction formulae of Chapter III we are able to define extension maps \( \text{ext}^{G,S}: M(G) \rightarrow M(GS) \) such that \( \text{res}^{GS} \circ \text{ext}^{G,S} \) is the identity, where \( M(G)^S \) denotes the subgroup of \( S \)-fixed elements in \( M(G) \) with respect to the conjugation action. Moreover, we define a map \( \text{gl}^{G,S}: M(G)^S \rightarrow M(G^S) \), where \( G^S \) denotes the subgroup of \( S \)-fixed points of \( G \). If \( M \) is the character ring Mackey functor and \( S \) is a \( p \)-group, then \( \text{gl}^{G,S} \) is related to the Glauberman correspondence. In Section 7 we construct canonical induction formulae for the projectification of a complex character and of a Brauer character. Both formulae have rational coefficients, and in the character ring case the denominators of these coefficients determine the defect of an irreducible character.

4.1 An extension theorem on morphisms of restriction functors

Throughout this section we fix the following situation:

Let \( k \) denote a commutative ring and \( G \) a finite group. We assume that we have a
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$k$-restriction functor $M$ on $G$ (not necessarily a Mackey functor) and a $k$-restriction subfunctor $A \subseteq M$ on $G$. Furthermore, let $a \in k-\text{Res}(G)(M,A)$ be such that $a_H(\varphi) = [H,\varphi]_H$ for $H \leq G$, $\varphi \in A(H)$. Let $p := \pi_A \circ a \in k-\text{Con}(G)(M,A)$ be the residue of $a$. Then we have $p_H(\varphi) = \varphi$ for $H \leq G$, $\varphi \in A(H)$. In fact, $p_H(\varphi) = \pi^A_H(a_H(\varphi)) = \pi^A_H([H,\varphi]_H) = \varphi$.

1.1 Definition For $M$, $A$, $a$ as above and $N \in k-\text{Mack}(G)$ we define a $k$-linear map

$$\Phi^M_{N,A} : k-\text{Res}(G)(M,N) \rightarrow k-\text{Res}(G)(A,N)$$

by restricting a morphism on $M$ to $A$. Conversely we define a $k$-linear map

$$\Sigma^M_{N,A,a} : k-\text{Res}(G)(A,N) \rightarrow k-\text{Res}(G)(M,N), \quad f \mapsto b^{N,N}_f \circ f_+ \circ a,$$

and call $\Sigma^M_{N,A,a}(f)$ the canonical extension of $f \in k-\text{Res}(G)(A,N)$ with respect to the morphism $a$.

The following theorem justifies this terminology.

1.2 Theorem Let $M$, $A$, and $a$ be as above, and let $N \in k-\text{Mack}(G)$. Then $\Phi^M_{N,A} \circ \Sigma^M_{N,A,a} = \text{id}$. In particular, the map

$$\Phi^M_{N,A} : k-\text{Res}(G)(M,N) \rightarrow k-\text{Res}(G)(A,N)$$

is split surjective.

Proof We show that $\Phi^M_{N,A} \circ \Sigma^M_{N,A,a} = \text{id}$, i.e. $f_H = b^{N,N}_H \circ f_+ \circ a_H$ on $A(H)$ for $H \leq G$. In fact, for $\varphi \in A(H)$ we have by assumption $a_H(\varphi) = [H,\varphi]_H$, and further by the definitions of $f_+$ and $b^{N,N}$ we obtain

$$(b^{N,N}_H \circ f_+ \circ a_H)(\varphi) = b^{N,N}_H(f_+([H,\varphi]_H)) = b^{N,N}_H([H,f_H(\varphi)]_H) = \text{ind}_H^H(f_H(\varphi)) = f_H(\varphi).$$

The following theorem describes a situation where $\Phi^M_{N,A}$ and $\Sigma^M_{N,A,a}$ are inverse isomorphisms.

1.3 Theorem Let $M$, $A$, and $a$ be as above and let $N \in k-\text{Mack}(G)$ such that $A(H) = M(H)$ for $H \in \mathcal{C}(N)$. Then the restriction map

$$\Phi^M_{N,A} : k-\text{Res}(G)(M,N) \rightarrow k-\text{Res}(G)(A,N)$$

is an isomorphism with inverse $\Sigma^M_{N,A,a}$. In particular, $\Sigma^M_{N,A,a}$ is independent of $a$.

Proof It suffices to show that $\Sigma^M_{N,A,a} \circ \Phi^M_{N,A} = \text{id}$, i.e. $b^{N,N}_H \circ f_+ \circ a_H = f_H$ for $f \in k-\text{Res}(G)(M,N)$ and $H \leq G$. Since $b$, $f_+$, and $a$ commute with restrictions, it suffices to prove this equation for $H \in \mathcal{C}(N)$. But for $H \in \mathcal{C}(N)$ we have $M(H) = A(H)$, and we obtain $(b^{N,N}_H \circ f_+ \circ a_H)(\varphi) = f_H(\varphi)$ for $\varphi \in M(H) = A(H)$ as in the proof of Theorem 1.2.
1.4 Definition  
Let \(M, A,\) and \(a\) be given as at the beginning of this section, and assume that \(M\) is a \(k\)-Mackey functor on \(G\).

(i) We define the \(k\)-module \(k\text{-}\text{Res}(G)(A \subseteq M, A \subseteq M)\) as the set of elements \(f \in k\text{-}\text{Res}(G)(M, M)\) satisfying \(f(A) \subseteq A\). Similarly, if \(M \in k\text{-}\text{Mack}_{\text{alg}}(G)\) and \(A \subseteq M\) is a \(k\)-algebra restriction subfunctor on \(G\), we define \(k\text{-}\text{Res}_{\text{alg}}(G)(A \subseteq M, A \subseteq M)\) as the set of elements \(f \in k\text{-}\text{Res}_{\text{alg}}(G)(M, M)\) with \(f(A) \subseteq A\).

(ii) Similar to Definition 1.1 we define the \(k\)-linear restriction function \(\Phi_{M,A}: k\text{-}\text{Res}(G)(A \subseteq M, A \subseteq M) \rightarrow k\text{-}\text{Res}(G)(A,A)\) and a \(k\)-linear map in the other direction, \(\Sigma_{M,A,a}: k\text{-}\text{Res}(G)(A,A) \rightarrow k\text{-}\text{Res}(G)(A \subseteq M, A \subseteq M), f \mapsto b^{M,A} \circ f_+ \circ a\).

We call \(\Sigma_{M,A,a}(f)\) again the canonical extension of \(f \in k\text{-}\text{Res}(G)(A,A)\) with respect to \(a\).

If \(M \in k\text{-}\text{Mack}_{\text{alg}}(G)\) and \(A \subseteq M\) is a \(k\)-algebra restriction subfunctor the \(k\)-linear map restricts to a function

\(\Phi^M_A: k\text{-}\text{Res}_{\text{alg}}(G)(A \subseteq M, A \subseteq M) \rightarrow k\text{-}\text{Res}_{\text{alg}}(G)(A,A)\).

1.5 Theorem  
Let \(M, A,\) and \(a\) be defined as at the beginning of this section, assume that \(M \in k\text{-}\text{Mack}(G)\), and let \(N \in k\text{-}\text{Mack}(G)\). Then we have \(\Phi^{M,A} \circ \Sigma^{M,A,a} = \text{id}\), in particular,

\(\Phi^{M,A}: k\text{-}\text{Res}(G)(A \subseteq M, A \subseteq M) \rightarrow k\text{-}\text{Res}(G)(A,A)\)

is split surjective.

Proof  This is proved in the same way as Theorem 1.2.

1.6 Theorem  
Let \(M, A,\) and \(a\) be defined as at the beginning of this section and assume that \(M\) is a \(k\)-Mackey functor on \(G\) with \(A(H) = M(H)\) for \(H \in \mathcal{C}(M)\). Then the restriction map

\(\Phi^{M,A}: k\text{-}\text{Res}(G)(A \subseteq M, A \subseteq M) \rightarrow k\text{-}\text{Res}(G)(A,A)\)

is an isomorphism with inverse \(\Sigma^{M,A,a}\).

Proof  This is proved in the same way as Theorem 1.3.

We have the following multiplicative versions of Theorems 1.5 and 1.6.

1.7 Theorem  
Let \(M, A, a,\) and \(p\) be defined as at the beginning of this section. Assume that \(M\) is a \(k\)-Green functor on \(G\) and that \(A \subseteq M\) is a \(k\)-algebra restriction subfunctor of \(M\) on \(G\).

(i) If \(p^A: A_+ \rightarrow A^+\) is injective, and \(p_H\) is a \(k\)-algebra homomorphism for \(H \in \mathcal{C}(M)\), then

\[\Sigma^{M,A,a}(k\text{-}\text{Res}_{\text{alg}}(G)(A,A)) \subseteq k\text{-}\text{Res}_{\text{alg}}(G)(A \subseteq M, A \subseteq M)\].
In particular,
\[ \Phi_{\text{alg}}^{M,A} : k - \text{Res}_{\text{alg}}(G)(A \subseteq M, A \subseteq M) \to k - \text{Res}_{\text{alg}}(G)(A, A) \]
is surjective.

(ii) If \( A(H) = M(H) \) for \( H \in \mathcal{C}(M) \), then \( \Phi_{\text{alg}}^{M,A} \) is bijective with inverse \( \Sigma^{M,A,a} \).

**Proof** Using Theorem 1.6 it suffices for both parts to show that the \( H \)-component \( b_{H}^{M,A} \circ f_{+H} \circ a_{H} : M(H) \to M(H) \) of \( \Sigma^{M,A,a}(f) \) is a \( k \)-algebra homomorphism for \( f \in k - \text{Res}_{\text{alg}}(G)(A, A) \). We have to prove that
\[ (b_{H}^{M,A} \circ f_{+H} \circ a_{H})(1_{M(H)}) = 1_{M(H)} \]
and
\[ (b_{H}^{M,A} \circ f_{+H} \circ a_{H})(x \cdot y) = (b_{H}^{M,A} \circ f_{+H} \circ a_{H})(x) \cdot (b_{H}^{M,A} \circ f_{+H} \circ a_{H})(y) \]
for \( H \leq G, x, y \in M(H) \). Since these equations of elements in \( M(H) \) can be verified by applying \( \text{res}_{K}^{H} \) for \( K \leq H, K \in \mathcal{C}(M) \), and since restriction maps are \( k \)-algebra homomorphisms and commute with \( b, f_{+}, \) and \( a \), it suffices to prove the above equations for \( H \leq G \) with \( H \in \mathcal{C}(M) \).

In the situation of part (i), Corollary I.4.2 (iv) implies that \( a_{H} \) is a \( k \)-algebra homomorphism for \( H \in \mathcal{C}(M) \), and the result follows, since \( b_{H}^{M,A} \) and \( f_{+H} \) are \( k \)-algebra homomorphisms.

In the situation of part (ii), the above equations hold for \( H \in \mathcal{C}(M) \), since \( (b_{H}^{M,A} \circ f_{+H} \circ a_{H})(x) = f_{H}(x) \) for \( x \in M(H) = A(H) \). \( \square \)

**1.8 Theorem** Let \( M, A, a, \) and \( p \) be defined as at the beginning of this section with \( M \in k - \text{Mack}(G) \) and assume that \( a \) is injective (which is the case if \( a \) is a canonical induction formula). Assume further that one of the following two conditions holds:

(i) The mark morphism \( \rho^{A} : A_{+} \to A^{+} \) is injective, and
\[ p_{H} \circ b_{H}^{M,A} \circ h_{+H} \circ a_{H} = h_{H} \circ p_{H} : M(H) \to A(H) \]
for \( h \in k - \text{Res}(G)(A, A) \) and \( H \in \mathcal{C}(M) \).

(ii) For \( H \in \mathcal{C}(M) \) we have \( M(H) = A(H) \).

Then we have
\[ \Sigma^{M,A,a}(f \circ g) = \Sigma^{M,A,a}(f) \circ \Sigma^{M,A,a}(g) \]
for \( f, g \in k - \text{Res}(G)(A, A) \).

**Proof** We have to show that
\[ b_{H}^{M,A} \circ f_{+H} \circ a_{H} \circ b_{H}^{M,A} \circ g_{+H} \circ a_{H} = b_{H}^{M,A} \circ (fg)_{+H} \circ a_{H} : M(H) \to M(H) \]
for \( H \leq G \). Since all maps in the above equation commute with restrictions, it suffices to prove the above equation for \( H \in \mathcal{C}(M) \). In situation (ii) we have \( (b_{H}^{M,A} \circ h_{+H} \circ a_{H})(x) = h_{H}(x) \) for all \( h \in k - \text{Res}(G)(A, A), H \leq \mathcal{C}(H) \), and \( x \in M(H) = A(H) \), and the result follows. In situation (i), we obtain from Diagram (2.2) that
the map \((p_K \circ \text{res}^H_K)_{K \subseteq H} = \rho^A_H \circ a_H\) is injective, since \(\rho^A_H\) and \(a_H\) are injective. Therefore it suffices to show that the above equation holds after having composed both sides with \(p_K \circ \text{res}^H_K\), where \(K \leq H\) and \(H \leq \mathcal{C}(M)\). Moreover we may also assume that \(K \in \mathcal{C}(M)\). But everything follows from the hypothesis in (i):

\[
\begin{align*}
  p_K \circ \text{res}^H_K \circ b^{M,A}_H \circ f_+ \circ a_H \circ b^{M,A}_H \circ g_+ \circ a_H &= p_K \circ b^{M,A}_K \circ f_+ \circ a_K \circ b^{M,A}_K \circ g_+ \circ a_K \circ \text{res}^H_K \\
  &= f_K \circ p_K \circ b^{M,A}_K \circ g_+ \circ a_K \circ \text{res}^H_K = f_K \circ g_K \circ p_K \circ \text{res}^H_K \\
  &= (f \circ g)_K \circ p_K \circ \text{res}^H_K = p_K \circ b^{M,A}_K \circ (f \circ g)_+ \circ a_K \circ \text{res}^H_K \\
  &= p_K \circ \text{res}^H_K \circ b^{M,A}_K \circ (f \circ g)_+ \circ a_H 
\end{align*}
\]

\[
\square
\]

1.9 Remark Recall from Chapter III the Examples 1.1 (\(R^{ab}_K \subseteq R_K\)), Example 2.1 (\(R^{ab}_F \subseteq R_F\)), Example 3.1 (\(P^{ab}_F \subseteq P_F\) or equivalently \(R^{ab}_{K,p} \subseteq R_{K,p}\)), Example 3.4 (\(R^{ab}_{K,p} \subseteq R_{K,p}\)), Example 4.2 (\(L^{ab}_G \subseteq L_G\)) and Example 4.9 (\(T^{ab}_G \subseteq T_G\)).

In all these examples we have \(p_H(x) = x\) and \(a_H(x) = [H,x]_H\) for \(H \leq G\) and \(x \in A(H)\), so that our basic assumptions at the beginning of this section hold.

For the examples 1.1, 2.1, 3.1, and 3.4 of Chapter III we have \(A(H) = M(H)\) for \(H \in \mathcal{C}(M)\) so that we have isomorphisms

\[
\begin{align*}
  \Theta^{M,A}: k\text{–}\text{Res}(G)(A \subseteq M, A \subseteq M) &\xrightarrow{\sim} k\text{–}\text{Res}(G)(A, A) \\
  \Theta_{\text{alg}}^{M,A}: k\text{–}\text{Res}_{\text{alg}}(G)(A \subseteq M, A \subseteq M) &\xrightarrow{\sim} k\text{–}\text{Res}_{\text{alg}}(G)(A, A)
\end{align*}
\] (4.1)

with inverse \(\Sigma^{M,A,a}_{\text{alg}}\).

In the situation of Examples III.4.2 and III.4.9, the map \(\Theta_{\text{alg}}^{M,A}\), and hence also the upper one, is not injective in general, but of course surjective by Theorem 1.4.

In fact, consider Example II.4.2 or II.4.9 with \(G\) the cyclic subgroup of order \(p\). Note that \(M(G) = A(G) \oplus \mathbb{Z} \cdot [OG]\) and \(M(1) = A(1)\). The identity in \(k\text{–}\text{Res}_{\text{alg}}(G)(A, A)\) has at least two extensions in \(k\text{–}\text{Res}_{\text{alg}}(G)(A \subseteq M, A \subseteq M)\), namely the identity and the morphism \(f\) with \(f_G([OG]) = p \cdot [O]\).

1.10 Remark Let \(M, A, \text{ and } a\) be given as at the beginning of this section with \(M \in k\text{–}\text{Mack}(G)\). Assume that \(M\) and \(A\) can be considered as contravariant functors from the category of finite groups to the category of abelian groups with respect to restrictions along arbitrary group homomorphisms. Then also \(A_+\) can be considered as such a functor, using the definition of \(\text{res}_{+,f}: A_+(G) \to A_+(G')\), given by Equation (3.3), for a group homomorphism \(f: G \to G'\). Assume further that \(a: M \to A_+\) and \(b^{M,A}: A_+ \to M\) are natural transformations. This is the case for the Examples III.1.1 (character ring), II.2.1 (Brauer character ring), II.4.2 (linear source ring), and III.4.9 (trivial source ring).

Let \(N\) be another \(k\text{–}\text{Mackey}\) functor on \(G\) which can also be regarded as a functor from the category of finite groups to the category of abelian groups by extending the definition of \(\text{res}_H^G\) (for \(H \leq G\)) to \(\text{res}_f\) (for \(f: G' \to G\)). If \(t: A \to N\) is a natural transformation, so is \(t_+\), and it follows that also \(\Sigma^{M,A,a}_N(t): M \to N\), defined by
\[ \Sigma^{M,A,a}_N(t)_G := g^{M,A}_G \circ t_{+G} \circ a_G : M(G) \to N(G) \] for any finite group, is a natural transformation. Hence, we obtain theorems which are similar to Theorems 1.1, 1.2, and 1.4–1.7. So, for instance, the modified version of Theorem 1.5 states for \( M := R, A := R^{ab}, \) using \( a \) as in Example III.1.1, that the restriction

\[ \text{Nat}(R^{ab} \subseteq R, R^{ab} \subseteq R) \to \text{Nat}(R^{ab}, R^{ab}) \]

between the ‘sets’ of natural transformations is a bijection. The inverse is given by \( \Sigma^{M,A,a}_N \) as in the proof of Theorem 1.5. It seems to be within reach to determine \( \text{Nat}(R^{ab}, R^{ab}) \) and thus obtain a nice description for \( \text{Nat}(R^{ab} \subseteq R, R^{ab} \subseteq R) \). The Adams operations \( \Psi^n, n \in \mathbb{Z}, \) cf. Section 3, form a set of \( \mathbb{Z} \)-linearly independent elements in the abelian group \( \text{Nat}(R^{ab}, R^{ab}) \). But we won’t go into this at the moment.

### 4.2 Determinants

For a commutative ring \( k \), a finite group \( G \), and a \( k \)-free \( kG \)-module \( V \) of finite \( k \)-rank the **determinant** \( \det(V) \in \text{Hom}(G, k^\times) \) is defined as the homomorphism

\[ G \xrightarrow{\rho} \text{GL}(V) \xrightarrow{\det} k^\times, \]

where \( \rho \) is the representation of \( G \) on \( V \) induced by the \( kG \)-module structure. First we show that determinants have values in a Mackey functor.

#### 2.1 Example

For a finite group \( G \) and a commutative ring \( k \) we define \( \hat{G}(k) \) as the multiplicative abelian group \( \text{Hom}(G, k^\times) \). For a fixed group \( G \), the abelian groups \( \hat{H}(k), H \leq G \), form a cohomological \( \mathbb{Z} \)-Mackey functor on \( G \) with the following structure maps:

\[
\begin{align*}
(\varphi)(h) &:= \varphi(h^g) \quad \text{for } h \in H \leq G, \ \varphi \in \hat{H}(k), \ g \in G, \\
(\text{res}^H_K(\varphi))(x) &:= \varphi(x) \quad \text{for } x \in K \leq H \leq G, \ \varphi \in \hat{H}(k), \\
(\text{ind}^H_K(\varphi))(h) &:= \varphi \circ \text{tr}_{H/K} \circ \nu_H \quad \text{for } K \leq H \leq G, \ \varphi \in \hat{K}(k), \ h \in H,
\end{align*}
\]

where \( \nu_H : H \to H^{ab} \) is the canonical surjection, \( \text{tr}_{H/K} : H^{ab} \to K^{ab} \) is the transfer map (see [CR81, 13.10] for a definition) and \( \varphi : K^{ab} \to k^\times \) is induced by \( \varphi : K \to k^\times \).

In fact, the Mackey functor axioms are verified by simple calculations, or by using the isomorphisms \( \hat{H}(k) \cong H^1(H, k^\times) \), where \( k^\times \) is regarded as trivial \( H \)-module for \( H \leq G \). These isomorphisms translate the three structure maps defined above to conjugation, restriction and corestriction of cohomology, where the Mackey functor axioms are well-known to hold. The set of coprimordial subgroups for \( \hat{H}(k), H \leq G \), is certainly contained in the set of cyclic subgroups of prime power order of \( G \), since \( k^\times \) being abelian) a group homomorphism \( G \to k^\times \) is uniquely determined by its values on elements of prime power order. If \( k^\times \) does not contain non-trivial \( p \)-elements for a prime \( p \), then the set of coprimordial groups is even contained in the set of cyclic \( p' \)-subgroups of prime power order.

The following proposition is well-known. A proof for \( k = \mathbb{C} \) can be found in [CR81, 13.15]. This proof generalizes without changes to arbitrary commutative rings \( k \).
2.2 Proposition Let $k$ be a commutative ring, $G$ a finite group, $H \leq G$, and $V$ a $k$-free $kH$-module of finite $k$-rank. Then $\text{ind}^G_H(V)$ is a $k$-free $kG$-module of finite $k$-rank and we have

$$\det(\text{ind}^G_H(V)) = \varepsilon^{\text{rk}_H(V)} \cdot (\overline{\det(V)} \circ \text{tr}_{G/H} \circ \nu_G) \in \hat{G}(k),$$

where $\nu_G: G \to G^{\text{ab}}$ is the canonical surjection, $\text{tr}_{G/H}: G^{\text{ab}} \to H^{\text{ab}}$ the transfer map, $\overline{\det(V)}: H^{\text{ab}} \to k^\times$ the map induced by $\det(V)$, and $\varepsilon_{G/H} \in \hat{G}(k)$ is defined by mapping an element $g \in G$ to the sign of the permutation by which $g$ acts on $G/H$ via left multiplication. (The dot in the above formula indicates multiplication in $\hat{G}(k)$.)

2.3 Next we are going to apply Theorems 1.1 and 1.2 to determinants. More precisely, we are going to define in different situations the following data for a finite group $G$: a $\mathbb{Z}$-Mackey functor $M$, a $\mathbb{Z}$-restriction subfunctor $A \subseteq M$, a canonical induction formula $a \in \mathbb{Z} - \text{Res}(G)(M, A_+)$, a commutative ring $k$, a $\mathbb{Z}$-Mackey functor $N$ on $G$ given by $N(H) := \hat{H}(k)$ for $H \leq G$ (cf. Example 2.1), and a morphism $\det \in \mathbb{Z} - \text{Res}(G)(A, N)$. Note that for $H \leq G$, the abelian group $M(H)$ is written additively and the abelian group $N(H)$ multiplicatively. We distinguish the following situations:

(i) Character ring: Let $k := K$ be a field for $G$ of characteristic zero which contains all $|G|$-th roots of unity, $M := R_K$, $A := R_K^{\text{ab}}$, $a: R_K \to R_K^{\text{ab}}$ as in Example III.1.1. Furthermore, for $H \leq G$, let $\det_H: R_K^{\text{ab}}(H) = \mathbb{Z} \cdot \hat{H}(K) \to \hat{H}(K) = N(H)$ be defined by $\det_H(\varphi) = \varphi$ for $\varphi \in \hat{H}(K)$.

(ii) Brauer character ring: Let $k := F$ be an algebraically closed field of characteristic $p > 0$, $M := R_F$, $A := R_F^{\text{ab}}$, $a: R_F \to R_F^{\text{ab}}$ as in Example III.2.1. Furthermore, for $H \leq G$, let $\det_H: A(H) = \mathbb{Z} \cdot \hat{H}(F) \to \hat{H}(F) = N(H)$ be defined by $\det(\varphi) = \varphi$ for $\varphi \in \hat{H}(F)$. Note that $\mathcal{C}(N)$ is the set of cyclic $p'$-groups of prime power order, since $F^\times$ contains no non-trivial $p$-elements.

(iii) $p$-projective character group: Let $k := F$ be as in (ii), $M := P_F$, $A := P_F^{\text{ab}}$, $a: P_F \to P_F^{\text{ab}}$ as in Example III.3.1. Furthermore, for a $p'$-group $H \leq G$, let $\det_H: A(H) = \mathbb{Z} \cdot \hat{H}(F) \to \hat{H}(F) = N(H)$ be defined as in (ii). Note that $\mathcal{C}(N)$ is the set of cyclic $p'$-subgroups of prime power order.

(iv) $\pi$-projective character group: Let $k := K$ be defined as for the character ring in (i), $M := R_{K, \pi}$, $A := R_{K, \pi}^{\text{ab}}$, $a: R_{K, \pi} \to R_{K, \pi}^{\text{ab}}$ as in Example III.3.4. Furthermore, for a $\pi'$-subgroup $H \leq G$, let $\det_H: A(H) = \mathbb{Z} \cdot \hat{H}(K) \to \hat{H}(K) = N(H)$ be defined as in situation (i).

(v) Linear source ring: Let $k := \mathcal{O}$ be a complete discrete valuation ring with algebraically closed residue field of characteristic $p > 0$, and a quotient field of characteristic zero which contains all $|G|$-th roots of unity. Let $M := L_\mathcal{O}$, $A := L_\mathcal{O}^{\text{ab}}$, and $a: L_\mathcal{O} \to L_\mathcal{O}^{\text{ab}}$ as in Example III.4.2. Furthermore, for $H \leq G$, let $\det_H: A(H) = \mathbb{Z} \cdot \hat{H}(\mathcal{O}) \to \hat{H}(\mathcal{O}) = N(H)$ be defined by $\det_H(\varphi) = \varphi$ for $\varphi \in \hat{H}(\mathcal{O})$. 

(vi) Trivial source ring: Let $k := \mathcal{O}$ be as for the linear source ring, $M := T_\mathcal{O}$, $A := T_\mathcal{O}^\text{ab}$, $a : T_\mathcal{O} \to T_\mathcal{O}^\text{ab}_+$ as in Example III.4.9. Furthermore, for $H \leq G$, let $\det_H : A(H) = \mathbb{Z} \cdot \hat{H}(\mathcal{O})_\varphi \to \hat{H}(\mathcal{O})_{\varphi'} = N(H)$ be defined by $\det_H(\varphi) = \varphi$ for $\varphi \in \hat{H}(\mathcal{O})_{\varphi'}$.

Note that in each of these situations there is an extension $\det \in \mathbb{Z} - \text{Res}(G)(M, N)$ of $\det \in \mathbb{Z} - \text{Res}(G)(A, N)$, given by the usual determinant.

### 2.4 Proposition
Assume that $G$, $M$, $A$, $a : M \to A_+$, $k$, $N$, and $\det \in \mathbb{Z} - \text{Res}(G)(A, N)$ are given as in one of the six preceding situations.

(i) The morphism $\det \in \mathbb{Z} - \text{Res}(G)(A, N)$ can be extended to a morphism $\tilde{\det} \in \mathbb{Z} - \text{Res}(G)(M, N)$ by defining $\det := \Sigma^M_{\mathcal{O},a}(\det) = b^M_{a} \circ \det + a$.

(ii) In the situations 2.3 (i) (character ring), 2.3 (ii) (Brauer character ring), and 2.3 (iii) ($p$-projective character group), the extension is unique, and therefore coincides with the usual morphism $\det \in \mathbb{Z} - \text{Res}(G)(M, N)$.

(iii) In situations 2.3 (iv)–2.3 (vi) ($\pi$-projective characters, linear source ring, trivial source ring), the morphisms $\det$ and $\tilde{\det}$ in $\mathbb{Z} - \text{Res}(G)(M, N)$ coincide, provided that $p \neq 2$, resp. $2 \not\in \pi$.

**Proof**

(i) This is an immediate consequence of Theorem 1.2.

(ii) This is an immediate consequence of Theorem 1.3, since $A(H) = M(H)$ for $H \in \mathcal{C}(N)$.

(iii) Note that in all three situations, the coprimordial subgroups on $N$ are the set of cyclic subgroups of prime power order. Hence it suffices to show that $\tilde{\det}_H = \det_H$ for a cyclic group of prime power order $q^n$, $q$ prime, $n \in \mathbb{N}$. If $q \neq p$, resp. $q \notin \pi$ in situation (iv), then $A(H) = M(H)$, and it follows easily that $\det_H = b^N_{\mathcal{O}^\text{ab}} \circ \det + a_H$. Next we consider situation 2.3 (iv) ($\pi$-projective character group) and assume that $q \in \pi$, hence $q \notin 2$. Then $M(H) = \mathbb{Z} \cdot \text{ind}_H^{H}(1)$, and we have $a_H(\text{ind}_H^{H}(1)) = [1, 1]_H$. Hence,

$$\tilde{\det}_H(\text{ind}_H^{H}(1)) = (b^N_{\mathcal{O}^\text{ab}} \circ \det + a_H)([1, 1]_H) = b^N_{\mathcal{O}^\text{ab}}([1, \det(1)]_H) = \text{ind}_H^{H}(1) = 1.$$  

On the other hand, $\det_H(\text{ind}_H^{H}(1)) = \varepsilon_{H/1} = 1$, since $q$ is odd.

Now we consider situation (2.3) (v) (linear source ring) and assume $q = p$. Then

$$L_\mathcal{O}(H) = \bigoplus_{U \leq H \not\in \mathcal{O}} \mathbb{Z} \cdot \text{ind}_H^{H}([\mathcal{O}]) = [U, \varphi]_H. \quad \text{Hence,}$$

$$\tilde{\det}(\text{ind}_H^{H}([\mathcal{O}])) = (b^N_{\mathcal{O}^\text{ab}} \circ \det + a_H)(\text{ind}_H^{H}([\mathcal{O}])) = \varphi \circ \text{tr}_{H/U}$$

On the other hand, by Proposition 2.2, $\det_H(\text{ind}_H^{H}([\mathcal{O}])) = \varepsilon_{H/U} \cdot (\varphi \circ \text{tr}_{H/U})$ with $\varepsilon_{H/U} = 1$, since $p$ is odd.

Finally we consider situation 2.3 (vi) (trivial source ring) and assume $q = p$. Then

$$T_\mathcal{O}(H) = \bigoplus_{U \leq H} \mathbb{Z} \cdot \text{ind}_H^{H}([\mathcal{O}]).$$
It is easy to show that $a_H(\text{ind}_H^U([\mathcal{O}])) = [U, 1]_H$. Hence,

$$\tilde{\det}(\text{ind}_H^U([\mathcal{O}])) = b_H^{N,N}([U, 1]_H) = 1 \circ \text{tr}_{H/U} = 1.$$ 

On the other hand,

$$\det_H(\text{ind}_U^H([\mathcal{O}])) = \varepsilon_{H/U} \cdot (1 \circ \text{tr}_{H/U}) = \varepsilon_{H/U} = 1,$$

since $p$ is odd.

2.5 Remark  
(i) First note that $\det \neq \tilde{\det}$ in situation 2.3 (iv)–(vi), for $G$ the cyclic group of order 2, and $p = 2$, resp. $2 \in \pi$ in situation 2.3 (iv). Consider the $KG$-module $KG$ with character $\text{ind}_1^G(1)$ in situation 2.3 (iv) with $2 \in \pi$. Then $\det(KG) = \det(\text{ind}_1^G(1)) = \varepsilon_{G/1}/(1 \circ \text{tr}_{G/1}) = \varepsilon_{G/1}$ which is the non-trivial element in the group $\hat{G}(K)$ of order 2. On the other hand it is easy to verify that $a_G(\text{ind}_1^G(1)) = [1, 1]_G$, hence $\det(\text{ind}_1^G(1)) = 1 \circ \text{tr}_{G/1} = 1$. Similar arguments show that in the situations 2.3 (v) and 2.3 (vi) we have $\det([\mathcal{O}G]) = 1$ and $\det([\mathcal{O}G]) = \varepsilon_{G/1} \neq 1$.

At a first glance, it is confusing that we obtain different results for $P_F$ and $R_{K,p}$. The reason is that the Mackey functor $N$ differs in the two cases. If we choose $P_O$ instead of $P_F$ in situation (2.3) (iii) and $N = \hat{?}(\mathcal{O})$ instead of $N = \hat{?}(F)$, then we obtain the same ambiguities as in example (2.3) (iv), since $\mathcal{C}(N)$ changes from the set of cyclic $p'$-subgroups of prime power order to the set of all cyclic subgroups of prime power order. But note that from an algebraic point of view none of the extensions det and $\tilde{\det}$ is distinguished. They can also both be considered as natural transformations in the sense of Remark 1.10.

(ii) The above proposition is certainly not surprising. The existence of an extension of the determinant from $A$ to $M$ is clear, since we have the representation theoretic interpretation of $M$ which allows to define determinants without effort. And the uniqueness part is also clear, because $\det_H(\chi)$ is a function on $H$, and hence determined by its restrictions to cyclic subgroups (resp. cyclic $p'$-subgroups). But on these subgroups $H$, the value is already defined, since $A(H) = M(H)$.

Nevertheless, we think that also this trivial application shows the power of the theorems in Section 1. Assume that by what reasons ever, we knew the character ring only as a certain lattice in the $\mathbb{C}$-vector space of class functions on a finite group, without having any module-theoretic interpretation. It would be very hard to extend the definition from $R^{ab}$ to $R$. Note that even without module theoretic interpretation the map $\det_H: R^{ab}(H) \to \hat{H}$, $\varphi \mapsto \varphi$, is very natural to be considered. The next section will provide an example, where because of the lack of a module theoretic interpretation of the Adams operations the definition in the literature is not straightforward.

(iii) Although the statements about uniqueness and existence in the last proposition are not surprising, the explicitness of the existence part has a surprising aspect. We write as in Equations (3.1) and (3.2), for $H \leq G$ and $\chi \in M(H)$,

$$a_H(\chi) = \sum_{(K, \psi) \in H \setminus M(H)} a_{(K, \psi)}^H(\chi)[K, \psi]_H \in A_+(H)$$
with \( \alpha^H_{(K,\psi)}(\chi) \in \mathbb{Z} \) and \( \mathcal{M}(H) \) according to the situation. Then we have

\[
\chi = \sum_{(K,\psi) \in H \setminus \mathcal{M}(H)} \alpha^H_{(K,\psi)}(\chi) \text{ind}_K^H(\chi) \in \mathcal{M}(H)
\]

(4.2) by applying \( \beta_H^{M,A} \). Hence, applying \( \det_H \) and using Proposition 2.2 we obtain

\[
\det_H(\chi) = \prod_{(K,\psi) \in H \setminus \mathcal{M}(H)} (\varepsilon_{H/K} \cdot (\overline{\psi} \circ \text{tr}_{H/K} \circ \nu_H))^\alpha^H_{(K,\psi)}(\chi)
\]

(4.3)

On the other hand, we have \( \det_H(\chi) = \widetilde{\det}_H(\chi) = (b_H^{M,A} \circ \text{det}_+ \circ a_H)(\chi) \) in the situations 2.3 (i)–(iii), which expands to

\[
\det_H(\chi) = \prod_{(K,\psi) \in H \setminus \mathcal{M}(H)} (\overline{\psi} \circ \text{tr}_{H/K} \circ \nu_H)^\alpha^H_{(K,\psi)}(\chi).
\]

(4.4)

This shows that we may forget about the extra factors \( \varepsilon_{H/K} \) if we use a canonical induction formula. This is obviously not true if we use an arbitrary induction formula. Another possible interpretation of Equations (4.3) and (4.4) is that starting with Equation 4.2 we may calculate \( \det_H(\chi) \) by letting \( \det_H \) commute with induction in each summand. Of course, \( \det_H \) does not commute with induction in general, but the error terms, namely \( \varepsilon_{H/K} \), just cancel over the whole sum (or product).

### 4.3 Adams operations

Throughout this section let \( G \) denote a finite group, and \( O \) a complete discrete valuation ring with algebraically closed residue field \( F \) of characteristic \( p > 0 \) and quotient field \( K \) of characteristic zero which contains the \( |G| \)-roots of unity.

We are going to apply the theorems of Section 1 to the morphisms \( \Psi^n \), \( n \in \mathbb{Z} \), which are known as Adams operations. We first give a partial definition in each situation and extend the definition using Theorem 1.4.

#### 3.1 In six situations we are going to specify the following data: A \( \mathbb{Z} \)-Mackey functor \( M \) on \( G \), a \( \mathbb{Z} \)-restriction subfunctor \( A \subseteq M \) which has a stable basis \( \mathcal{B}(H) \), \( H \leq G \), carrying the structure of a multiplicative group in its own right (i.e. \( A(H) \) is a group ring) unless \( \mathcal{B}(H) = \emptyset \), and a canonical induction formula \( a \in \mathbb{Z} - \text{Res}(G)(M, A_+) \). Moreover, we define in each of these situations the \( n \)-th \textit{Adams operation} \( \Psi^n \in \mathbb{Z} - \text{Res}(G)(A, A) \) for \( n \in \mathbb{Z} \) by

\[
\Psi^n_H(\varphi) = \varphi^n
\]

for \( H \leq G, \varphi \in \mathcal{B}(H) \). Here the six situations:

(i) Character ring: \( M := R_K, A := R_K^{ab}, \mathcal{B}(H) := \hat{H}(K) \) for \( H \leq G, a: R_K \to R_K^{ab} \) as in Example III.1.1.
(ii) Brauer character ring: \( M := R_F, A := R^\text{ab}_F, \mathcal{B}(H) := \hat{H}(F) \) for \( H \leq G \), 
\( a: R_F \to R^\text{ab}_F \) as in Example III.2.1.

(iii) \( p \)-projective character group: \( M := P_F, A := P^\text{ab}_F, \mathcal{B}(H) := \hat{H}(F) \) for \( H \leq G \)
\( a \) \( p \)-subgroup, \( \mathcal{B}(H) = \emptyset \) otherwise, \( a: P_F \to P^\text{ab}_F \) as in Example III.3.1.

(iv) \( \pi \)-projective character group: \( M := R^K, A := R^\text{ab}_{K^\pi}, \mathcal{B}(H) := \hat{H}(K) \) for \( H \leq G \) a \( \pi \)-group, \( a: R^K \to R^\text{ab}_{K^\pi} \) as in Example III.3.4.

(v) Linear source ring: \( M := L, A := L^\text{ab}, \mathcal{B}(H) := \hat{H}(O) \) for \( \leq G, a: L \to L^\text{ab} \) as in Example III.4.2.

(vi) Trivial source ring: \( M := T, A := T^\text{ab}, \mathcal{B}(H) := \hat{H}(O)_{p^\prime} \) for \( H \leq G, \)
\( a: T \to T^\text{ab} \) as in Example III.4.9.

Note that in situation (i), (ii), (v), and (vi), we have \( M \in \mathbb{Z} - \text{Mack}_{\text{alg}}(G), A \in \mathbb{Z} - \text{Res}_{\text{alg}}(G) \), and \( \Psi^n \in \mathbb{Z} - \text{Res}_{\text{alg}}(G)(A, A) \). In situation (ii) and (iv), \( A(H) \subseteq M(H) \) are still rings (but possibly without unity) and \( \Psi^n \) is still multiplicative.

**3.2 Proposition** Let \( M, A, \Psi^n \) for \( n \in \mathbb{Z} \), be defined as in one of the six situations in 3.1.

(i) The morphisms \( \Psi^n \in k - \text{Res}(G)(A, A), n \in \mathbb{Z} \), can be extended to morphisms \( \tilde{\Psi}^n \in \mathbb{Z} - \text{Res}(G)(A \subseteq M, A \subseteq M) \) by the definition \( \tilde{\Psi}^n := \Sigma^{M,A,a}(\Psi^n) = \iota^{M,A} \circ \Psi^n + a \).

(ii) In situation 3.1 (i)–(iv) the extensions are unique.

(iii) In situation 3.1 (i), (ii), and (vi), we have \( \tilde{\Psi}^n \in \mathbb{Z} - \text{Res}_{\text{alg}}(G)(A \subseteq M, A \subseteq M) \) for \( n \in \mathbb{Z} \).

(iv) In situation 3.1 (i)–(iv) and (vi) we have \( \tilde{\Psi}^n \circ \tilde{\Psi}^m = \tilde{\Psi}^{nm} \) for \( m, n \in \mathbb{Z} \).

(v) In situation 3.1 (i), (ii), (v), and (vi), the morphisms \( \tilde{\Psi}^n, n \in \mathbb{Z} \), can be regarded as natural transformations from \( M \) to \( M \) in the sense of Remark 1.10, where \( M \) is considered as a contravariant functor from the category of finite groups to the category of commutative rings.

**Proof** (i) This is immediate from Theorem 1.5.

(ii) This follows from Theorem 1.6, since \( A(H) = M(H) \) for \( H \in \mathcal{C}(M) \), in these situations.

(iii) This follows from Theorem 1.7 (i), since \( \rho^A: A_+ \to A^+ \) is injective (cf. Propositions I.3.2 and I.2.4), and \( p_H: M(H) \to A(H) \) is a \( k \)-algebra homomorphism for \( H \in \mathcal{C}(M) \) in these situations. In fact, in situation 3.1 (i) and (ii), \( p_H \) is the identity, and in situation 3.1 (vi) this is stated in Lemma III.4.11 (iv).

(iv) We apply Theorem 1.8 (i) in situation 3.1 (vi) (cf. Lemma III.4.11 (iii)) and Theorem 1.8 (ii) in situation 3.1 (i)–(iv), and observe that \( \Psi^n \circ \Psi^m = \Psi^{nm} \) holds obviously on \( A \) for \( m, n \in \mathbb{Z} \).

(v) This follows from Remark 1.10, since \( \Psi: A \to A \) can be considered as a natural transformation.

**3.3 Remark** Note that in the situation 3.1 (i), (ii), (iii), and (vi) (with \( T_F \) instead of \( T_0 \)) Adams operations

\[
\Psi^n_G: M(G) \to M(G)
\]
are defined in the literature. In situation 3.1 (i) the definition is well-known:

\[(\Psi^n_G(\chi))(g) := \chi(g^n)\]

for \(n \in \mathbb{Z}, \chi \in R_K(G), g \in G\) (see [Se78, 9.1, Exercise 3] for example) The same definition holds for the ring of Brauer characters if they are interpreted as class functions on \(p\)-regular elements. But note that it is not at all trivial to show that \(\Psi^n_G(\chi) \in R_K(G)\), resp. \(\Psi^n_G(\chi) \in R_F(G)\). Kervaire gives a treatment of the situations 3.1 (i), (ii), and (iii) in [Ke75]. Note that his definition of \(\Psi^n_G\) is complicated, and so are the proofs of the properties of \(\Psi^n_G\). Benson defines \(\Psi^n_G: Gr_F(G) \rightarrow Gr_F(G)\) for the Green ring in [Be84a]. Again the definition seems complicated, using tensor induction, and also the proofs of the main properties are not trivial. But it is clear from the definition that the subgroups \(P_F(G) \leq T_F(G) \leq Gr_F(G)\) are invariant under \(\Psi^n\). It is also clear from the definitions in all four situations that the Adams operations are morphisms of \(\mathbb{Z}\)-restriction functors on \(G\), and that they coincide with the definitions in 3.1 on \(A\).

Next we show that our definitions coincide with the definitions in the literature in situations 3.1 (i), (ii), (iii), and (vi).

3.4 Proposition Let \(M, A, a, \Psi^n \in \mathbb{Z} - \text{Res}(G)(A, A)\), for \(n \in \mathbb{Z}\), be defined as in one of the situations 3.1 (i), (ii), (iii), or (vi), and let \(\Psi^n \in \mathbb{Z} - \text{Res}(G)(A \subseteq M, A \subseteq M)\) be the canonical extension as described in Proposition 3.2. Then \(\Psi^n = \Psi^n\).

Proof In situation 3.1 (i)–(iii) this follows from the uniqueness statement of Proposition 3.2 (ii). In situation 3.1 (vi) it suffices to show that

\[s_{(H,h)}(\Psi^n_G(x)) = s_{(H,h)}(\Psi^n_G(x))\]

for \(H \leq G\) \(p\)-hypo-elementary, \(h \leq G\), \(n \in \mathbb{Z}\), \(x \in T_G(G)\), cf. Proposition III.4.14. Benson showed in [Be84a, Lemma 2] that

\[s_{(H,h)}(\Psi^n_G(x)) = s_{(H[n],h^n)}(x),\]

where \(H[n]\) is the unique subgroup of \(H\) of index \(\gcd(n,|H|_{p'})\). Hence, we obtain from the definition (see Definition III.4.13) of \(s_{(H,h)}\) that

\[s_{(H,h)}(\Psi^n_G(x)) = (s_{h^n} \circ p_{H[n]} \circ \text{Res}_{H[n]}^G)(x)\]

On the other hand we have by Lemma III.4.11 (iii),

\[s_{(H,h)}(\Psi^n_G(x)) = \left(s_h \circ p_H \circ \text{Res}_H^G \circ b_{G}^{M,A} \circ \Psi^n_G \circ a_G\right)(x)\]

and it suffices to show that

\[
(s_h \circ \Psi^n_H \circ p_H)([V]) = (s_{h^n} \circ p_{H[n]} \circ \text{Res}_{H[n]}^H)([V])
\]
for $H \leq G$ $p$-hypo-elementary, $h \in H$, $V \in \text{OH-triv}$ indecomposable, $n \in \mathbb{Z}$. If $\text{rk}_O V = 1$, i.e. $V \cong O_\varphi$ for some $\varphi \in \bar{H}(O)_{e'}$, then both sides are equal to $\varphi^n(h) = \varphi(h^n)$. If $\text{rk}_O V > 1$, then $p_H([V]) = 0$, and $V$ has vertex $Q < O_p(H)$ by Lemma III.4.11 (i). Therefore,

$$
\text{res}^H_{H^{[n]}}(V) \mid \text{res}^H_{H^{[n]}}(\text{ind}^H_{O}([O])) \cong \bigoplus_{h \in H/H^{[n]}} \text{ind}^H_{Q}([O]),
$$

and $\text{res}^H_{H^{[n]}}(V)$ has no indecomposable summand with vertex $O_p(H)$. Now part (i) of Lemma III.4.11 implies

$$
p_H^{[n]}(\text{res}^H_{H^{[n]}}([V])) = 0.
$$

Note that we can use the above proposition as proof that the Adams operations carry the character ring $R_K(G)$ to $R_K(G)$. In fact, if we tensor the character ring Mackey functor with $K$, we can be sure that this extended Mackey functor is stable under the Adams operations. But now the two constructions $\Psi^n$ and $\tilde{\Psi}^n$ are extensions of the same morphism on $K \otimes R^{ab}$. Hence they coincide, and with $\tilde{\Psi}^n$ also $\Psi$ stabilizes $R_K$.

From now on we will denote $\tilde{\Psi}^n$ by $\Psi^n$, since whenever both are defined, they coincide.

Recall that the Cartan map $c_G: P_F(G) \to R_F(G)$ is defined by $c_G([V]) = [V]$ for a projective $FG$-module $V$. This defines a morphism $c: P_F \to R_F$ of $Z$-Mackey functors on $G$ which restricts to a morphism $c: P_F^{ab} \to R_F^{ab}$ of $Z$-restriction functors on $G$.

### 3.5 Proposition (cf. [Ke75, §5])

For $n \in \mathbb{Z}$ the diagram

$$
\begin{array}{ccc}
P_F & \xrightarrow{\Psi^n} & P_F \\
c \downarrow & & c \downarrow \\
R_F & \xrightarrow{\Psi^n} & R_F
\end{array}
$$

in $Z - \text{Res}(G)$ is commutative.

**Proof** The two morphisms $\Psi^n \circ c$ and $c \circ \Psi^n$ in $Z - \text{Res}(G)(P_F, R_F)$ coincide on $P_F^{ab}$. Hence they have to be equal by Theorem 1.3, since we have the canonical induction formula $a$ from Example III.3.1. □

### 3.6 Remark

(i) Note that $\Psi^n$ does not commute with the decomposition morphism $d: R_K \to R_F$ or the morphism $e: P_F \to R_K$ from the $cde$-triangle (cf. [Se78, §15]) which is given as the composition of the canonical morphisms $P_F \cong P_O \cong R_{K,p} \subseteq R_K$. It is easy to show that the Adams operations $\Psi^n: P_F \to P_F$ and $\tilde{\Psi}^n: R_{K,p} \to R_{K,p}$ correspond to each other via the canonical isomorphism. But the Adams operations on $R_K$ do not restrict to the Adams operations on $R_{K,p}$, since the power of a $p$-singular element in $G$ is in general not $p$-singular.

(ii) For two sets $\pi_1 \subseteq \pi_2$ of primes we have an inclusion $R_{K,\pi_2} \subseteq R_{K,\pi_1}$ of $Z$-Mackey functors on $G$ which restricts to an inclusion $R_{K,\pi_2}^{ab} \subseteq R_{K,\pi_1}^{ab}$ of $Z$-restriction
functors on $G$. As in the above proposition one shows easily that the Adams operations on $R_{K,p_1}$ restrict to those of $R_{K,p_2}$. In particular, Adams operations on $R_{K,p_1}$ and $R_{K,p_2}$ coincide on the intersection $R_{K,(p_1,p_2)}$.

### 3.7 Proposition

The diagram

$$
\begin{array}{ccc}
P_O & \subseteq & T_O & \subseteq & L_O \\
\psi^n & & \psi^n & & \psi^n \\
P_O & \subseteq & T_O & \subseteq & L_O
\end{array}
$$

commutes for $n \in \mathbb{Z}$.

**Proof**

The commutativity of the diagram

$$
\begin{array}{cccccc}
L_O & \overset{p}{\rightarrow} & L^a_O & \overset{\psi^n}{\rightarrow} & L^a_O & \subseteq & L_O \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
T_O & \overset{p}{\rightarrow} & T^a_O & \overset{\psi^n}{\rightarrow} & T^a_O & \subseteq & T_O \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
P_O & \overset{p}{\rightarrow} & P^a_O & \overset{\psi^n}{\rightarrow} & P^a_O & \subseteq & P_O
\end{array}
$$

implies by Proposition III.3.11 the commutativity of the diagram

$$
\begin{array}{cccccc}
L_O & \overset{a}{\rightarrow} & L^a_O & \overset{\psi^n}{\rightarrow} & L^a_O & \overset{\psi^n}{\rightarrow} & L_O \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
T_O & \overset{a}{\rightarrow} & T^a_O & \overset{\psi^n}{\rightarrow} & T^a_O & \overset{\psi^n}{\rightarrow} & T_O \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
P_O & \overset{a}{\rightarrow} & P^a_O & \overset{\psi^n}{\rightarrow} & P^a_O & \overset{\psi^n}{\rightarrow} & P_O
\end{array}
$$


### 3.8 Remark

Let $M$, $A$, $a$, and $\Psi^n \in \mathbb{Z} - \text{Res}(G)(A \subseteq M, A \subseteq M)$ for $n \in \mathbb{Z}$ be given in one of the situations 3.1 (i)–(vi), and write for $\chi \in M(G)$:

$$
a_G(\chi) = \sum_{(H,\varphi) \in G \setminus M(G)} \alpha^G_{(H,\varphi)}(\chi)[H,\varphi]_G
$$

with $\alpha^G_{(H,\varphi)}(\chi) \in \mathbb{Z}$ and $M$ as in Definition II.3.2. Then, applying $b^{M,A}$ one obtains

$$
\chi = \sum_{(H,\varphi) \in G \setminus M(G)} \alpha^G_{(H,\varphi)}(\chi) \text{ind}^G_H(\varphi). \quad (4.5)
$$

Now the definition of $\Psi^n_G$ as canonical extension is given by

$$
\Psi^n_G(\chi) = \sum_{(H,\varphi) \in G \setminus M(G)} \alpha^G_{(H,\varphi)}(\chi) \text{ind}^G_H(\varphi^k). \quad (4.6)
$$

If we compare Equation (4.5) and (4.6) we see that we may commute induction with $\Psi^n$ in each summand without changing the result. But in general, $\Psi^n$ does not commute with induction, and there is an error term in each summand, if one commutes them by force. However, the error terms cancel over the whole sum.
4.4 Chern classes

Hence, using the canonical induction formula (4.5) we may assume that $\Psi^n$ commutes with induction, as long as we are only interested in the result of the whole sum.

Now we are able to prove Proposition III.3.2 (vi) and III.3.5 (v).

3.9 Remark Assume that we are in situation 3.1 (iv) ($\pi$-projective character group). In order to prove Proposition III.3.5 (v) we have to show that

$$\sum_{(H,\varphi)\in G\setminus M(G)} \alpha^G_{(H,\varphi)}(\chi) = \frac{|G_{\pi'}|}{|G|}$$

for $\chi \in R_{K,\pi}(G)$. We consider Equation (4.6) with $n = 0$ and obtain

$$\Psi^0_G(\chi) = \sum_{(H,\varphi)\in G\setminus M(G)} \alpha^H_{(H,\varphi)}(\chi) \text{ind}^G_H(1).$$

Note that by the compatibility with restrictions, $\Psi^0_G(\chi)$ is the class function which is constantly $\chi(1)$ on $\pi$-regular elements, i.e. $\pi'$-elements, and zero on $\pi$-singular elements. Taking scalar products with the trivial character yields the result in Proposition III.3.5 (v).

With $\pi = \{p\}$ we obtain Proposition III.3.2 (vi) as a special case.

4.4 Chern classes

Let $G$, $O$, $F$, and $K$ be given as at the beginning of the previous section.

In this section we are going to apply Theorem 1.2 to construct Chern classes for characters, Brauer characters, projective characters, trivial source modules, and linear source modules. We start with the description of the Mackey functor $N$, the Chern classes live in.

4.1 For a finite group $G$ define an abelian group

$$H^*_e(G, \mathbb{Z}) := \prod_{n \geq 0} H^{2n}(G, \mathbb{Z}).$$

The upper index ** indicates that we take the direct product (not the direct sum as for the cohomology ring), and the index ‘$e$’ indicates that we consider only even degrees. It is easy to see that $H^*_e(G, \mathbb{Z})$ is a commutative ring with respect to the cup product. We will write the elements $x$ of $H^*_e(G, \mathbb{Z})$ as infinite series $x = x_0 + x_1 + x_2 + \ldots$ with $x_i \in H^{2i}(G, \mathbb{Z})$. For a group homomorphism $f: G' \to G$ there is an induced ring homomorphism

$$H^*_e(f, \mathbb{Z}) : H^*_e(G, \mathbb{Z}) \to H^*_e(G', \mathbb{Z}).$$

With this definition we may consider $H^*_e(-, \mathbb{Z})$ as a contravariant functor from the category of finite groups to the category of commutative rings. Taking units, we obtain a contravariant functor $H^*_e(-, \mathbb{Z})^\times$ from the category of finite groups to
the category of abelian groups. In particular, for \( H \leq G \), \( g \in G \), we have ring homomorphisms

\[
e_{g,H} : H^*_e(H, \mathbb{Z}) \to H^*_e(gH, \mathbb{Z}),
\]

\[
\text{res}_H^G : H^*_e(G, \mathbb{Z}) \to H^*_e(H, \mathbb{Z})
\]

induced by the conjugation map \( gH \to H \) and by the inclusion \( H \leq G \). For \( H \leq G \), Evens defined in [Ev63] a function

\[
\text{ind}_H^G : H^*_e(H, \mathbb{Z}) \to H^*_e(G, \mathbb{Z})
\]

which is multiplicative with respect to the cup product, but not additive. Hence we also obtain a group homomorphism

\[
\text{ind}_H^G : H^*_e(H, \mathbb{Z})^\times \to H^*_e(G, \mathbb{Z})^\times.
\]

**4.2 Proposition**  
For a finite group \( G \), the abelian groups \( H^*_e(H, \mathbb{Z})^\times \), \( H \leq G \), together with the maps defined above, define a \( \mathbb{Z} \)-Mackey functor on \( G \).

**Proof**  
The Mackey functor axioms, where induction is not involved are obviously satisfied by functoriality. The Mackey functor axioms involving induction maps follow immediately from the properties of \( \text{ind}_H^G \) proved in [Ev63].

**4.3** For an integral domain \( k \) we may indentify the torsion subgroup of \( k^\times \) with the group of roots of unity in the quotient field of \( k \). Hence, it may be identified (non-canonically) with a subgroups of \( \mathbb{Q}/\mathbb{Z} \). The exact sequence

\[
0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0
\]

of \( G \)-modules with trivial action gives rise to a long exact cohomology sequence, which yields an isomorphism

\[
H^1(G, \mathbb{Q}/\mathbb{Z}) \cong H^2(G, \mathbb{Z}).
\]

Using the above identification \( k^\times \leq \mathbb{Q}/\mathbb{Z} \) we may regard \( \text{Hom}(G, k^\times) \) as a subgroup of \( \text{Hom}(G, \mathbb{Q}/\mathbb{Z}) = H^1(G, \mathbb{Q}/\mathbb{Z}) \), and obtain an embedding

\[
\text{Hom}(G, k^\times) \to H^2(G, \mathbb{Z})
\]

which we keep fixed for \( k \). Now we define a group homomorphism

\[
ch_G : \text{Hom}(G, k^\times) \to H^*_e(G, \mathbb{Z})^\times, \quad \varphi \mapsto 1 + \varphi,
\]

where we consider \( 1 \in H^0(G, \mathbb{Z}) = \mathbb{Z} \) and \( \varphi \in H^2(G, \mathbb{Z}) \) via the embedding in (4.7).

The groups \( \text{Hom}(H, k^\times) \) and \( H^*_e(H, \mathbb{Z})^\times \), \( H \leq G \) form \( \mathbb{Z} \)-Mackey functors on \( G \) with the definitions in Example 2.1 and in 4.1. It is easy to check that \( ch_H \), \( H \leq G \) form a morphism of \( \mathbb{Z} \)-restriction functors on \( G \). Now we extend \( ch_H \) for \( H \leq G \), linearly to

\[
\mathbb{Z} \cdot \text{Hom}(H, k^\times) \to H^*_e(H, \mathbb{Z})^\times
\]
and obtain, with suitable choices for $k$, morphisms

\[
\begin{align*}
ch: R_{ab}^K & \rightarrow H\hat{e}^*(-, \mathbb{Z})^\times \\
ch: R_{ab}^F & \rightarrow H\hat{e}^*(-, \mathbb{Z})^\times \\
ch: R_{ab}^{K,\pi} & \rightarrow H\hat{e}^*(-, \mathbb{Z})^\times \\
ch: P_{ab}^O & \rightarrow H\hat{e}^*(-, \mathbb{Z})^\times \\
ch: T_{ab}^O & \cong T_{ab}^F \rightarrow H\hat{e}^*(-, \mathbb{Z})^\times \\
ch: L_{ab}^O & \rightarrow H\hat{e}^*(-, \mathbb{Z})^\times
\end{align*}
\]

(4.8)

of $\mathbb{Z}$-restriction functors on $G$ with $R_{ab}^K$, $R_{ab}^F$, $P_{ab}^O$, $T_{ab}^O$, $T_{ab}^F$, and $L_{ab}^O$ as defined in Chapter III. Note that in all these cases, except for $R_{ab}^{K,\pi}$ and $P_{ab}^O$, the morphism $ch$ can be regarded as a natural transformation between contravariant functors from the category of finite groups to the category of abelian groups.

4.4 Proposition The morphisms in (4.8) can be extended to morphisms

\[
\begin{align*}
ch: R_K & \rightarrow H\hat{e}^*(-, \mathbb{Z})^\times \\
ch: R_F & \rightarrow H\hat{e}^*(-, \mathbb{Z})^\times \\
ch: R_{K,\pi} & \rightarrow H\hat{e}^*(-, \mathbb{Z})^\times \\
ch: P_O & \rightarrow H\hat{e}^*(-, \mathbb{Z})^\times \\
ch: T_O & \cong T_F \rightarrow H\hat{e}^*(-, \mathbb{Z})^\times \\
ch: L_O & \rightarrow H\hat{e}^*(-, \mathbb{Z})^\times
\end{align*}
\]

of $\mathbb{Z}$-restriction functors on $G$. These morphisms, except the ones defined on $R_{K,\pi}$ and $P_O$, can be considered as natural transformations in the sense of Remark 1.10.

Proof This is immediate from Theorem 1.2.

We call $ch_G(\chi) \in H\hat{e}^*(G, \mathbb{Z})^\times$ the total Chern class of $\chi \in R_K(G)$ (resp. $\chi \in R_F, R_{K,\pi}, P_O, T_O, T_F, L_O$). We write

\[
ch_G(\chi) = ch_G^0(\chi) + ch_G^1(\chi) + ch_G^2(\chi) + \ldots
\]

with $ch_G^i(\chi) \in H^{2i}(G, \mathbb{Z})$ and $ch_G^0(\chi) = 1$, and call $ch_G^i(\chi)$ the $i$-th Chern class of $\chi$.

4.5 Remark (i) Classically, Chern classes were defined for ordinary complex characters, or on $R_K(G)$ (see [At61] for example), by using the theory of vector bundles on the classifying space $BG$ of $G$. On one-dimensional characters this definition coincides with the one in (4.8). It was first observed by Symonds (cf. [Sy91]) that the canonical induction formula $a: R_K \rightarrow R_{ab}^+$ of Example III.1.1 can be used to give an algebraic definition of the Chern classes. Since the set of coprimalorial subgroups of the cohomology Mackey functors is the set of $p$-subgroups for all primes $p$, and since $R_K(H) \neq R_{ab}^H(H)$ for arbitrary $p$-groups $H$, Theorem 1.3 cannot be applied to show that the two constructions coincide. Fortunately, there is an axiomatic description of Chern classes by Kroll in [Kr87] with states that Chern classes are characterized by three properties, namely functoriality, additivity, and
the definition on one-dimensional characters. Obviously our construction has these properties, and therefore coincides with the original definition.

Using the explicit version (cf. Proposition III.1.2 (v)) of the canonical induction formula, we obtain the explicit formula

$$ch_G(\chi) = \prod_{(H_0, \varphi_0) \ldots (H_n, \varphi_n) \in G \cap (\Gamma(M(G))} ind_{H_0}^G(1 + \varphi_0)^{-1}(\text{res}_{H_0}^G(\chi), \varphi_n)_{H_n}$$

for the total Chern class of $\chi \in R_K(G)$.

(ii) As far as the author knows, there are no definitions of Chern classes for $R_F(G), R_K, \pi(G), P_F(G) \sim_\sim PO(G), TF(G) \sim_\sim TO(G)$, or $L_O(G)$. Theorem 1.3 cannot guarantee the uniqueness of the extension, since we can’t say more about the set of coprimordial subgrops of $H^*_e(-, \mathbb{Z})^\times$ than that it is a subset of the set of subgroups of prime power order. But it might be possible that there are characterizations as the one of Kroll for the Chern classes on $R_F(G), TF(G) \sim_\sim TO(G)$, and $L_O(G)$, which can be interpreted as contravariant functors in the sense of Remark 1.10.

(iii) Similar to the proof of Proposition 3.7 for Adams operations we can show that the diagram

$$\begin{align*}
P_O & \longrightarrow T_O & \longrightarrow & L_O \\
ch & \downarrow & ch & \downarrow \text{ch} \\
H^*_e(-, \mathbb{Z})^\times & \longrightarrow & H^*_e(-, \mathbb{Z})^\times & \longrightarrow & H^*_e(-, \mathbb{Z})^\times,
\end{align*}$$

where the upper horizontal arrows denote the inclusions, is commutative.

(iv) In general, the diagram

$$\begin{align*}
P_O & \xrightarrow{e} R_K \\
ch & \downarrow \text{ch} \\
H^*_e(-, \mathbb{Z})^\times & \longrightarrow H^*_e(-, \mathbb{Z})^\times
\end{align*}$$

is not commutative. Let $G$ be the cyclic group of order 2 and consider $OG \in OG-\text{mod}$. Then $ch_G([OG]) = \text{ind}_1^G(ch_1([O])) = \text{ind}_1^G(1) = 1$, since $a_G([O]) = [1, 1]_G$. On the other hand $ch_G(\text{ind}_1^G(1)) = ch_G(1 + \tau)$, if $\tau$ denotes the non-trivial one-dimensional character of $G$. Hence, $ch_G(\text{ind}_1^G(1)) = ch_G(1) \cdot ch_G(\tau) = 1 \cdot (1 + \tau) = 1 + \tau$.

4.5 Virtual extensions of characters and modules

In this section we will use some of the canonical induction formulae from Chapter III to define virtual extensions of characters or modules.

Throughout this section let $X = G \rtimes S$ be the semidirect product of a finite group $G$ with a finite group $S$ satisfying $\gcd(|G|, |S|) = 1$. We consider both $G$ and $S$ as subgroups of $G$ and denote by $G^S$ the fixed points of $G$ under $S$. Some readers may prefer the notation $C_G(S)$ for $G^S$. Let $\pi$ denote the set of prime divisors of $|S|$.

First we recall a Lemma on coprime action, cf. [Is76, 13.8 and 13.9].
5.1 Lemma  Let $D$ be a finite left $X$-set such that $G$ acts transitively on $D$. Then we have:

(i) There exists an $S$-stable element in $D$, i.e. $D^S \neq \emptyset$.
(ii) The set $D^S$ is a transitive $G^S$-set.

5.2 We adopt the notation of Section III.1. Our first aim is to define a map $\text{ext}^{G,S}: R(G)^S \to R(GS)$ with $\text{res}^{G,S} \circ \text{ext}^{G,S} = \text{id}_{R(G)^S}$, where $R(G)^S$ denotes the fixed points of $R(G)$ under the conjugation action of $S$. We will do this by defining

$$\text{ext}^{G,S}: R(G)^S \xrightarrow{a_G} R(ab)_G^{+} \xrightarrow{\text{ext}^{G,S}} R(ab)_S^{+} \xrightarrow{\beta_{R,ab}} R(GS),$$

with a map $\text{ext}^{G,S}$ which will be defined in a moment. Note that $a_G$ maps $S$-fixed points to $S$-fixed points, since it commutes with conjugation.

Note that the basis elements $[H, \varphi]|_G$ of $R^{ab}_+(G)$, which correspond to $G \setminus \mathcal{M}(G)$, are permuted by the $S$-conjugation maps. Therefore, the $S$-orbit sums

$$\sum_{s \in S/\text{stab}_G([H, \varphi]|_G)} [sH, s\varphi]|_G, \quad (H, \varphi) \in GS \setminus \mathcal{M}(G),$$

form a $\mathbb{Z}$-basis of $R^{ab}_+(G)^S$. For $(H, \varphi) \in \mathcal{M}(G)$ let $T := \text{stab}_S([H, \varphi]|_G)$ be the stabilizer of $[H, \varphi]|_G$ in $S$. Then $GT$ acts $G$-transitively on the $G$-orbit of $(H, \varphi)$. Hence, by Lemma 5.1, there exists a $T$-fixed point $(H', \varphi')$ in the $G$-orbit of $(H, \varphi)$, i.e. $T = \text{stab}_G([H', \varphi']|_G) = \text{stab}_S((H', \varphi'))$, and the set of $T$-fixed points forms a $G^T$-orbit. Now we define

$$\text{ext}^{G,S}_+ \left( \sum_{s \in S/T} [sH, s\varphi]|_G \right) := [H' \cdot T, \varphi' \cdot 1]_{GS}$$

with $\varphi' \cdot 1 \in \text{Hom}(H' \cdot T, C^\times)$ defined by $(\varphi' \cdot 1)(h t) := \varphi'(h')$ for $h' \in H', t \in T$. This is a homomorphism, since $\varphi'$ is $T$-stable. It is easy to verify that $[H' \cdot T, \varphi' \cdot 1]|_{GS}$ does not depend on the choice of $[H, \varphi]$ in the $S$-orbit, and of $(H', \varphi')$ in the $G$-orbit of $(H, \varphi)$. Here Lemma 5.1 (ii) is needed.

Note that the notation $\text{ext}^{G,S}_+$ is misleading, since it is not the result of some morphism under the functor $-^S$. All the same we use this notation just to distinguish $\text{ext}^{G,S}: R(G)^S \to R(GS)$ and $\text{ext}^{G,S}_+: R^{ab}_+(G)^S \to R^{ab}_+(GS)$.

5.3 Proposition  For $U \leq S$ the diagram

$$\begin{array}{ccc}
R(G)^S & \xrightarrow{\text{ext}^{G,S}} & R(GS) \\
\cap & & \downarrow \text{res}^{G,S}_{GU} \\
R(G)^U & \xrightarrow{\text{ext}^{G,U}} & R(GU)
\end{array}$$

is commutative.

Proof  It suffices to prove that the three squares in the diagram

$$\begin{array}{cccc}
R(G)^S & \xrightarrow{a_G} & R^{ab}_+(G)^S & \xrightarrow{\text{ext}^{G,S}} & R^{ab}_+(GS) & \xrightarrow{\beta_{R,ab}} & R(GS) \\
\cap & & \cap & & \downarrow \text{res}^{G,S}_{GU} & & \downarrow \text{res}^{G,S}_{GU} \\
R(G)^U & \xrightarrow{a_G} & R^{ab}_+(G)^U & \xrightarrow{\text{ext}^{G,U}} & R^{ab}_+(GU) & \xrightarrow{\beta_{R,ab}} & R(GU)
\end{array}$$
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commute. The left hand square commutes obviously, and the right hand square also, since \( b \) is a morphism of Mackey functors. We will show that also the middle square commutes.

Let \((H, \varphi) \in \mathcal{M}(G)\) such that \( T := \text{stab}_S([H, \varphi]_G) = \text{stab}_S((H, \varphi))\). Then we have

\[
\begin{align*}
(res_+^{GS} \circ \text{ext}_+^{G,S})(\sum_{s \in S/T} [^sH, ^s\varphi]_G) &= res_+^{GS}([HT, \varphi \cdot 1]_GS) \\
&= \sum_{x \in GU \setminus GS/HT} [GU \cap ^x(HT), res_{GU \cap ^x(HT)}^{^x(\varphi \cdot 1)}]_{GU}.
\end{align*}
\]

It is easy to verify that a set of representatives for the double cosets in \( U \setminus S/T \) is also a set of representatives for the double cosets in \( GU \setminus GS/HT \). For \( s \in S \) we have \( GU \cap ^s(HT) = GU \cap (^sH \cdot ^sT) = ^sH \cdot (U \cap ^sT) \) and \( res_{GU \cap ^x(HT)}^{^x(\varphi \cdot 1)} = ^x\varphi \cdot 1 \). Hence we obtain

\[
(res_+^{GS} \circ \text{ext}_+^{G,S})(\sum_{s \in S/T} [^sH, ^s\varphi]_G) = \sum_{s \in U \setminus S/T} [^sH \cdot (U \cap ^sT), ^s\varphi \cdot 1]_{GU}.
\]

On the other hand, if we want to apply \( \text{ext}_+^{G,U} \), we first have to decompose the element \( \sum_{s \in S/T} [^sH, ^s\varphi]_G \) into \( U \)-orbit sums, namely

\[
\sum_{s \in S/T} [^sH, ^s\varphi]_G = \sum_{s \in U \setminus S/T} \sum_{u \in U \cap U \cap ^sT} [^u s^sH, ^u s^s \varphi]_G.
\]

Note that \( \text{stab}_U(^sH, ^s\varphi) = U \cap ^sT = \text{stab}_U([^sH, ^s\varphi]_G) \), and therefore we may use \((^sH, ^s\varphi)\) as representative in each \( U \)-orbit sum in order to apply \( \text{ext}_+^{G,U} \) to the \( U \)-orbit sum \( \sum_{u \in U \cap U \cap ^sT} [^u s^sH, ^u s^s \varphi]_G \), and we obtain

\[
\text{ext}_+^{G,U}(\sum_{s \in S/T} [^sH, ^s\varphi]_G) = \sum_{s \in U \setminus S/T} \text{ext}_+^{G,U}(\sum_{u \in U \setminus U \cap ^sT} [^u s^sH, ^u s^s \varphi]_G) = \sum_{s \in U \setminus S/T} [^sH \cdot (U \cap ^sT), ^s\varphi \cdot 1]_{GU}.
\]

\[\square\]

5.4 Corollary The composition

\[
R(G)^{S \text{ext}_+^{G,S}} \xrightarrow{R(G)^{\text{res}_+^{GS}}} R(GS) \xrightarrow{R(G)^{\text{res}_+^{GS}}} R(G)^{S}
\]

is the identity map.

Proof This follows from Proposition 5.3 with \( U = 1 \). In fact, the map \( \text{ext}_+^{G,1} : R_{ab}^b(G) \to R_{ab}^a(G) \) is the identity by definition, and \( b \circ a \) is the identity, since \( a \) is a canonical induction formula. \[\square\]

5.5 Proposition The map \( \text{ext}_+^{G,S} : R(G)^{S} \to R(GS) \) is Galois invariant, i.e.

\[
\sigma(\text{ext}_+^{G,S}(\chi)) = \text{ext}_+^{G,S}(\sigma \chi)
\]
for $\chi \in R(G)^S$ and $\sigma$ a field automorphism of the algebraic closure of $Q$.

**Proof** Since $a_G$ and $b_G$ are Galois invariant, it suffices to show that $\text{ext}^G_S$ is Galois invariant. This in turn follows easily from the fact that Galois action and conjugation action commute. \hfill \square

### 4.5. Virtual Extensions

#### 5.6 Remark

(i) An analysis of the definition of $\text{ext}^G_S : R^ab(G)^S \rightarrow R^ab(GS)$ shows that we may define $\text{ext}^G_S : A_+(G)^S \rightarrow A_+(GS)$ for $A \in \mathbb{Z} - \text{Res}(GS)$ with stable basis $B$ containing the units $1_{A(H)}$, $H \leq G$, such that for $HT \leq GS$ with $H \leq G$, $T \leq N_S(H)$, and $T$-stable $\varphi \in B(H)$, there exists a unique $\tilde{\varphi} \in B(HT)$ with $\text{res}^HT(\varphi) = \varphi$ and $\text{res}^HT(\tilde{\varphi}) = 1_{A(T)}$ (extendibility property for $B$). Note that $R^ab_K$ (cf. Section III.1), $R^ab$ (cf. III.2), $T^ab_G \cong T^ab_F$ (cf. III.4), and $L^ab_G$ (cf. III.4) have this property, and that $P^ab_G \cong P^ab_F$ and $R^ab_K$ (cf. III.3) have this property when $\text{char}(F) \notin \pi$ resp. $\pi \cap \pi_1 = \emptyset$. Moreover, if $M \in \mathbb{Z} - \text{Mack}(G)$ with $A \subseteq M$ a $\mathbb{Z}$-restriction subfunctor on $G$, and $a : M \rightarrow A_+$ is a canonical induction formula, then we can define $\text{ext}^G_S : M(G)^S \rightarrow M(GS)$ by $\text{ext}^G_S := b^M_G \circ \text{ext}_+ \circ a_G$. Proposition 5.3 and Corollary 5.4 hold in this general situation by the same proofs. If there is a Galois action which commutes with the conjugation action, then also Proposition 5.5 holds. Note that all this applies to $M = R_K$, $R_F$, $T_O \cong T_F$, $L_O$ always, and to $P_O \cong P_F$, resp. $R_{K, \pi_1}$, when $\text{char}(F) \nmid |S|$, resp. $|S|_\pi_1 = 1$.

(ii) Note that this extendibility property is well-known for complex irreducible characters $\chi \in \text{Irr}(G)^S$, cf. [Hu67, V.17.12] for example. There it is shown that one has even an irreducible extension. More generally, Isaacs showed in [Is81] that each irreducible $S$-stable $LG$-module is the restriction of an irreducible $L[GS]$-module, where $L$ is an arbitrary field. Note that for $\chi \in \text{Irr}(G)^S$ there is a canonical extension $\tilde{\chi} \in \text{Irr}(GS)$ which is uniquely determined by the property $\det_S(\text{res}^GS_G(\tilde{\chi})) = 1$, cf. [Gla68, Thm. 1]. Since our map $\text{ext}^G_S$ is defined in a natural way, one would assume that it produces the canonical extension for irreducible $S$-fixed characters. But this is not true in general. Even worse, the next example shows that $\text{ext}^G_S$ does not map $\text{Irr}(G)^S$ to $\text{Irr}(GS)$, but may result in virtual characters. However, it seems that the prime 2 is responsible for this ‘bad’ behaviour in Example 5.7, a phenomenon encountered also in other situations, cf. Proposition 2.4 or Remark 2.5 (i). In fact, in all examples with odd $|S|$ we know so far, the map $\text{ext}^G_S$ produces the canonical irreducible extension in the sense of [Gla68].

#### 5.7 Example

Let $G = \langle x, y, z \mid x^p = y^p = z^p = 1, [x, y] = z, [x, z] = [y, z] = 1 \rangle$ be the extraspecial $p$-group of order $p^3$ with exponent $p$ for an odd prime $p$, and let $S = \{1, s\}$ act on $G$ by $\overset{s}{x} = x^{-1}$, $\overset{s}{y} = y^{-1}$, $\overset{s}{z} = z$. $G$ has $p - 1$ irreducible characters of degree $p$. Let $\chi$ be one of them, and denote by $\tilde{\chi} \in \text{Irr}(GS)$ the canonical extension and $\hat{\chi}' \in \text{Irr}(GS)$ the other extension of $\chi$. Then

$$\text{ext}^G_S(\chi) = \begin{cases} \frac{p+1}{2} \hat{\chi} - \frac{p-1}{2} \chi', & p \equiv 1 \pmod{4}, \\ \frac{p+1}{2} \chi' - \frac{p-1}{2} \hat{\chi}, & p \equiv 3 \pmod{4}. \end{cases}$$

This can be derived easily from the explicit formula for $a_G(\chi)$ in Proposition III.7.2.
4.6 The Glauberman correspondence

Throughout this section we assume the notation of the previous section. In particular, X is the semidirect product of a normal subgroup G and a subgroup S with relatively prime orders.

Following a suggestion of Puig we will construct a map

\[ gl^{G,S}: R(G)^S \to R(G^S), \]

using the canonical induction formula \( a_G: R(G) \to R_{ab}^G(G) \) from Example III.1.1. This map is related to the Glauberman correspondence. Before we define \( gl^{G,S} \), we will recall the definition and some basic facts of the Glauberman correspondence, cf. [Gla68] or [Is76, §13].

First we state a lemma which is of fundamental importance for the Glauberman correspondence.

6.1 Lemma (cf. [Is76, 13.14]) Assume that S is a p-group and let \( \chi \in \text{Irr}(G)^S \) be an S-invariant irreducible character of G. Then there is a unique constituent \( \lambda \in \text{Irr}(G^S) \) of \( \text{res}^G_S(\chi) \) whose multiplicity is not divisible by p. Moreover, this multiplicity is congruent to 1 or -1 modulo p.

6.2 Theorem (cf. [Is76, 13.1]) There is a unique bijection \( \pi^{G,S}: \text{Irr}(G)^S \to \text{Irr}(G^S) \), called the Glauberman correspondence, satisfying the following conditions

(i) If \( T \leq S \), then \( \pi^{G,T}(\text{Irr}(G)^T) \subseteq \text{Irr}(G^T)^S \).

(ii) In the situation of (i), \( \pi^{G,S} = \pi^{G,T} \circ \pi^{G,T} \).

(iii) If \( S \) is a p-group and \( \chi \in \text{Irr}(G)^S \), then \( \pi^{G,S}(\chi) \) is the unique irreducible constituent \( \lambda \) of \( \text{res}^G_S(\chi) \) with \( p \nmid (\lambda, \text{res}^G_S(\chi))_{G^S} \) according to Lemma 6.1.

For \( \chi \in \text{Irr}(G)^S \), the irreducible character \( \lambda := \pi^{G,S}(\chi) \in \text{Irr}(G^S) \) is called the Glauberman correspondent of \( \chi \), and \( \chi \) is called the Glauberman correspondent of \( \lambda \). Note that it follows directly from Theorem 6.2 that \( \pi^{G,S}(\varphi) = \text{res}^G_S(\varphi) \) for \( \varphi \in \text{Irr}(G)^S \) with \( \varphi(1) = 1 \).

For \( \chi \in \text{Irr}(G) \) we will denote by \( T_rS(\chi) \in R(G)^S \) the S-orbit sum of \( \chi \), i.e. if \( T := \text{stabs}(\chi) \), then \( T_rS(\chi) = \sum_{s \in S/T} s^*\chi \). Note that the set \( T_rS(\chi) \), where \( \chi \) runs through a set of representatives of the S-orbits \( S \setminus \text{Irr}(G) \) of \( \text{Irr}(G) \), is a Z-basis of \( R(G)^S \). Moreover, we write \( R(G)^S_\leq S \) for the Z-span of the elements \( T_rS(\chi) \in R(G)^S \) with \( \text{stabs}(\chi) < S \). Then we have \( R(G)^S = Z \cdot \text{Irr}(G)^S \oplus R(G)^S_\leq S \).

We will need the following consequence of Lemma 6.1 and Theorem 6.2 later.

6.3 Proposition Let S be a p-group.

(i) For \( \chi \in \text{Irr}(G)^S \) with \( \text{stabs}(\chi) < S \) we have

\[ \text{res}^G_S(T_rS(\chi)) \equiv 0 \pmod{pR(G^S)}. \]

(ii) For \( \lambda \in \text{Irr}(G^S) \) we have \( \text{ind}^G_{G^S}(\lambda) \in R(G)^S \), and there is a unique \( \chi \in \text{Irr}(G)^S \) with \( p \nmid (\chi, \text{ind}^G_{G^S}(\lambda))_{G^S} \); Moreover, \( (\chi, \text{ind}^G_{G^S}(\lambda))_G \equiv \pm 1 \pmod{p} \) for this \( \chi \).
(iii) For \( \lambda \in R(G^S) \) we have

\[
\text{res}^G_{G^S}(\text{ind}^G_{G^S}(\lambda)) \equiv \lambda \pmod{pR(G^S)}.
\]

(iv) For \( \chi \in R(G)^S \) we have

\[
\text{ind}^G_{G^S}(\text{res}^G_{G^S}(\chi)) \equiv \chi \pmod{pR(G)^S + R(G)^S_{<S}}.
\]

**Proof**  

(i) Let \( T := \text{stab}_S(\chi) \), then

\[
\text{res}^G_{G^S}(Tr_S(\chi)) = \sum_{s \in S/T} \text{res}^G_{G^S}(s\chi) = \sum_{s \in S/T} s(\text{res}^G_{G^S}(\chi)) = |S/T| \cdot \text{res}^G_{G^S}(\chi).
\]

(ii) Let \( s \in S \), then \( s(\text{ind}^G_{G^S}(\chi)) = \text{ind}^G_{G^S}(s\chi) = \text{ind}^G_{G^S}(\chi) \). Furthermore, since \( \pi^{G,S} : \text{Irr}(G)^S \rightarrow \text{Irr}(G^S) \) is surjective, Theorem 6.2 implies that there is some \( \chi \in \text{Irr}(G)^S \) with \( p \nmid \lambda, \text{res}^G_{G^S}(\chi) = (\text{ind}^G_{G^S}(\lambda), \chi)_G \). Moreover, since \( \pi^{G,S} \) is injective, Theorem 6.2 (iii) implies that \( \chi \) is unique.

(iii) It suffices to prove the congruence for \( \lambda \in \text{Irr}(G^S) \). If \( \lambda = \pi^{G,S}(\chi) \) for \( \chi \in \text{Irr}(G)^S \), then by part (ii) we have

\[
\text{ind}^G_{G^S}(\lambda) \equiv \pm \chi \pmod{pR(G)^S + R(G)^S_{<S}}.
\]

Hence \( \text{res}^G_{G^S}(\text{ind}^G_{G^S}(\lambda)) \equiv \lambda \pmod{pR(G^S)} \) by part (i).

(iv) It suffices to prove the congruence for \( Tr_S(\chi), \chi \in \text{Irr}(G) \). If \( \text{stab}_S(\chi) < S \), then (i) implies \( \text{ind}^G_{G^S}(\text{res}^G_{G^S}(Tr_S(\chi))) \in pR(G)^S + R(G)^S_{<S} \), and we also have \( Tr_S(\chi) \in pR(G)^S + R(G)^S_{<S} \). If \( \text{stab}_S(\chi) = S \) and \( \lambda \in \text{Irr}(G^S) \) is the Glauberman correspondent of \( \chi \), then \( \text{res}^G_{G^S}(\chi) \equiv \pm \lambda \pmod{pR(G^S)} \) and \( \text{ind}^G_{G^S}(\pm \lambda) \equiv \chi \pmod{pR(G)^S + R(G)^S_{<S}} \) by part (ii). This yields the result for \( \chi \).

**6.4 Definition**  

We define the map

\[
gl^{G,S} : R(G)^S \xrightarrow{a_G} R^ab(G)^S \xrightarrow{g^{G,S} \cdot} R^ab(G^S) \xrightarrow{b_G} R(G^S),
\]

where \( gl^{G,S}_+ \) is defined for a basis element \( \sum_{s \in S/T} [sH, s\varphi]_G \), \( (H, \varphi) \in \mathcal{M}(G) \) with \( T := \text{stab}_S([H, \varphi]_G) = \text{stab}_S(H, \varphi) \), by

\[
gl^{G,S}_+ \left( \sum_{s \in S/T} [sH, s\varphi]_G \right) := \begin{cases} 
[G^S \cap H, \text{res}^H_{G^S \cap H}(\varphi)]_{G^S}, & \text{if } T = S, \\
0, & \text{if } T < S.
\end{cases}
\]

If \( S = T \), we can write \( H^S \) instead of \( G^S \cap H \), since \( S \) acts on \( H \). Note that \( gl^{G,S}_+ \) is well-defined by Lemma 6.1.

**6.5 Proposition**  

For \( \varphi \in \text{Irr}(G) \) with \( \varphi(1) = 1 \) we have

\[
gl^{G,S}_+(Tr_S(\varphi)) = \begin{cases} 
\text{res}^G_{G^S}(\varphi), & \text{if } \text{stab}_S(\varphi) = S, \\
0, & \text{if } \text{stab}_S(\varphi) < S.
\end{cases}
\]
Proof  This is immediate from Proposition III.1.2 (v) which states \( a_G(\varphi) = [G, \varphi]_G \). \( \square \)

6.6 Proposition  Assume that \( S \) is a \( p \)-group.
(i) For \( \chi \in R(G)^S \) we have
\[
gl^{G,S}(\chi) \equiv \res^{G,S}_S(\chi) \pmod{pR(G^S)}.
\]
(ii) For \( \chi \in R(G)^S \leq S \) we have
\[
gl^{G,S}(\chi) \equiv 0 \pmod{pR(G^S)}.
\]

Proof  (i) Since \( \gl^{G,S} = b_{GS} \circ \gl^{G,S}_+ \circ a_G \) and \( \res^{G,S}_S = b_{GS} \circ \res^{G}_+ \circ a_G \), it suffices to show that
\[
b_{GS} \circ \gl^{G,S}_+ \equiv b_{GS} \circ \res^{G}_S.
\]
Let \((H, \varphi) \in \mathcal{M}(G)\) with \( T := \stab_S([H, \varphi]_G) = \stab_S(H, \varphi)\). If \( T < S \), then
\[
\gl^{G,S}_+(\sum_{s \in S/T} [^sH, ^s\varphi]_G) = 0
\]
so that the congruence holds in this case. From now on we assume that \((H, \varphi)\) is \( S \)-stable. Then we have by Proposition 6.3 (iii):
\[
(b_{GS} \circ \gl^{G,S})([H, \varphi]_G) \equiv \left( \res^{G}_S \circ \ind^{G}_S \circ b_{GS} \right) \left( [H^S, \res^{H^S}_{H^S}(\varphi)]_G \right) \pmod{pR(G^S)}
\]
\[
= \left( \res^{G}_S \circ \ind^{G}_S \circ \ind^{H^S}_{H^S} \right) \left( \res^{H^S}_{H^S}(\varphi) \right)
\]
\[
= \left( \res^{G}_S \circ \ind^{G}_S \right) \left( \ind^{H^S}_{H^S}(\res^{H^S}_{H^S}(\varphi)) \right).
\]
By Proposition 6.3 (iv) we have
\[
\ind^{H^S}_{H^S}(\res^{H^S}_{H^S}(\varphi)) \equiv \varphi \pmod{pR(H) + R(H)^S}.
\]
Since \( \ind^{G}_H(R(H)^S \leq S) \subseteq pR(G)^S + R(G)^S \), we obtain from Proposition 6.3 (i) that
\[
(b_{GS} \circ \gl^{G,S})([H, \varphi]_G) \equiv \res^{G}_S(\ind^{G}_H(\varphi)) \pmod{pR(G^S)}.
\]
On the other hand
\[
(b_{GS} \circ \res^{G}_S)([H, \varphi]_G) = (\res^{G}_S \circ b_{GS})([H, \varphi]_G) = \res^{G}_S(\ind^{G}_H(\varphi)).
\]
(ii) This follows from part (i) and Proposition 6.3 (i). \( \square \)

6.7 Remark  (i) Proposition 6.6 (i) together with Theorem 6.2 shows that our map \( \gl^{G,S} \) can be used as a definition for the Glauberman correspondence in the case, where \( S \) is a \( p \)-group. But note, that \( \gl^{G,S} \) is defined for arbitrary \( S \) and \( G \) with relatively prime order. It might be that the Glauberman correspondence can be defined directly from \( \gl^{G,S} \) without going step by step through a composition
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series of $S$. But we have no good evidence for this at the moment. Interesting examples tend to be too big even for computers. And without interesting examples we don’t even know what we would like to prove about $gl^{G,S}$ in the general case. The few examples we know, still allow that $gl^{G,S}(\chi)$ is a multiple of a single $\lambda \in \text{Irr}(G^S)$ for $\chi \in \text{Irr}(G)^S$, but we don’t believe in this property in general.

(ii) Note that in the case, where $S$ is not solvable, $|S|$ is even by a theorem of Feit and Thompson. Hence $G$ is odd. Generally, in the case, where $G$ is odd (and hence solvable), Isaacs defined in ([Is73]) a bijection $\pi^{G,S} : \text{Irr}(G)^S \rightarrow \text{Irr}(G^S)$ with coincides with the Glaubermann correspondence, whenever both are defined, which was proved by Wolf in [Wo78]. Isaacs’ correspondence is defined by totally different methods as Glauberman’s. It would be very nice if the map $gl^{G,S}$ allowed a unified construction of these two correspondences. But again we don’t have any serious results relating Isaacs’ correspondence to the map $gl^{G,S}$.

(iii) Note that $gl^{G,S}$ can be defined similarly for the various other Grothendieck goups we considered in Chapter III. But we didn’t examine their properties so far.

6.8 Example We adopt the notation of Section III.7. Let $G$ be the extraspecial $p$-group of order $p^{2n+1}$ and exponent $p$ for an odd prime $p$. In Proposition III.7.2 we have shown that, for $\chi \in \text{Irr}(G)$ with $\chi(1) = p^n$, we have

$$a_G(\chi) = \sum_{d=0}^{n} \sum_{\substack{Z \leq H \text{ abelian} \\ |H/Z| = p^{n-d}}} (-1)^d p^{d(d-1)} [H, \phi]_G,$$

where $\phi$ is an arbitrary extension of $\lambda \in \text{Irr}(Z)$, and $\lambda$ is given by $\text{res}_Z^G(\chi) = p^n \cdot \lambda$. The automorphism group of $G$ has a normal subgroup $\widetilde{Sp}_{2n}$ which is an extension of the symplectic group $Sp_{2n}(F_p)$ by the normal subgroup $\text{Inn}(G) \cong G/Z \cong F_p^{2n}$. The group $\widetilde{Sp}_{2n}$ acts trivially on $Z$ and therefore stabilizes the irreducible characters $\chi \in \text{Irr}(G)$ of degree $p^n$. Let $S$ be a $p'$-subgroup of $\widetilde{Sp}_{2n}$. A basis element $[H, \phi]_G$ is $S$-stable if an only if $H$ is $S$-stable, since $\phi$ is again an extension of $\lambda \in \text{Irr}(Z)$, and therefore $G$-conjugate to $\phi$, cf. III.7.1. Since $gl^{G,S}_+$ is trivial on $S$-orbit sums of $[H, \phi]_G$ with $\text{stab}_S([H, \phi]_G) < S$, we obtain

$$gl^{G,S}(\chi) = \sum_{d=0}^{n} \sum_{\substack{Z \leq H \text{ abelian} \\ H \text{ $S$-invariant} \\ |H/Z| = p^{n-d}}} (-1)^d p^{d(d-1)} \text{ind}^{G^S}_{H^S}(\phi),$$

where $\phi \in \text{Irr}(H^S)$ is an extension of $\lambda \in \text{Irr}(Z)$, $\text{res}_Z^G(\chi) = p^n \cdot \lambda$.

In the special case where $G^S = Z$, we obtain

$$gl^{G,S}(\chi) = \left( \sum_{d=0}^{n} \sum_{\substack{Z \leq H \text{ abelian} \\ H \text{ $S$-invariant} \\ |H/Z| = p^{n-d}}} (-1)^d p^{d(d-1)} \right) \cdot \lambda.$$

If we specialize further, and assume that $Z$ is the only $S$-stable abelian subgroup above $Z$, then we obtain

$$gl^{G,S}(\chi) = (-1)^n p^{n(n-1)} \lambda.$$
If \( G^S = Z \) and all abelian subgroups of \( G \) containing \( Z \) are \( S \)-invariant, we obtain
\[
gl^{G,S}(\chi) = p^n \lambda.
\]
In fact,
\[
\sum_{d=0}^{n} (-1)^d p^{d(d-1)} \# \{ Z \leq H \text{ abelian } | |H/Z| = p^{n-d} \} =
\]
\[
\sum_{(H,\varphi) \in G \setminus M(G)} \alpha_{(H,\varphi)}^G(\chi) = \chi(1) = p^n
\]
by Proposition III.1.2 (x), where \( \alpha_{(H,\varphi)}^G(\chi) \in \mathbb{Z} \) is given as in Equation (3.1).

### 4.7 Projectification

Throughout this section \( G \) denotes a finite group, \( K \) denotes a field of characteristic zero containing all \( |G| \)-th roots of unity, and \( F \) denotes an algebraically closed field of characteristic \( p > 0 \). We will define morphisms (see III.1, III.2 and III.3 for the notation)
\[
a: \mathbb{Q} \otimes R_K \longrightarrow \mathbb{Q} \otimes (R_{K,p}^{ab})^+, \quad a: \mathbb{Q} \otimes R_F \longrightarrow \mathbb{Q} \otimes P_{F,+}^{ab},
\]
such that
\[
pr: \mathbb{Q} \otimes a \longrightarrow \mathbb{Q} \otimes (R_{K,p}^{ab})^+ \longrightarrow \mathbb{Q} \otimes R_{K,p}
\]
is the multiplication with the characteristic function on \( p' \)-elements, and
\[
pr: \mathbb{Q} \otimes R_F \longrightarrow \mathbb{Q} \otimes P_{F,+}^{ab} \longrightarrow \mathbb{Q} \otimes P_F
\]
is the inverse of the Cartan morphism \( c: P_F \to R_F \).

#### 7.1 Recall from Section III.3 that
\[
R_{K,p}^{ab}(H) = \begin{cases} 
R_{K,p}^{ab}(H), & \text{if } H \text{ is a } p'\text{-group,} \\
0, & \text{otherwise,}
\end{cases}
\]
for \( H \leq G \). We can regard \( R_{K,p}^{ab} \subseteq R_K \) as a \( \mathbb{Z} \)-restriction subfunctor on \( G \), and define \( p \in \mathbb{Z} - \text{Con}(G)(R_K, R_{K,p}^{ab}) \) for \( H \leq G \) and \( \chi \in \text{Irr}_K(H) \) by
\[
p_H(\chi) := \begin{cases} 
\chi, & \text{if } H \text{ is a } p'\text{-group and } \chi(1) = 1,
0, & \text{otherwise.}
\end{cases}
\]
Note that \( R_K, R_{K,p}^{ab}, \) and \( B \), where \( B(H) := \text{Hom}(H, K^\times) \) for a \( p' \)-subgroup \( H \leq G \) and \( B(H) := \emptyset \) otherwise, satisfy Hypothesis II.3.1 so that we may define
\[
a := a^{R_K, R_{K,p}^{ab}} \in \mathbb{Q} - \text{Res}(G)(\mathbb{Q} \otimes R_K, \mathbb{Q} \otimes (R_{K,p}^{ab})^+).
\]
Furthermore, let \( pr \in \mathbb{Q} - \text{Res}(G)(\mathbb{Q} \otimes R_K, \mathbb{Q} \otimes R_{K,p}) \) be defined as the composition
\[
pr: \mathbb{Q} \otimes R_K \longrightarrow \mathbb{Q} \otimes (R_{K,p}^{ab})^+ \longrightarrow \mathbb{Q} \otimes R_{K,p}.
\]
4.7. PROJECTIFICATION

We call \( pr \) the \textit{projectification morphism}. Note that \( R_K, R_{K,p}^{ab}, B \), and \( p \) satisfy condition \((*_{p'})\) of Theorem II.4.5 by the same argument as in the proof of Theorem III.1.2.

7.2 Proposition  Assume the notation of 7.1. The morphism \( pr : \mathbb{Q} \otimes R_K \to \mathbb{Q} \otimes R_{K,p} \) is given by multiplication with the characteristic function on \( p' \)-elements, i.e.

\[
(pr_H(\chi))(h) = \begin{cases} h, & \text{if } h \text{ is a } p' \text{-element,} \\ 0, & \text{otherwise} \end{cases}
\]

for \( H \leq G, \chi \in R_K(H) \). Moreover we have the explicit formula

\[
pr_H(\chi) = \sum_{\sigma = (H_0, \varphi_0) \ldots (H_n, \varphi_n) \in H^\Gamma(M(H))} (-1)^n \times \left| \frac{\left( \left( N_H(\sigma) / H_0 \right)_{p'} \right)}{|N_H(\sigma) / H_0|} \right| \left( \text{res}^H_{H_n}(\chi) / \chi_{H_n} \right)_{\text{ind}^H_{H_0}(\varphi_0)}
\]

for \( H \leq G \) and \( \chi \in R_K(H) \), where \( M(H) \) is the poset of monomial pairs associated to \( B \) as in 7.1, cf. Definition II.3.2. Furthermore, \( |H|_p \cdot a_H(\chi) \) is integral for \( H \leq G \) and \( \chi \in R_K(H) \).

Proof Multiplication by the characteristic function on \( p' \)-elements induces a morphism \( pr' : K \otimes R_K \to K \otimes R_{K,p} \) of \( K \)-restriction functors on \( G \), and we may also extend \( pr \) to \( pr : K \otimes R_K \to K \otimes R_{K,p} \). In order to prove that \( pr' \) and \( pr \) coincide it suffices to show \( pr'_H = pr_H \) for cyclic \( p' \)-subgroups of \( G \). But in this case it is obvious, since, for \( \varphi \in \text{Irr}_K(H) \), we have \( pr'_H(\varphi) = \varphi \), and \( pr_H(\varphi) = b_H([H, \varphi])_G \), because \( a_H(\varphi) = [H, \varphi]_G \) by Lemma II.3.7. The explicit formula for \( pr_H, H \leq G, \) follows from Theorem II.4.5, since \( R_K, R_{K,p}^{ab}, B \), and \( p \) satisfy condition \((*_{p'})\). The integrality of \( |H|_p \cdot a_H(\chi) \) for \( H \leq G \) and \( \chi \in R_K(H) \) follows from Corollary 4.7.

For \( \chi \in \text{Irr}_K(G) \) let the \textit{p-defect} \( d_p(\chi) \in \mathbb{N}_0 \) be defined by \( \chi(1)_p \cdot p^{d_p(\chi)} = |G|_p \).

In [BF59] Brauer and Feit prove the following result about the morphism \( pr \).

7.3 Proposition  Assume the notation of 7.1. If \( \chi \in \text{Irr}_K(G) \), then

\[
p^{d_p(\chi)} \cdot pr_G(\chi) \in R_{K,p}(G) \quad \text{and} \quad p^{d_p(\chi) - 1} \cdot pr_G(\chi) \notin R_{K,p}(G).
\]

\( \square \)

We obtain a similar result for \( a_G \).

7.4 Proposition  Assume the notation of 7.1.

(i) For \( \chi \in R_{K,p}(G) \) we have \( a_G(\chi) \in (R_{K,p}^{ab})_+(G) \), i.e. \( a_G(\chi) \) is integral.

(ii) For \( \chi \in \text{Irr}_K(G) \) we have

\[
p^{d_p(\chi)} \cdot a_G(\chi) \in (R_{K,p}^{ab})_+(G) \quad \text{and} \quad p^{d_p(\chi) - 1} \cdot a_G(\chi) \notin (R_{K,p}^{ab})_+(G).
\]
CHAPTER 4. APPLICATIONS

Proof (i) Let \( p' \in \mathbb{Z} - \text{Con}(G)(R_{K,p}, R_{ab}^{K,p}) \) denote the morphism from Example III.3.4 which led to an integral canonical induction formula

\[ a' \in \mathbb{Z} - \text{Res}(G)(R_{K,p}, (R_{ab}^{K,p})^+). \]

Then we have a commutative diagram

\[
\begin{array}{ccc}
R_K & \xrightarrow{p} & R_{ab}^{K,p} \\
\cup & & \\ R_{K,p} & \xrightarrow{p'} & R_{ab}^{K,p}
\end{array}
\]

Hence, we obtain a commutative diagram

\[
\begin{array}{ccc}
\mathbb{Q} \otimes R_K & \xrightarrow{a} & \mathbb{Q} \otimes (R_{ab}^{K,p})^+ \\
\cup & & \\
\mathbb{Q} \otimes R_{K,p} & \xrightarrow{a'} & \mathbb{Q} \otimes (R_{ab}^{K,p})^+
\end{array}
\]

by Proposition II.3.11. Since \( a' \) is integral, the result follows.

(ii) Let \( \chi \in R_K(G) \) with \( 0 = pr_G(\chi) = (b_G \circ a_G)(\chi) \). Then \( a_G(\chi) = 0 \), since \( \text{res}_H^G(\chi) = 0 \) for all \( p' \)-subgroups \( H \leq G \), and since \( a_G \) is determined by Diagram (2.2). Therefore, \( b_G \) is injective on \( a_G(\mathbb{Q} \otimes R_K(G)) \). Together with \( pr \circ pr = pr \) this implies

\[ a_G \circ b_G \circ a_G = a_G. \]

Now let \( \chi \in \text{Irr}_K(G) \). We already know from Proposition 7.2 that \( |G|_p \cdot a_G(\chi) \in (R_{ab}^{K,p})^+(G) \). For \( n \in \mathbb{N}_0 \) we have

\[ p^n \cdot a_G(\chi) = a_G(p^n \cdot pr_G(\chi)) \]

by the above equation. Hence, if \( p^n \cdot pr_G(\chi) \) is integral, so is \( p^n \cdot a_G(\chi) \) by part (i). Conversely, if \( p^n \cdot a_G(\chi) \) is integral, so is \( b_G(p^n \cdot a_G(\chi)) = p^n \cdot pr_G(\chi) \). Therefore, Proposition 7.3 yields the result.

Next we turn to the case of positive characteristic \( p \).

7.5 Recall from Section II.3 that

\[ P_{ab}^F(H) = \begin{cases} R_{ab}^F(H), & \text{if } H \text{ is a } p'\text{-group}, \\ 0, & \text{otherwise}, \end{cases} \]

for \( H \leq G \). We can regard \( P_{ab}^F \subseteq R_F \) as a \( \mathbb{Z} \)-restriction subfunctor on \( G \), and define \( p \in \mathbb{Z} - \text{Con}(G)(R_F, P_{ab}^F) \) for \( H \leq G \) and irreducible \( S \in FH - \text{mod} \) by

\[ p_H([S]) := \begin{cases} [S], & \text{if } H \text{ is a } p'\text{-group and } \dim_FS = 1, \\ 0, & \text{otherwise}. \end{cases} \]

Note that \( R_F, P_{ab}^F, \) and \( B \), where \( B(H) := \text{Hom}(H, F^+) \) for a \( p' \)-subgroup \( H \leq G \) and \( B(H) := \emptyset \) otherwise, satisfy Hypothesis II.3.1 so that we can define

\[ a := a^{R_F \cdot P_{ab}^F} \in \mathbb{Q} - \text{Res}(G)(\mathbb{Q} \otimes R_F, \mathbb{Q} \otimes P_{ab}^F^+). \]
Furthermore, we define \( pr \in \mathbb{Q} - \text{Res}(G)(\mathbb{Q} \otimes R_F, \mathbb{Q} \otimes P_F) \) as the composition
\[
pr : \mathbb{Q} \otimes R_F \xrightarrow{a} \mathbb{Q} \otimes P_{F+}^{ab} \xrightarrow{b^F_{P},r_{F}^{ab}} \mathbb{Q} \otimes P_F.
\]

We call \( pr \) the projectification morphism. Note that \( R_F, P_{F}^{ab}, B, \) and \( p \) satisfy condition \( (*_p') \) of Theorem II.4.5 by the same argument as in the proof of Theorem III.3.2. Hence, we have \( |H|_{|p} \cdot a_H([V]) \in P_{F+}^{ab}(H) \) for \( H \leq G \) and \( S \in FH-\text{mod} \) by Corollary II.4.7.

7.6 Proposition Assume the notation of 7.5. Let \( V \in FH-\text{mod} \) be projective. Then \( a_G([V]) \) is integral, i.e. \( a_G([V]) \in P_{F+}^{ab}(G) \). In particular, \( pr_G([V]) \) is integral, i.e. \( pr_G([V]) \in P_F(G) \).

Proof Let \( p' \in \mathbb{Z} - \text{Con}(G)(P_F, P_{F}^{ab}) \) denote the morphism of Example III.3.1 which led to an integral canonical induction formula \( \alpha' \in \mathbb{Z} - \text{Res}(G)(P_F, P_{F+}^{ab}) \). Then we have a commutative diagram
\[
\begin{array}{ccc}
P_F & \xrightarrow{c} & R_F \\
p' \downarrow & & \downarrow p \\
P_{F}^{ab} & \xrightarrow{id} & P_{F}^{ab},
\end{array}
\]
where \( c \) denotes the Cartan morphism defined by \( c_H([V]) = [V] \) for \( H \leq G \) and a projective \( FH \)-module \( V \). By Proposition II.3.11 we obtain a commutative diagram
\[
\begin{array}{ccc}
\mathbb{Q} \otimes P_F & \xrightarrow{c} & \mathbb{Q} \otimes R_F \\
a' \downarrow & & \downarrow a \\
\mathbb{Q} \otimes P_{F+}^{ab} & \xrightarrow{id} & \mathbb{Q} \otimes P_{F+}^{ab}.
\end{array}
\]
Since \( a' \) is integral, the result follows.

7.7 Proposition Keep the notation of 7.5, and let \( c \in \mathbb{Z} - \text{Mack}(G)(P_F, R_F) \) be the Cartan morphism, which is given by \( c_H([V]) = [V] \) for \( H \leq G \) and a projective \( FH \)-module \( V \). Then we have \( pr \circ c = \text{id}_{\mathbb{Q} \otimes P_F} \) and \( c \circ pr = \text{id}_{\mathbb{Q} \otimes R_F} \). Moreover, we have the explicit formula
\[
pr_H([V]) = \sum_{\sigma = ((H_0,\phi_0)<...<(H_n,\phi_n)) \in H/\Gamma(M(H))} (-1)^n \times 
\left|\frac{(N_H(\sigma)/H_0)_{p'}}{N_H(\sigma)/H_0}\right| m_{\phi_n}(\text{res}_{H_n}^H([V])) \text{ind}_{H_0}^H([F_{\phi_0}])
\]
for \( H \leq G \) and \( V \in FH-\text{mod} \), where \( M(H) \) is the poset of monomial pairs associated to \( B \) as in 7.5, cf. Definition II.3.2. Furthermore, \( |H|_{|p} \cdot a_H(\chi) \in P_{F+}^{ab}(H) \) and \( |H|_{|p} \cdot pr_H([V]) \in P_F(H) \) for \( H \leq G \) and \( V \in FH-\text{mod} \).

Proof Since the coprimordial subgroups for \( R_F \) and \( P_F \) are precisely the \( p' \)-subgroups of \( G \), it suffices to show that \( pr_H(c_H([F_{\phi}])) = [F_{\phi}] \) and \( c_H(pr_H([F_{\phi}])) = [F_{\phi}] \) for cyclic \( p' \)-subgroups \( H \) and \( \phi \in \text{Hom}(H, F^\times) \). From Lemma II.3.7 we
have \( a_H([F\varphi]) = [H,\varphi]_H \), and that implies \( pr_H([F\varphi]) = [F\varphi] \). Moreover, we have \( c_H([F\varphi]) = [F\varphi] \), and the above equations follow.

The explicit formula follows from Theorem II.4.5, since condition \((*)_{\varphi'}\) is satisfied for \( R_F, P_F^{ab}, B, \) and \( p \).

For \( H \leq G \) and \( V \in FH-\text{mod} \), the element \( |H|_p \cdot a_H([V]) \) is integral by Corollary II.4.7, as already stated in 7.5.

7.8 Corollary The Cartan map \( c_G : P_F(G) \to R_F(G) \) is injective and

\[ R_F(G)/c_G(P_F(G)) \cong pr_G(R_F(G))/P_F(G). \]

In particular, \( R_F(G)/c_G(P_F(G)) \) is a finite \( p \)-group.

Proof The isomorphism \( R_F(G)/c_G(P_F(G)) \cong pr_G(R_F(G))/P_F(G) \) is induced by the map \( pr_G \). Since \( |G|_p \cdot pr_G([V]) \in P_F(G) \) for an \( FG \)-module \( V \), the latter group is a finite \( p \)-group.

7.9 Remark There are obvious generalizations of 7.1 and 7.5 to the case of relative projective modules instead of projective modules.
Chapter 5

Monomial Resolutions

In this chapter we are going to lift the canonical induction formula \(a_G: R(G) \rightarrow R^b(G)\) to a categorical level, by interpreting \(R(G)\) as the Grothendieck group of \(\mathbb{C}G\mod\), and \(R^b(G)\) as the Grothendieck group of a category \(\mathbb{C}G\mon\), consisting of monomial \(\mathbb{C}G\)-modules with an additional structure. The homomorphism \(a_G\) is induced by a functor from \(\mathbb{C}G\mod\) to the homotopy category of chain complexes in \(\mathbb{C}G\mon\), which we call the monomial resolution. The monomial resolution can be defined more generally for suitable commutative rings \(k\) instead of \(\mathbb{C}\). However these general monomial resolutions are not in relation to other canonical induction formulae from Chapter III. Following a suggestion of Rickard we interpret the monomial resolution in Section 3 as a projective resolution by embedding the whole setup into a bigger category.

Throughout this chapter \(G\) denotes a finite group, \(E\) its trivial subgroup, and \(k\) a noetherian integral domain. Let \(\hat{G}(k) = \text{Hom}(G, k^\times)\) be the multiplicative group of group homomorphisms from \(G\) to \(k^\times\). By \(kG\mod\) (resp. \(k\mod\)) we denote the full subcategory of \(kG\mod\) (resp. \(k\mod\)) whose objects are projective as \(k\)-modules (resp. projective \(k\)-modules).

Furthermore, let \(\mathcal{M}_k(G)\) denote the set of all pairs \((H, \varphi)\) where \(H \leq G\) and \(\varphi \in \hat{H}(k)\). \(\mathcal{M}_k(G)\) is a poset by setting \((K, \psi) \leq (H, \varphi)\) if and only if \(K \leq H\) and \(\psi = \varphi|_K\). The group \(G\) acts from the left on \(\mathcal{M}_k(G)\) by poset automorphisms: \(s(H, \varphi) := (sH, s\varphi)\) for \(s \in G\), \((H, \varphi) \in \mathcal{M}_k(G)\), where \(sH := sHs^{-1}\) and \(s\varphi \in \hat{sH}(k)\) is defined by \(s\varphi(g) := \varphi(s^{-1}gs)\) for \(g \in sH\). We denote the \(G\)-orbit of \((H, \varphi)\) by \([H, \varphi]_G\) and the set of \(G\)-orbits of \(\mathcal{M}_k(G)\) by \(G\backslash\mathcal{M}_k(G)\). \(\mathcal{R}_k(G) \subseteq \mathcal{M}_k(G)\) always denotes a set of representatives of the \(G\)-orbits \(G\backslash\mathcal{M}_k(G)\). The set \(G\backslash\mathcal{M}_k(G)\) is again a poset by the following definition: \([K, \psi]_G \leq [H, \varphi]_G\) if and only if there is some \(s \in G\) with \((K, \psi) \leq (sH, s\varphi)\). Note that \(\mathcal{R}_k(G)\) inherits the structure of a poset from \(\mathcal{M}_k(G)\), but in general it is not clear that \(\mathcal{R}_k(G)\) can be chosen such that \(\mathcal{R}_k(G)\) and \(G\backslash\mathcal{M}_k(G)\) are isomorphic posets. More precisely, the canonical map \(\mathcal{R}_k(G) \rightarrow G\backslash\mathcal{M}_k(G)\) is always a bijective map of posets, but not an isomorphism of posets, since there may be orbits \([K, \psi]_G \leq [H, \varphi]_G\) in \(G\backslash\mathcal{M}_k(G)\) such that their representatives in \(\mathcal{R}_k(G)\) are not in relation.

We will omit the argument \(k\) in \(\hat{G}(k), \mathcal{M}_k(G), \mathcal{R}_k(G)\), if \(k = \mathbb{C}\).
5.1 The category of $kG$-monomial modules

In this section we define the category $kG$-mon of $kG$-monomial modules. We study the morphisms in this category and determine its indecomposable objects. Moreover, we give various equivalent conditions for $M, N \in kG$-mon to be isomorphic. At the end this results in the determination of the Grothendieck group $\text{Pic}_k(G)$ of $kG$-mon, which coincides with the construction defined in I.2.2 applied to a $\mathbb{Z}$-restriction functor $R^\text{ab}_k$ on $G$.

1.1 Definition Let $V \in kG$-mod and $\varphi \in \hat{G}(k)$. $V$ is called $\varphi$-homogeneous if $gv = \varphi(g)v$ for all $v \in V$ and all $g \in G$. $V$ is called abelian if $V$ is the sum of $\varphi$-homogenous $kG$-submodules, where $\varphi$ runs through $\hat{G}(k)$.

For each pair $(H, \varphi) \in \mathcal{M}_k(G)$ we define the $(H, \varphi)$-homogeneous component $V^{(H, \varphi)}$ of $V$ by

$$V^{(H, \varphi)} = \{ v \in V \mid hv = \varphi(h)v \text{ for all } h \in H \}.$$ 

Obviously $V^{(H, \varphi)}$ is the largest $\varphi$-homogeneous $kH$-submodule of the $kH$-module $V$. If $V \in kG$-mod pr, then the sum of the submodules $V^{(G, \varphi)}$, $\varphi \in \hat{G}(k)$, is always a direct sum, and $V$ is abelian if and only if $V$ is the direct sum of its different $(G, \varphi)$-homogeneous components $V^{(G, \varphi)}$.

We denote the full subcategory of $kG$-mod which consists of abelian $kG$-modules by $kG$-mod ab, and we define $kG$-mod pr := $kG$-mod ab $\cap kG$-mod pr. For each $V \in kG$-mod the abelian part $\mathcal{P}(V)$ of $V$ is defined to be the largest abelian $kG$-submodule of $V$, i.e. $\mathcal{P}(V) = \sum_{\varphi \in \hat{G}(k)} V^{(G, \varphi)}$. This defines a functor $\mathcal{P} : kG$-mod $\rightarrow$ $kG$-mod ab, since for $V, W \in kG$-mod, each $f \in \text{Hom}_{kG}(V, W)$ maps $V^{(G, \varphi)}$ to $W^{(G, \varphi)}$ for all $\varphi \in \hat{G}(k)$.

$V$ is called a monomial $kG$-module, if $V$ is isomorphic to a direct sum of $kG$-modules $\text{ind}_{H_i}^G(V_i)$, where $H_i \leq G$ are arbitrary subgroups and $V_i \in kH_i$-mod pr. This extends the definition preceding Proposition III.4.1.

1.2 Remark The definition of $V^{(H, \varphi)}$ provides a $G$-equivariant $\mathcal{M}_k(G)$-filtration on each $V \in kG$-mod, i.e. we have:

(i) If $(K, \psi) \leq (H, \varphi) \in \mathcal{M}_k(G)$, then $V^{(H, \varphi)} \subseteq V^{(K, \psi)}$.

(ii) $V^{s(H, \varphi)} = sV^{(H, \varphi)}$ for all $(H, \varphi) \in \mathcal{M}_k(G)$, $s \in G$.

(iii) $V^{(E, 1)} = V$.

In view of (ii), $V^{(H, \varphi)}$ is a $kN_G(H, \varphi)$-module, where $N_G(H, \varphi) := \{ s \in G \mid s(H, \varphi) = (H, \varphi) \}$ denotes the stabilizer of $(H, \varphi)$ in $G$. The set $\{ V^{(H, \varphi)} \mid (H, \varphi) \in \mathcal{M}_k(G) \}$ of submodules of $V$ is again a $G$-poset (ordered by inclusion). This $G$-poset was used as a main ingredient in [Bo94, Thm. 3.2(c)]. In Section 2 it will play an important role in the definition of a monomial resolution and in upper bounds for the monomial homological dimension of $V$ (cf. Theorem 2.14).

1.3 Definition A $kG$-monomial module $M$ is an object $M \in kG$-mod pr (hence finitely generated and projective over $k$) with the additional structure of a
fixed decomposition $M = M_1 \oplus \ldots \oplus M_m$ into $k$-submodules $M_1, \ldots, M_m$ (which are thus again finitely generated and projective over $k$), such that the two following conditions are satisfied:

(i) The $G$-action on $M$ makes $\{M_1, \ldots, M_m\}$ into a $G$-set, i.e. for each $s \in G$ and for each $i \in \{1, \ldots, m\}$ there is some $j \in \{1, \ldots, m\}$ such that $sM_i = M_j$.

(ii) If for each $i \in \{1, \ldots, m\}$ we denote the stabilizer $\{s \in G \mid sM_i = M_i\}$ of $M_i$ by $H_i$, then the $kH_i$-module $M_i$ is $\varphi$-homogeneous for some $\varphi_i \in \hat{H}_i(k)$.

Note that $\varphi_i \in \hat{H}_i(k)$ is uniquely determined, since $M_i$ is $k$-torsion free. We call $(H_i, \varphi_i)$ the stabilizing pair of $M_i$.

Let $M = M_1 \oplus \ldots \oplus M_m$ and $N = N_1 \oplus \ldots \oplus N_n$ be $kG$-monomial modules. A $kG$-monomial homomorphism from $M = M_1 \oplus \ldots \oplus M_m$ to $N = N_1 \oplus \ldots \oplus N_n$ is a $k$-linear combination of proper $kG$-monomial homomorphisms which are defined to be $kG$-linear maps $f: M \to N$ respecting the decomposition of $M$ and $N$, i.e. for each $i \in \{1, \ldots, m\}$ there is some $j \in \{1, \ldots, n\}$ such that $f(M_i) \subseteq N_j$.

The $kG$-monomial modules and their homomorphisms form an additive category which we denote by $kG\text{-}\text{mon}$. For $M, N \in kG\text{-}\text{mon}$ we denote the $k$-module of $kG$-monomial homomorphisms from $M$ to $N$ by $\text{mon}_{kG}(M, N)$. Note that $kG\text{-}\text{mon}$ is not an abelian category. In general kernels and cokernels don’t carry the structure of a $kG$-monomial module.

An object $M = M_1 \oplus \ldots \oplus M_m \in kG\text{-}\text{mon}$ is called $\varphi$-homogeneous for some $\varphi \in \hat{G}(k)$, if all the stabilizing pairs $(H_i, \varphi_i)$ are equal to $(G, \varphi)$, i.e. the underlying $kG$-module of $M$ is $\varphi$-homogeneous. $M$ is called abelian, if $H_i = G$ for all $i \in \{1, \ldots, m\}$ (the $\varphi_i$ need not necessarily to be the same). We denote the full subcategory of $kG\text{-}\text{mon}$ consisting of abelian $kG$-monomial modules by $kG\text{-}\text{mon}_{ab}$.

1.4 Remark (a) We will often write $M \in kG\text{-}\text{mon}$ if there is no need for giving the fixed decomposition which is part of an object in $kG\text{-}\text{mon}$.

(b) Note that the above definition is different from the definition given in [Bo89, I.1] and [Bo90, 1.4], where we worked over the field of complex numbers only and required that the components $M_i$ are one-dimensional, and where we only considered proper monomial homomorphisms. With this new definition we have more freedom with respect to the choice of the base ring $k$, $kG\text{-}\text{mon}$ becomes an additive category by allowing $k$-linear combinations of proper $kG$-monomial homomorphisms, and the Grothendieck group remains the same for $k = \mathbb{C}$ (cf. Subsection 1.22 and [Bo90, 1.4]). We also should mention that alternatively to the above definition one could impose on each $M_i$ the additional restriction to be free over $k$ of rank one. In the case where all objects in $k\text{-}\text{mod}^{\text{pr}}$ are even free (as for $k$ local or a principal ideal domain), the embedding of this alternative category into $kG\text{-}\text{mon}$ is actually an equivalence. In the meanwhile we learnt that W. F. Reynolds has introduced in [Re71, Sect. 3] the same category as we did in [Bo89] and [Bo90], but for different reasons.

(c) By definition, each $kG$-monomial module is a $kG$-module and each $kG$-monomial homomorphism is $kG$-linear, which gives rise to the forgetful functor

$$\mathcal{V}: kG\text{-}\text{mon} \to kG\text{-}\text{mod}^{\text{pr}}.$$
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Note that \(\mathcal{V}\) restricts to a functor on the abelian subcategories

\[\mathcal{V}: kG\text{-}\mathbf{mon}_{\text{ab}} \to kG\text{-}\mathbf{mod}^\text{pr}_{\text{ab}}.\]

(d) The two notions ‘monomial \(kG\text{-module}’ \text{ and ‘}kG\text{-monomial module}’ should not be confused. They are objects in different categories: The first one is an object in \(kG\text{-mon}\) and the second one in \(kG\text{-mod}\). Note also that non-isomorphic objects in \(kG\text{-mon}\) may become isomorphic after applying \(\mathcal{V}\), cf. Example 1.5. The confusing similarity of the two notions has the following justification: The monomial \(kG\text{-modules in }kG\text{-mod}\) form precisely the image of \(kG\text{-mon}\) under \(\mathcal{V}\). In fact, let \(M = M_1 \oplus \ldots \oplus M_m \in kG\text{-mon}\) and let us fix some \(M_i\), then the sum of all \(M_j\) in the \(G\)-orbit of \(M_i\) is a \(kG\text{-submodule of }M\) which is isomorphic to \(\text{ind}_{G/H}^G(M_i)\). Hence, we have

\[\mathcal{V}(M) \cong \bigoplus_{M_i \in \{M_1, \ldots, M_m\}/G} \text{ind}_{G/H}^G(M_i),\]

where the sum runs over a set of representatives of the \(G\)-orbits of \(\{M_1, \ldots, M_m\}\). Since \(M_i\) is a \(k\)-projective and an (even \(\varphi_i\)-homogeneous) abelian \(kH_i\text{-module by definition, }\mathcal{V}(M)\) is a monomial \(kG\text{-module.}\)

Conversely, let \(H \leq G\) and let \(U \in kH\text{-mod}^\text{pr}\) be \(\varphi\)-homogeneous for some \(\varphi \in \hat{H}(k)\). Then \(\text{ind}_{H}^G(U) = kG \otimes_{kH} U\) is the image under \(\mathcal{V}\) of the same underlying \(kG\text{-module with the decomposition }\oplus_{s \in G/H}(s \otimes_{kH} U),\) where \(s\) runs over a set of representatives of \(G/H\). Note that \(G\) permutes the summands \(s \otimes_{kH} U\) in the same way as it permutes the \(G\)-set \(G/H\), that the stabilizer of \(s \otimes_{kH} U\) is \(sH\) and that \(s \otimes_{kH} U\) is an \(\varphi\)-homogeneous \(kH\text{-module, i.e. the stabilizing pair of }s \otimes_{kH} U\) is \(s(H, \varphi)\). This construction can be extended to direct sums. Hence, the notion of a \(kG\text{-monomial module can be seen as the notion of a monomial }kG\text{-module together with a specified way of writing it as a sum of induced homogeneous modules, where both the subgroups }H\text{ we induce from and the }\varphi\text{-homogeneous modules (}\varphi \in \hat{H}(k)\text{)}\) which we induce are specified up to \(G\)-conjugacy.

1.5 Example  Let \(G\) be the quaternion group of order 8 and let \(k = \mathbb{C}\). Let \(V\) be an irreducible \(kG\text{-module of dimension 2. Then }V\) is a monomial \(kG\text{-module.}\) In fact, if \(H_1, H_2, H_3\) denote the three cyclic subgroups of order 4, and if \(\varphi_i\) denotes a faithful one-dimensional character of \(H_i\), \(i = 1, 2, 3\), then the character \(\chi\) of \(V\) equals \(\text{ind}_{H_i}^G(\varphi_i)\) and also \(\text{ind}_{H_i}^G(\varphi_i^{-1})\) for all \(i = 1, 2, 3\). In order to provide \(V\) with the structure of a \(kG\text{-monomial module we have to choose a suitable decomposition of }V.\)

Let us choose \(i \in \{1, 2, 3\}\), then \(\text{res}_{H_i}^G(\chi) = \varphi_i + \varphi_i^{-1}\), and this gives rise to a unique decomposition \(V = V_i' \oplus V_i''\) as \(kH_i\text{-modules, such that }V_i', \text{ resp. } V_i''\), are one-dimensional \(\varphi_i\text{-, resp. }\varphi_i^{-1}\text{-homogeneous }kH_i\text{-submodules. Together with this decomposition, }V\text{ is a }kG\text{-monomial module which we denote by }V_i.\) The stabilizing pair of \(V_i'\), resp. \(V_i''\), is \((H_i, \varphi_i), \text{ resp. } (H_i, \varphi_i^{-1})\).

It is easy to see that \(V_1, V_2, V_3\) are pairwise non-isomorphic as \(kG\text{-monomial modules. In Example 1.13 we will even see that }\text{mon}_{kG}(V_i, V_j) = 0\) for \(i \neq j.\)

1.6 Remark  The following is a list of constructions in the category \(kG\text{-mon}\) which can be lifted from well-known constructions in \(kG\text{-mod}, \text{i.e. these con-}
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structures commute with the forgetful functor $\mathcal{V}$. Let $M = M_1 \oplus \ldots \oplus M_m$ and $N = N_1 \oplus \ldots \oplus N_n$ be in $kG\text{-mon}$.

(a) Direct sum: This is the $kG$-module $M \oplus N$ with the decomposition $M_1 \oplus \ldots \oplus M_m \oplus N_1 \oplus \ldots \oplus N_n$. Together with the canonical inclusions $M \to M \oplus N$ and $N \to M \oplus N$ this is also a coproduct in the categorical sense.

(b) Tensor product: This is the $kG$-module $M \otimes N$ (unadorned tensor products are taken over $k$) with the decomposition $\bigoplus_{i,j} (M_i \otimes N_j)$.

(c) Homomorphisms: This is the $kG$-module $\text{Hom}_k(M, N)$ with the decomposition $\bigoplus_{i,j} \text{Hom}_k(M_i, N_j)$. Here the $G$-action is given by $(sf)(m) := sf(s^{-1}m)$, for $f \in \text{Hom}_k(M, N)$, $s \in G$, $m \in M$.

(d) Dual: Let $M^* := \text{Hom}_k(M, k)$ be endowed with the induced decomposition $\text{Hom}_k(G, k) = \bigoplus_i \text{Hom}_k(M_i, k)$. Considering the trivial $kG$-module $k$ with one-term decomposition as an object in $kG\text{-mon}$, this is a special case of (c).

(e) Restriction: Let $f : G' \to G$ be a homomorphism of finite groups. Then we have a functor $\text{res}_f : kG\text{-mon} \to kG'\text{-mon}$ given by $\text{res}_f(M) = M$ as a $k$-module and the old decomposition, where $g' \in G'$ acts on $\text{res}_f(M) = M$ via $f(g')$. If $f$ is the inclusion of a subgroup $H \leq G$, then we write $\text{res}^G_H(M)$ instead of $\text{res}_f(M)$.

(f) Induction: If $H \leq G$, then we have a functor $\text{ind}_H^G : kH\text{-mon} \to kG\text{-mon}$ given by the following construction. Let $L = L_1 \oplus \ldots \oplus L_l \in kH\text{-mon}$, then $\text{ind}_H^G L \in kG\text{-mon}$ is defined to be $kG \otimes_{kH} L$ as a $kG$-module together with the decomposition into $k$-submodules $s \otimes_{kH} L_i$, where $s$ runs through a set of representatives of $G/H$ and $i$ through $\{1, \ldots, l\}$.

(g) Note that the constructions (a)—(e) applied to abelian $kG$-monomial modules result again in abelian $kG$- (or $kG'$-) monomial modules.

(h) All the well-known canonical isomorphism in $kG\text{-mod}$, as for example $M \oplus N \cong N \oplus M$, $M \otimes N \cong N \otimes M$, $M \otimes (N \oplus P) \cong (M \otimes N) \oplus (M \otimes P)$, $\text{Hom}_k(M, N) \cong M^* \otimes N$, etc., hold also in $kG\text{-mon}$, since the underlying isomorphisms in $kG\text{-mod}$ are proper $kG$-monomial homomorphisms. Moreover we mention that the constructions (b)—(f) commute with taking direct sums, and (d) and (e) commute with taking tensor products. The following proposition lists three more canonical isomorphisms.

1.7 Proposition

(a) Frobenius reciprocity: Let $H \leq G$, then the functor $\text{res}_H^G : kG\text{-mon} \to kH\text{-mon}$ is right adjoint to $\text{ind}_H^G : kH\text{-mon} \to kG\text{-mon}$, i.e. for $M \in kH\text{-mon}$ and $N \in kG\text{-mon}$ there is a $k$-linear isomorphism

$$\text{mon}_{kG}(\text{ind}_H^G(M), N) \cong \text{mon}_{kH}(M, \text{res}_H^G(N))$$

which is natural in $M$ and $N$.
(b) Let $H \leq G$, $M \in kG\text{-}\mathbf{mon}$ and $N \in kH\text{-}\mathbf{mon}$. Then we have a canonical isomorphism in $kG\text{-}\mathbf{mon}$
\[ M \otimes \text{ind}^G_H(N) \cong \text{ind}^G_H(\text{res}^G_H(M) \otimes N). \]

(c) Let $U, H \leq G$ and let $M \in kH\text{-}\mathbf{mon}$. Then we have a canonical isomorphism in $kU\text{-}\mathbf{mon}$
\[ \text{res}^G_U \text{ind}^G_H(M) \cong \bigoplus_{s \in U \setminus G/H} \text{ind}^U_{U \cap sH} \text{res}^s_H(M), \]
where $sM$ denotes the $k^sH$-monomial module $\text{res}_f(M)$ with $f: sH \to H$, $g \mapsto s^{-1}gs$, and where $s$ runs through a set of representatives of the double cosets $U \setminus G/H$ in $G$.

**Proof** (a) Let $M \in kH\text{-}\mathbf{mon}$ and $N \in kG\text{-}\mathbf{mon}$, then it is easily checked that the well-known inverse $k$-linear isomorphisms
\[ \text{Hom}_{kG}(\text{ind}^G_H(M), N) \cong \text{Hom}_{kH}(M, \text{res}^G_H(N)), \]
\[ f \mapsto (m \mapsto f(1 \otimes_{kH} m)), \]
\[ (g \otimes_{kH} m \mapsto gf'(m)) \leftrightarrow f', \]
map proper $kG$- (resp. $kH$-) monomial homomorphisms to proper ones. Moreover, these isomorphisms are natural in $M$ and $N$.

(b) The well-known $kG$-linear inverse isomorphisms
\[ M \otimes (kG \otimes_{kH} N) \cong kG \otimes_{kH} (\text{res}^G_H(M) \otimes N), \]
\[ m \otimes (g \otimes_{kH} n) \mapsto g \otimes_{kH} (g^{-1}m \otimes n), \]
\[ gm \otimes (g \otimes_{kH} n) \leftrightarrow g \otimes_{kH} (m \otimes n), \]
are obviously proper $kG$-monomial homomorphisms.

(c) The well-known $kU$-linear inverse isomorphisms
\[ kG \otimes_{kH} M \cong \bigoplus_{s \in U \setminus G/H} kU \otimes_{k(U \cap sH)^s} sM \]
\[ g \otimes_{kH} m \overset{g=ush}{\mapsto} u \otimes_{k(U \cap sH)^s} hm \quad \in s\text{-component}, \]
\[ us \otimes_{kH} m \leftrightarrow u \otimes_{k(U \cap sH)^s} m \quad \in s\text{-component}, \]
are obviously proper $kG$-monomial homomorphisms. \qed

Similar to the $\mathcal{M}_k(G)$-filtration of any $V \in kG\text{-mod}$ we a have an $\mathcal{M}_k(G)$-filtration and also an $\mathcal{M}_k(G)$-grading on any $M \in kG\text{-mon}$ by the following definition.

1.8 Definition For $M = M_1 \oplus \ldots \oplus M_m \in kG\text{-mon}$ and $(H, \varphi) \in \mathcal{M}_k(G)$ we define the $(H, \varphi)$-homogeneous component $M(H, \varphi)$ of $M$ to be the (direct)
sum of those $M_i$ whose stabilizing pair $(H_i, \varphi_i)$ equals $(H, \varphi)$. More generally we define for $\mathcal{M} \subseteq \mathcal{M}_k(G)$ the following $k$-submodule of $M$:

$$M(\mathcal{M}) := \bigoplus_{(H, \varphi) \in \mathcal{M}} M(H, \varphi) = \bigoplus_{i \in \{1, \ldots, m\}} M_i.$$ 

If $\mathcal{M} = [H, \varphi]_G$ is the $G$-orbit of some $(H, \varphi) \in \mathcal{M}_k(G)$, then we call

$$M([H, \varphi]_G) = \bigoplus_{s \in G/N_G(H, \varphi)} M(\ast(H, \varphi)) = \bigoplus_{i \in \{1, \ldots, m\}} M_i$$

the $[H, \varphi]_G$-homogeneous component of $M$. The $kG$-module $M([H, \varphi]_G)$ is a $kG$-monomial module in its own right with the decomposition inherited from $M$.

For $(H, \varphi) \in \mathcal{M}_k(G)$ we define

$$\mathcal{M}_{\geq (H, \varphi)} := \{(H', \varphi') \in \mathcal{M}_k(G) \mid (H, \varphi) \leq (H', \varphi')\}$$

and

$$\mathcal{M}_{\geq [H, \varphi]_G} := \{(H', \varphi') \in \mathcal{M}_k(G) \mid [H, \varphi]_G \leq [H', \varphi']_G\},$$

and we set

$$M^{(H, \varphi)} := M(\mathcal{M}_{\geq (H, \varphi)}) \text{ and } M^{[H, \varphi]_G} := M(\mathcal{M}_{\geq [H, \varphi]_G}).$$

The following remarks are immediate consequences of the last definition.

**1.9 Remark** Let $M \in kG$-mon.

(a) $\mathcal{M} \subseteq \mathcal{M}' \subseteq \mathcal{M}_k(G) \implies M(\mathcal{M}) \subseteq M(\mathcal{M}')$.

(b) $(K, \psi) \leq (H, \varphi) \in \mathcal{M}_k(G) \implies M^{(H, \varphi)} \subseteq M^{(K, \psi)}$.

(c) $M = \bigoplus_{(H, \varphi) \in \mathcal{M}_k(G)} M(H, \varphi) = M^{(E, 1)}$.

(d) $M(\mathcal{M}) = sM(\mathcal{M})$ for all $\mathcal{M} \subseteq \mathcal{M}_k(G), s \in G$. In particular, $M(\mathcal{M})$ is a $kN_G(\mathcal{M})$-monomial module, where $N_G(\mathcal{M}) := \{s \in G \mid \mathcal{M} = s\mathcal{M}\}$. 

(e) $[K, \psi]_G \leq [H, \varphi]_G \in G \setminus \mathcal{M}_k(G) \implies M^{[H, \varphi]_G} \subseteq M^{[K, \psi]_G}$.

(f) $M = \bigoplus_{(H, \varphi) \in \mathcal{R}_k(G)} M([H, \varphi]_G) = M^{[E, 1]_G}$ as $kG$-monomial modules.

(g) For $(H, \varphi) \in \mathcal{M}_k(G)$ we have $M^{(H, \varphi)} = (\text{res}^G_H(M))(H, \varphi) = (\text{res}^G_H(M))^{(H, \varphi)}$.

(h) $M^{(H, \varphi)} \subseteq V(M)^{(H, \varphi)}$ for all $(H, \varphi) \in \mathcal{M}_k(G)$. But in general (cf. Remark 1.15) the other inclusion is false.

(i) Let $M \in kG$-mon$_{ab}$, then $M(G, \varphi) = M^{(G, \varphi)} = V(M)^{(G, \varphi)}$ for all $\varphi \in \hat{G}(k)$.  

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Let \( M \in kG\text{-mon} \) and \((H, \varphi) \in \mathcal{M}_k(G)\). If we consider \( M(H, \varphi) \) as a \( kN_G(H, \varphi)\)-monomial module by part (d), then \( \text{ind}^G_{N_G(H, \varphi)} (M(H, \varphi)) \) and \( M([H, \varphi]_G) \) are isomorphic as \( kG\)-monomial modules via the proper isomorphism \( s \otimes_{kN_G(H, \varphi)} m \mapsto sm \).

**1.10 Remark**  By the decomposition in 1.9 (e) each \( M \in kG\text{-mon} \) is an \( \mathcal{M}_k(G)\)-graded \( k \)-module, and by 1.9 (f) each \( M \in kG\text{-mon} \) is a \( G\backslash \mathcal{M}_k(G)\)-graded \( kG\)-monomial module. Each \( kG\)-monomial module is the direct sum (in \( kG\text{-mon} \)) of its \([H, \varphi]_G\)-homogeneous components. Note that \( M \in kG\text{-mon} \) is \( \varphi\)-homogeneous for \( \varphi \in \hat{G}(k) \) if and only if \( M = M([G, \varphi]_G) = M(G, \varphi) = M(G, \varphi) \), and note that \( M \) is abelian if and only if \( M = \bigoplus_{\varphi \in \hat{G}(k)} M(G, \varphi) \). Note also that if \( M \in kH\text{-mon} \) is \( \varphi\)-homogeneous for some \( \varphi \in \hat{H}(k) \), then \( \text{ind}^G_H M \) is \([H, \varphi]_G\)-homogeneous, i.e. \( \text{ind}^G_H M \) equals its \([H, \varphi]_G\)-homogeneous component.

In general, the \( \mathcal{M}_k(G)\)- (resp. \( G\backslash \mathcal{M}_k(G)\)-) grading is not preserved under \( kG\)-monomial homomorphisms. However, as the following proposition shows, the \( \mathcal{M}_k(G)\)- (resp. \( G\backslash \mathcal{M}_k(G)\)-) filtration on \( M \) given by the \( k \)-submodules \( M^{[H, \varphi]} \) (resp. \( kG\)-monomial submodules \( M^{[H, \varphi]_G} \)) is compatible with the notion of \( kG\)-monomial homomorphisms.

**1.11 Proposition**  Let \( M, N \in kG\text{-mon} \) and let \( f \in \text{Hom}_{kG}(M, N) \). Then the statements (a)—(c) are equivalent:

(a) \( f \in \text{mon}_G(M, N) \),
(b) \( f(M([H, \varphi])) \subseteq N([H, \varphi]) \) for all \((H, \varphi) \in \mathcal{M}_k(G)\).
(c) \( f(M(H, \varphi)) \subseteq N(H, \varphi) \) for all \((H, \varphi) \in \mathcal{M}_k(G)\).
(d) If (a)—(c) hold, then we have \( f(M([H, \varphi]_G)) \subseteq f(M^{[H, \varphi]_G}) \subseteq N^{[H, \varphi]_G} \) for all \([H, \varphi]_G \in G\backslash \mathcal{M}_k(G)\). In general (d) does not imply (a), (b) or (c).

**Proof**  Let \( M = M_1 \oplus \ldots \oplus M_m \) and \( N = N_1 \oplus \ldots \oplus N_m \) denote the respective decompositions and let \((H_i, \varphi_i) \in \mathcal{M}_k(G)\) be the stabilizing pair of \( M_i \), \( i \in \{1, \ldots, m\} \).

(a) \( \Rightarrow \) (b): Since \( N^{[H, \varphi]} \) is a \( k \)-submodule of \( N \), we may assume that \( f \) is proper. It suffices to show that for each \( i \in \{1, \ldots, m\} \) with \((H_i, \varphi_i) \leq (H, \varphi)\) we have \( f(M_i) \subseteq N^{[H, \varphi]} \). Obviously we may assume that \( f(M_i) \neq \{0\} \). Since \( f \) is proper, there is some \( j \in \{1, \ldots, n\} \) such that \( f(M_i) \subseteq N_j \). Let \((U, \mu) \in \mathcal{M}_k(G)\) be the stabilizing pair of \( N_j \). It suffices to show that \((H_i, \varphi_i) \leq (U, \mu)\). So let \( h \in H_i \) and \( v \in M_i \) with \( f(v) \neq 0 \). Then

\[
0 \neq hf(v) = f(hv) \in hN_j \cap N_j,
\]

which forces \( hN_j = N_j \), and hence \( h \in U \), showing that \( H_i \leq U \). Now we have for each \( h \in H_i \):

\[
\mu(h)f(v) = hf(v) = f(hv) = f(\varphi_i(h)v) = \varphi_i(h)f(v),
\]

and therefore \( \mu(h) = \varphi_i(h) \), hence \( \mu|_{H_i} = \varphi_i \).

(b) \( \Rightarrow \) (c): This is trivial, since \( M([H, \varphi]_G) \subseteq M^{[H, \varphi]} \).

(c) \( \Rightarrow \) (a): Since \( f \) is the sum of \( kG\)-linear maps which are zero on all but one of the \( G \)-orbits of the \( G \)-set \( \{M_1, \ldots, M_m\} \), we may assume that \( G \) acts transitively on
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\{M_1, \ldots, M_m\}. Let J \subseteq \{1, \ldots, n\} be the subset of those j such that \(N_j \subseteq N^{(H_1, \varphi_1)}\), and let \(f_1 : M_1 \to N\) be the restriction of f to \(M_1\). Then \(f_1\) is \(kH_1\)-linear and by the hypothesis in (c), \(f_1 = \sum_{j \in J} e_j f_1\), where \(e_j : N \to N_j \to N\) is the projection with respect to the decomposition \(N = N_1 \oplus \cdots \oplus N_n\). Each \(e_j, j \in J\), is \(kH_1\)-linear, hence each \(e_j f_1, j \in J\), is \(kH_1\)-linear. Since \(M \cong \text{ind}_{H_1}^G(M_1)\) as \(kG\)-modules, each \(e_j f_1\) extends uniquely to a \(kG\)-linear map \(f_j : M \to N\). By the same uniqueness property we have \(f = \sum_{j \in J} f_j\), since both of them extend \(f_1\). Finally, for any \(j \in J\), \(f_j\) is a proper \(kG\)-monomial homomorphism, since for each \(i \in \{1, \ldots, m\}\) there is some \(s \in G\) with \(M_i = s M_1\), and hence \(f_j(M_i) = f_j(s M_1) = s f_j(M_1) = s e_j f_1(M_1) \subseteq s N_j\).

(d) The first inclusion holds always, since \(M([H, \varphi]_G) \subseteq M^{[H, \varphi]}_G\). Using the obvious equation \(M^{[H, \varphi]}_G = \sum_{s \in G} M^{s[H, \varphi]}\) we obtain

\[
f(M^{[H, \varphi]}_G) = f(\sum_{s \in G} M^{s[H, \varphi]}) = \sum_{s \in G} f(M^{s[H, \varphi]}) \subseteq \sum_{s \in G} N^{s[H, \varphi]} = N^{[H, \varphi]_G}
\]

from the hypothesis in (b).

Finally we show the last assertion in (d) by giving an example. Let \(k = \mathbb{C}\) for simplicity, and let \(G\) be a finite group which has a non-normal subgroup \(H\). Let \(M \in kG\text{-mon}\) be defined by \(M := \text{ind}_H^G(k), \) where \(k \in kH\text{-mon}\) is the trivial \(kH\)-module with one-term decomposition. We claim that any \(f \in \text{Hom}_{kG}(M, M)\) satisfies the condition in (d). In fact, let \([K, \psi]_G \in G \setminus \mathcal{M}_k(G)\). If \([K, \psi]_G \not\subseteq [H, \varphi]_G\), then \(M^{[K, \psi]}_G = 0\), and if \([K, \psi]_G \subseteq [H, \varphi]_G\), then \(M^{[K, \psi]}_G = M\). Applying Frobenius reciprocity and the Mackey decomposition formula (Proposition 1.7 (c)) one sees immediately that \(\text{Hom}_{kG}(M, M)\) has k-rank \(|H \setminus G|\). Whereas in Prop. 1.20 (b) we will see that \(\text{mon}_{kG}(M, M)\) has k-rank \(#\{s \in H \setminus G \mid H \not\subseteq sH\}\) which is different from \(|H \setminus G|\), if \(H\) is not normal in \(G\).

1.12 Corollary

(a) Let \(M \in kG\text{-mon}\) and \(N \in kG\text{-mon}_{ab}\), then

\[
\text{mon}_{kG}(M, N) = \text{Hom}_{kG}(M, N).
\]

(b) The forgetful functor \(\mathcal{V} : kG\text{-mon}_{ab} \to kG\text{-mod}_{ab}^\text{pr}\) is a category equivalence with inverse functor given by

\[
V \mapsto M = \bigoplus_{\varphi \in G(k)} V^{(G, \varphi)}.
\]

**Proof** (a) This follows from Proposition 1.11 (b) \(\implies\) (a) and Remark 1.9 (g), (h), and (i): For \(f \in \text{Hom}_{kG}(M, N)\) and \((H, \varphi) \in \mathcal{M}_k(G)\) we have

\[
f(M^{(H, \varphi)}) = f((\text{res}_H^G M)^{(H, \varphi)}) \subseteq f(\mathcal{V}(\text{res}_H^G M)^{(H, \varphi)}) \subseteq \mathcal{V}((\text{res}_H^G N)^{(H, \varphi)})
\]

\[
= (\text{res}_H^G N)^{(H, \varphi)} = N^{(H, \varphi)},
\]

since \(\text{res}_H^G(N)\) is again abelian.

(b) This follows from part (a).
1.13 Example With the notation of Example 1.5 we have, for the quaternion group $G$ of order 8, three $kG$-monomial modules $V_1, V_2, V_3$ with $\mathcal{V}(V_1) \cong \mathcal{V}(V_2) \cong \mathcal{V}(V_3)$, but $\text{mon}_kG(V_i, V_j) = 0$ for $i \neq j$. In fact, let $f \in \text{mon}_kG(V_i, V_j)$, then by Proposition 1.11 (d) we have $f(V_i) = f(V'_i)^{[H_i, \varphi_i]} \subseteq V'_i^{[H_i, \varphi_i]}G$. But this last submodule of $V_j$ is 0, since the stabilizing pairs of the components $V'_j$ and $V''_j$ of $V_j$ are $(H_j, \varphi_j)$ and $(H_j, \varphi_j^{-1})$ respectively, and since $[H_i, \varphi_i]G \not\subseteq (H_j, \varphi_j)^G$ for $i \neq j$.

1.14 Definition For each $M \in kG\text{–mon}$ the abelian part $\mathcal{P}(M)$ of $M$ is defined by

$$\mathcal{P}(M) := \bigoplus_{\varphi \in \hat{G}(k)} M(G, \varphi) \subseteq M.$$  

Note that $\mathcal{P}(M) \in kG\text{–mon}_{ab}$, with the above decomposition, and that $M \in kG\text{–mon}_{ab}$ if and only if $\mathcal{P}(M) = M$. For arbitrary $M, N \in kG\text{–mon}$, $f \in \text{mon}_kG(M, N)$ and $\varphi \in \hat{G}(k)$ we have by Proposition 1.11 $f(M(G, \varphi)) \subseteq N^{(G, \varphi)} = N(G, \varphi)$, showing that $\mathcal{P}$ defines a functor

$$\mathcal{P} : kG\text{–mon} \rightarrow kG\text{–mon}_{ab}.$$  

1.15 Remark Although the concepts of taking abelian parts of objects in $kG\text{–mon}$ and $kG\text{–mod}$ seem to be very similar, they do not commute with the forgetful functor, i.e. in general,

$$\forall \mathcal{P} \not= \mathcal{P} \mathcal{V} : kG\text{–mon} \rightarrow kG\text{–mod}_{ab}$$

are not naturally isomorphic. Let for example $G$ be non-trivial, $M := \text{ind}_E^G(k) \in kG\text{–mon}$, $k$ considered as a $kE$-monomial module with one-term decomposition. Then $M = M_1 \oplus \ldots \oplus M_{|G|}$ and $G$ acts freely on $\{M_1, \ldots, M_{|G|}\}$ showing that all the stabilizing pairs are equal to $(E, 1)$, and therefore $\mathcal{P}(M) = 0$. On the other hand $\mathcal{V}(M) \cong kG$ has a non-trivial abelian part, since $kG^{(G, 1)} \not= 0$. This example also shows that in general $M^{(H, \varphi)} \not= \mathcal{V}(M)^{(H, \varphi)}$, cf. 1.9 (h).

1.16 Definition We call $M \in kG\text{–mon}$ indecomposable, if $M$ is not isomorphic to a direct sum $M' \oplus M''$ of non-zero $kG$-monomial modules $M'$ and $M''$. For each $(H, \varphi) \in \mathcal{M}_k(G)$ we denote by $k_{\varphi}$ the $\varphi$-homogeneous abelian $kH$-module with underlying $k$-module $k$. By Corollary 1.12 (b) we may consider $k_{\varphi}$ as element in $kH\text{–mon}_{ab}$, namely with the one-term decomposition. We define

$$S^G_{(H, \varphi)} := \text{ind}_H^G(k_{\varphi}) \in kG\text{–mon}.$$  

By the definition of induction, the decomposition of $S^G_{(H, \varphi)}$ is given by $S^G_{(H, \varphi)} = \bigoplus_{s \in \mathcal{G}/H} (s \otimes_{kH} k_{\varphi})$. Each summand is isomorphic to $k$ as $k$-module and $G$ acts transitively on these summands. The stabilizing pair of $s \otimes_{kH} k_{\varphi}$ is $^s(H, \varphi)$, $s \in \mathcal{G}/H$. Hence, $S^G_{(H, \varphi)} = S^G_{(H, \varphi)}([H, \varphi]_G)$ is homogeneous in degree $[H, \varphi]_G \in \mathcal{G}/\mathcal{M}_k(G)$.

We will often need the following hypothesis for the base ring $k$, and will refer to it as condition $(\ast)$:

Each finitely generated projective $k$-module is free.  

$(\ast)$
1.17 Proposition
(a) For \((H, \varphi), (K, \psi) \in \mathcal{M}_k(G)\) we have: \(S^G_{(H, \varphi)} \cong S^G_{(K, \psi)} \iff [H, \varphi]_G = [K, \psi]_G\). 
(b) \(S^G_{(H, \varphi)}\) is indecomposable for \((H, \varphi) \in \mathcal{M}_k(G)\).
(c) Let \(M \in kG\text{-mon}\) be homogeneous in degree \([H, \varphi]_G \in G \setminus \mathcal{M}_k(G)\). There is some \(N \in kG\text{-mon}\), homogeneous in degree \([H, \varphi]_G\), and some \(n \in \mathbb{N}\), such that \(M \oplus N\) is isomorphic to the direct sum of \(n\) copies of \(S^G_{(H, \varphi)}\). If \(k\) satisfies condition \((*)\), then \(M\) is isomorphic to the direct sum of \(\text{rk}_k M/(G : H)\) copies of \(S^G_{(H, \varphi)}\), where \(\text{rk}_k\) denotes the \(k\)-rank.
(d) Let \(M, N \in kG\text{-mon}\). The following are equivalent:

(i) \(M \) and \(N\) are isomorphic as \(kG\)-monomial modules.

(ii) \(M(H, \varphi)\) and \(N(H, \varphi)\) are isomorphic as \(kN_G(H, \varphi)\)-modules, for all \((H, \varphi) \in \mathcal{M}_k(G)\).

(iii) \(M(H, \varphi)\) and \(N(H, \varphi)\) are isomorphic as \(kN_G(H, \varphi)\)-monomial modules, for all \((H, \varphi) \in \mathcal{M}_k(G)\).

(iv) \([M(H, \varphi)_G]\) and \([N(H, \varphi)_G]\) are isomorphic as \(kG\)-monomial modules, for all \((H, \varphi) \in \mathcal{M}_k(G)\).

Proof (a) Let \(f: S^G_{(H, \varphi)} \to S^G_{(K, \psi)}\) be an isomorphism of \(kG\)-monomial modules. Then by part (a) \(\Rightarrow\) (d) of Proposition 1.11 we have

\[
S^G_{(K, \psi)} = f(S^G_{(H, \varphi)}) = f((S^G_{(H, \varphi)})^{[H, \varphi]_G}) \subseteq (S^G_{(K, \psi)})^{[H, \varphi]_G} \subseteq S^G_{(K, \psi)}.
\]

Hence, equality must hold, and from \((S^G_{(K, \psi)})^{[H, \varphi]_G} = S^G_{(K, \psi)} = S^G_{(K, \psi)}([K, \psi]_G)\) we can deduce \([H, \varphi]_G \leq [K, \psi]_G\). By symmetry we also get the other relation.

Conversely, let \((K, \psi) = \varphi(H, \varphi)\) for some \(s \in G\). Then the map

\[
S^G_{(H, \varphi)} \to S^G_{(K, \psi)}, \quad g \otimes_{kH} \alpha \mapsto gs^{-1} \otimes_{kK} \alpha
\]

for \(g \in G, \alpha \in k\) is a \(kG\)-monomial isomorphism.

(b) Assume that \(S^G_{(H, \varphi)} \cong M \oplus N\) with non-trivial \(M, N \in kG\text{-mon}\). Then \(M\) and \(N\) are homogeneous in degree \([H, \varphi]_G\). It is clear that the \(k\)-rank of an \([H, \varphi]_G\)-homogeneous \(kG\)-monomial module is a multiple of \((G : H)\). Since \(M\) and \(N\) are non-trivial, the \(k\)-rank of \(M \oplus N\) is at least \(2(G : H)\) which contradicts the fact that the \(k\)-rank of \(S^G_{(H, \varphi)}\) is \((G : H)\).

(c) If we know the assertion for two \(kG\)-monomial modules satisfying the hypothesis, we also know it for their direct sum. Hence, we may assume that \(G\) acts transitively on \(\{M_1, \ldots, M_m\}\), where \(M = M_1 \oplus \ldots \oplus M_m\) is the decomposition of \(M\). Certainly, one of the \(M_i\) has stabilizing pair \((H, \varphi)\), and we may assume that this is the case for \(M_1\). Then \(M \cong \text{ind}_G^H(M_1)\) as \(kG\)-monomial modules, where \(M_1\) can be viewed as a \(\varphi\)-homogeneous \(kH\)-monomial module with one-term decomposition by Corollary 1.12 (b). In fact,

\[
kG \otimes_{kH} M_1 \to M, \quad s \otimes_{kH} m \mapsto sm
\]
provides such an isomorphism.

If \( M_1 \) (which is \( k \)-projective in general) is \( k \)-free, then \( M_1 \) is isomorphic to a direct sum of \( \text{rk}_k M_1 \) copies of \( k \varphi \) as \( kH \)-monomial module by Corollary 1.12 (b). Since induction commutes with taking direct sums, we obtain that \( M \) is isomorphic to the direct sum of \( \text{rk}_k M_1 \) copies of \( S^G_{(H, \varphi)} \). Finally, it is clear that \( (G : H) \cdot \text{rk}_k M_1 = \text{rk}_k M \).

In general, there is \( N_1 \in k-\text{mod}^{\text{pt}} \) such that \( M_1 \oplus N_1 \) is \( k \)-free. We may consider \( N_1 \) as a \( \varphi \)-homogeneous \( kH \)-module and view it via Corollary 1.12 (b) as object in \( kH-\text{mon}_{\text{ab}} \). We define \( N := \text{ind}^G_H(N_1) \). Then \( M \oplus N \cong \text{ind}^G_H(M_1 \oplus N_1) \), and since \( M_1 \oplus N_1 \) is \( k \)-free and since its isomorphism type in \( kH-\text{mon} \) doesn’t change if we view it endowed with the one-term decomposition (cf. Corollary 1.12 (b)), we are in the previous case which shows that \( M \oplus N \) is isomorphic to a direct sum of copies of \( S^G_{(H, \varphi)} \).

(d) (i) \( \implies \) (ii): Let \( f \in \text{mon}_{kG}(M, N) \) be an isomorphism and \((H, \varphi) \in \mathcal{M}_k(G)\). Proposition 1.11 (b) implies that \( f \) induces \( kN_G(H, \varphi) \)-linear isomorphisms \( M^{(H, \varphi)} \rightarrow N^{(H, \varphi)} \) and \( \sum_{(H, \varphi) < (H', \varphi')} M^{(H', \varphi')} \rightarrow \sum_{(H, \varphi) < (H', \varphi')} N^{(H', \varphi')} \). Therefore we get \( kN_G(H, \varphi) \)-linear isomorphisms between the quotients which are \( kN_G(H, \varphi) \)-isomorphic to \( M(H, \varphi) \) and \( N(H, \varphi) \).

(ii) \( \implies \) (iii): Let \( f_{(H, \varphi)} : M(H, \varphi) \rightarrow N(H, \varphi) \) be a \( kN_G(H, \varphi) \)-linear isomorphism for each \((H, \varphi) \in \mathcal{M}_k(G)\). We may consider \( M(H, \varphi) \) and \( N(H, \varphi) \) as \( kN_G(H, \varphi) \)-monomial modules with the original decomposition. Then each \( f_{(H, \varphi)} \) is a \( kN_G(H, \varphi) \)-monomial isomorphism by Proposition 1.11 (c), since \( M(H, \varphi) \) and \( N(H, \varphi) \) are homogeneous in degree \((H, \varphi) \) with respect to the \( \mathcal{M}_k(N_G(H, \varphi)) \)-grading.

(iii) \( \implies \) (iv): By the hypothesis in (iii) we have \( kN_G(H, \varphi) \)-monomial isomorphisms \( f_{(H, \varphi)} : M(H, \varphi) \rightarrow N(H, \varphi) \) for all \((H, \varphi) \in \mathcal{R}_k(G)\). From Remark 1.19 (j) we know that \( M([H, \varphi]_G) \cong \text{ind}^G_{N_G(H, \varphi)} M(H, \varphi) \) as \( kG \)-monomial modules, and the same holds for \( N \). Hence, \( \text{ind}^G_{N_G(H, \varphi)}(f_{(H, \varphi)}) \) is an isomorphism \( M([H, \varphi]_G) \rightarrow N([H, \varphi]_G) \) for all \((H, \varphi) \in \mathcal{R}_k(G)\).

(iv) \( \implies \) (i): Each \( kG \)-monomial module is the direct sum of its \([H, \varphi]_G \)-homogeneous components. \( \square \)

1.18 Corollary
(a) Let \( k \) satisfy condition (\( k \)). Each indecomposable \( kG \)-monomial module is isomorphic to some \( S^G_{(H, \varphi)} \), \((H, \varphi) \in \mathcal{R}_k(G)\). Hence \( \mathcal{R}_k(G) \) (or as well \( G \cdot \mathcal{M}_k(G) \)) parametrizes the set of isomorphism classes of indecomposable \( kG \)-monomial modules.

(b) Let \( k \) satisfy condition (\( k \)) and let \( M \in kG-\text{mon} \). Then

\[
M \cong \bigoplus_{(H, \varphi) \in \mathcal{R}_k(G)} r_{[H, \varphi]_G}(M) \cdot S^G_{(H, \varphi)},
\]

where \( r_{[H, \varphi]_G}(M) := \text{rk}_k M([H, \varphi]_G)/(G : H) \), and \( n \cdot S^G_{(H, \varphi)} \) denotes the direct sum of \( n \) copies of \( S^G_{(H, \varphi)} \).

(c) Let \( k \) satisfy condition (\( k \)). For \( M, N \in kG-\text{mon} \) the following statements are equivalent:

(i) \( M \cong N \) as \( kG \)-monomial modules.
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(ii) \( r_{[H,\varphi]_G}(M) = r_{[H,\varphi]_G}(N) \) for all \([H,\varphi]_G \in G\backslash \mathcal{M}_k(G)\).

(iii) \( \text{rk}_k M([H,\varphi]_G) = \text{rk}_k N([H,\varphi]_G) \) for all \([H,\varphi]_G \in G\backslash \mathcal{M}_k(G)\).

(iv) \( \text{rk}_k M([H,\varphi]_G) = \text{rk}_k N([H,\varphi]_G) \) for all \([H,\varphi]_G \in G\backslash \mathcal{M}_k(G)\).

(v) \( \text{rk}_k M(H,\varphi) = \text{rk}_k N(H,\varphi) \) for all \((H,\varphi) \in \mathcal{M}_k(G)\).

(vi) \( \text{rk}_k M(H,\varphi) = \text{rk}_k N(H,\varphi) \) for all \((H,\varphi) \in \mathcal{M}_k(G)\).

(d) For each \( M \in kG\text{--mon} \) we have

\[
M \cong \bigoplus_{(H,\varphi) \in \mathcal{R}_k(G)} \text{ind}_H^G L(H,\varphi),
\]

where \( L(H,\varphi) \in kH\text{--mon}_{ab} \) is a \( \varphi \)-homogeneous \( kH \)-monomial module with one-term decomposition. In particular the \([H,\varphi]_G\)-homogeneous component of \( M \) is isomorphic to \( \text{ind}_H^G(L(H,\varphi)) \) by Proposition 1.17 (d). If \( k \) satisfies condition (\(*\)), then \( L(H,\varphi) \) can be chosen \( k \)-free of \( k \)-rank \( r_{[H,\varphi]_G}(M) \).

**Proof** Part (a) and (b) follow immediately from Remark 1.9 (f) and from Proposition 1.17 (c).

(c) The statements (i), (ii), and (iii) are equivalent by part (b) and the definition of \( r_{[H,\varphi]_G}(M) \).

(i) \( \implies \) (iv),(vi): This follows from Proposition 1.11 (a) \( \implies \) (d) resp. (a) \( \implies \) (b).

(vi) \( \implies \) (v): Since \( M(H,\varphi) = \bigoplus_{(H,\varphi) \leq (H',\varphi')} M(H',\varphi') \) for all \((H,\varphi) \in \mathcal{M}_k(G)\), we get by Möbius inversion

\[
\text{rk}_k M(H,\varphi) = \sum_{(H,\varphi) \leq (H',\varphi')} \mu_{\mathcal{M}_k(G)}((H,\varphi),(H',\varphi')) \cdot \text{rk}_k M(H',\varphi'),
\]

where \( \mu_{\mathcal{M}_k(G)} \) is the Möbius function of the poset \( \mathcal{M}_k(G) \), see Appendix B.

(iv) \( \implies \) (ii): Similarly we obtain

\[
\text{rk}_k M([H,\varphi]_G) = \sum_{[H,\varphi]_G \leq [H',\varphi']_G} \mu_{G\backslash \mathcal{M}_k(G)}([H,\varphi]_G,[H',\varphi']_G) \cdot \text{rk}_k M([H',\varphi']_G),
\]

where \( \mu_{G\backslash \mathcal{M}_k(G)} \) is the Möbius function of the poset \( G\backslash \mathcal{M}_k(G) \).

(v) \( \implies \) (ii): This follows immediately from the equation

\[
\text{rk}_k M([H,\varphi]_G) = (G : N_G(H,\varphi)) \cdot \text{rk}_k M(H,\varphi)
\]

for all \((H,\varphi) \in \mathcal{M}_k(G)\).

(d) By Remark 1.9 (f) we may assume that \( M \) is \([H,\varphi]_G\)-homogeneous for some \((H,\varphi) \in \mathcal{R}_k(G)\), and we have to show that \( M \) is isomorphic to \( \text{ind}_H^G(L(H,\varphi)) \) for some \( L(H,\varphi) \) as in the assertion. We also may assume that \( G \) acts transitively on \( \{M_1,\ldots,M_m\} \), if \( M = M_1 \oplus \ldots \oplus M_m \) is the decomposition of \( M \), since induction commutes with taking direct sums, and since two \( \varphi \)-homogeneous \( kH \)-monomial modules are isomorphic if and only if their underlying \( k \)-modules are isomorphic, cf. part (b) of Corollary 1.12. By renumeration we may assume that \( M_1 \) has stabilizing pair \((H,\varphi)\). Then we have \( M \cong \text{ind}_H^G(M_1) \), as already observed in the proof of Proposition 1.17 (c). Hence we may choose \( L(H,\varphi) = M_1 \).
If \( k \) satisfies condition \((\ast)\), then the special choice for \( L_{(H,\varphi)} \) follows from part (b) and the fact that \( S^G_{(H,\varphi)} \) is homogeneous in degree \([H,\varphi]_G\). \(\square\)

1.19 If \( k \) satisfies condition \((\ast)\), there is a well-known pairing for \( V,W \in kG\mod^{pr} \) defined by

\[
(V,W)_G := \text{rk}_k \text{Hom}_{kG}(V,W).
\]

Similarly we define a pairing on \( kG\mon \) by

\[
[M,N]_G := \text{rk}_k \text{mon}_{kG}(M,N),
\]

for \( M,N \in kG\mon \). Note that \([-,-]_G \) is bilinear with respect to direct sums, but in general \([-,-]_G \) is not symmetric, cf. Proposition 1.20 (b). However, Frobenius reciprocity holds by Proposition 1.7 (a):

\[
[ind^G_H(M),N]_G = [M,\text{res}^G_H(N)]_H
\]

for all \( H \leq G, M \in kH\mon, N \in kG\mon \). We extend the definition of \([M,N]_G \) for arbitrary \( k \), provided that \( \text{mon}_{kG}(M,N) \) is a free \( k \)-module.

1.20 Proposition

(a) For \((K,\psi) \in \mathcal{M}_k(G)\) and \( N \in kG\mon \) we have a \( k \)-linear isomorphism

\[
\text{mon}_{kG}(S^G_{(K,\psi)},N) \cong N^{(K,\psi)},
\]

\[
f \mapsto f(1 \otimes_{kK} 1),
\]

\[
(s \otimes_{kK} \alpha \mapsto \alpha sn) \leftrightarrow n,
\]

where \( \alpha \in k, s \in G \). In particular,

\[
[S^G_{(K,\psi)},N]_G = \text{rk}_k N^{(K,\psi)}.
\]

(b) Let \((K,\psi),(H,\varphi) \in \mathcal{M}_k(G)\). The \( k \)-module \( \text{mon}_{kG}(S^G_{(K,\psi)},S^G_{(H,\varphi)}) \) is free with \( k \)-basis \( \{ f_s \mid s \in G/H, (K,\psi) \leq ^{s}(H,\varphi) \} \) given by \( f_s(g \otimes_{kK} \alpha) = gs \otimes_{kH} \alpha \) for \( g \in G \) and \( \alpha \in k \). Moreover,

\[
[S^G_{(K,\psi)},S^G_{(H,\varphi)}]_G = \# \{ s \in K \backslash G/H \mid (K,\psi) \leq ^{s}(H,\varphi) \}
\]

\[
= \# \{ s \in G/H \mid (K,\psi) \leq ^{s}(H,\varphi) \}.
\]

In particular, \([S^G_{(K,\psi)},S^G_{(H,\varphi)}]_G = 0 \), unless \([K,\psi]_G \leq [H,\varphi]_G \), and

\[
[S^G_{(H,\varphi)},S^G_{(H,\varphi)}]_G = (N_G(H,\varphi) : H).
\]

(c) The \( k \)-module \( \text{mon}_{kG}(M,N) \) is finitely generated and projective for \( M,N \in kG\mon \).

Proof (a) By Frobenius reciprocity (Proposition 1.7 (a)) we have an isomorphism

\[
\text{mon}_{kG}(S^G_{(K,\psi)},N) \cong \text{mon}_{kK}(k,\text{res}^G_K N).
\]
By Proposition 1.11 (a) \(\implies\) (c), we have
\[
\text{mon}_{kK}(k_{\psi}, \text{res}_{K}^{G}N) = \text{Hom}_{kK}(k_{\psi}, (\text{res}_{K}^{G}N)^{(K,\psi)}) ,
\]
since \(k_{\psi}\) is homogeneous in degree \((K, \psi)\) with respect to the \(\mathcal{M}_{K}(k)\)-grading. But obviously
\[
\text{Hom}_{kK}(k_{\psi}, (\text{res}_{K}^{G}N)^{(K,\psi)}) = \text{Hom}_{k}(k_{\psi}, (\text{res}_{K}^{G}N)^{(K,\psi)}) \cong (\text{res}_{K}^{G}N)^{(K,\psi)}
\]
and the latter \(k\)-module equals \(N^{(K,\psi)}\) by Remark 1.9 (g). The composition of all these identifications yields exactly the map described in the proposition.

(b) The isomorphism in part (a) applied to \(N = S_{(H,\varphi)}^{G}\) yields an isomorphism
\[
\text{mon}_{kG}(S_{(K,\psi)}^{G}, S_{(H,\varphi)}^{G}) \cong (S_{(H,\varphi)}^{G})^{(K,\psi)}.
\]
The latter submodule of \(S_{(H,\varphi)}^{G}\) is the direct sum of those components \(s \otimes_{kH} k_{\varphi}, s \in G/H\), with \((K, \psi) \leq s(H, \varphi)\). The \(k\)-basis \(\{s \otimes_{kH} 1 \mid s \in G/H, (K, \psi) \leq s(H, \varphi)\}\) of the right hand side of the above isomorphism then corresponds exactly to the set \(\{f_{s}\}\), described in the proposition.

Note that whenever \(s \in G\) satisfies \((K, \psi) \leq s(H, \varphi)\), then the coset \(sH = (s \cdot H)s = KSsH\) is equal to its double coset. This proves the equation of the cardinalities, and everything else is obvious.

(c) Since \(M\) and \(N\) are the direct sums of their \([H, \varphi]_{G}\)-homogeneous components, \([H, \varphi]_{G} \subset G \setminus \mathcal{M}_{k}(G)\), we may assume that \(M\) and \(N\) are homogeneous with respect to the \(G \setminus \mathcal{M}_{k}(G)\)-grading. Then, by Proposition 1.17 (c), we may assume that \(M = S_{(K,\psi)}^{G}\) and \(N = S_{(H,\varphi)}^{G}\) for some \((K, \psi), (H, \varphi) \in \mathcal{M}_{k}(G)\). In this case the assertion follows from part (b).

For \((K, \psi), (H, \varphi) \in \mathcal{M}_{k}(G)\) we define \(\gamma_{(K,\psi),(H,\varphi)}^{G} := [S_{(K,\psi)}^{G}, S_{(H,\varphi)}^{G}]_{G}\). This definition coincides with the definition in [Bo90, 1.16]. Proposition 1.17 (a) shows that \(\gamma_{(K,\psi),(H,\varphi)}^{G}\) depends only on the \(G\)-orbits of \((K, \psi)\) and \((H, \varphi)\), and therefore we will also write \(\gamma_{[K,\psi], [H,\varphi]}^{G}\).

In addition to the characterizations of isomorphic objects of \(kG\text{-mon}\) in Corollary 1.18 (c) we have the following proposition:

1.21 Proposition Let \(k\) satisfy condition \((*)\). For \(M, N \in kG\text{-mon}\) the following statements are equivalent:

(a) \(M \cong N\) as \(kG\text{-monomial modules}.

(b) \([L, M]_{G} = [L, N]_{G}\) for all \(L \in kG\text{-mon}\).

(c) \([M, L]_{G} = [N, L]_{G}\) for all \(L \in kG\text{-mon}\).

Proof Obviously (a) implies (b) and (c) by the definition of \([-,-]_{G}\).

(b) \(\implies\) (a): Let \((H, \varphi) \in \mathcal{M}_{k}(G)\). Setting \(L := S_{(H,\varphi)}^{G}\) we obtain from the hypothesis in (b) and from Proposition 1.20 (a) that \(\text{rk}_{k} M_{(H,\varphi)} = \text{rk}_{k} N_{(H,\varphi)}\). This implies \(M \cong N\) by Corollary 1.18 (c).
(c) $\implies$ (a): From Corollary 1.18 (b) we know
\[ M \cong \bigoplus_{(H,\varphi)\in R_k(G)} r_{[H,\varphi]_G}(M) S^G_{(H,\varphi)} \quad \text{and} \quad N \cong \bigoplus_{(H,\varphi)\in R_k(G)} r_{[H,\varphi]_G}(N) S^G_{(H,\varphi)}. \]

Let $(U,\mu)\in\mathcal{M}_k(G)$, then setting $L = S^G_{(U,\mu)}$ we obtain from the hypothesis in (c) and from Proposition 1.20 (b) that
\[ \sum_{[H,\varphi]_G\in G\setminus\mathcal{M}_k(G)} r_{[H,\varphi]_G}(M) \gamma_{[H,\varphi]_G;[U,\mu]_G}^G = \sum_{[H,\varphi]_G\in G\setminus\mathcal{M}_k(G)} r_{[H,\varphi]_G}(N) \gamma_{[H,\varphi]_G;[U,\mu]_G}^G. \]

Using the last part of Proposition 1.20 (b) we can show by induction on $|U|$ that $r_{[U,\mu]_G}(M) = r_{[U,\mu]_G}(N)$ for all $[U,\mu]_G \in G\setminus\mathcal{M}_k(G)$. This implies $M \cong N$ by Corollary 1.18 (c).

1.22 Let $R^{ab}_{k+}(G)$ be the Grothendieck group of the category $kG\text{-}\text{mon}$, i.e. the free abelian group over the isomorphism classes $\{M\}$ of $kG$-monomial modules $M$, modulo the subgroup generated by the elements $\{M\} + \{N\} - \{M \oplus N\}$, where $M, N \in kG\text{-}\text{mon}$. The class of $\{M\}$ in the Grothendieck group will be denoted by $[M]$.

We will determine $R^{ab}_{k+}(G)$ in the case where $k$ satisfies condition (*), and we will keep this assumption for the rest of this section. If $k = \mathbb{C}$ we just write $R^{ab}_{+}(G)$ instead of $R^{ab}_{k+}(G)$. The following description of $R^{ab}_{k+}(G)$ justifies the coincidence of notation with Section III.1.

Since each $M \in kG\text{-}\text{mon}$ is isomorphic to a direct sum of some $S^G_{(H,\varphi)}$, $(H,\varphi) \in R_k(G)$, their images $[S^G_{(H,\varphi)}]$, $(H,\varphi) \in R_k(G)$, generate the Grothendieck group. Moreover, since $kG\text{-}\text{mon}$ has the cancellation property, i.e. $M \oplus L \cong N \oplus L \implies M \cong N$ for $L, M, N \in kG\text{-}\text{mon}$ (cf. Corollary 1.18 (b) and (c) (i) $\iff$ (ii)), the elements $[S^G_{(H,\varphi)}]$, $(H,\varphi) \in R_k(G)$, form a $\mathbb{Z}$-basis of the Grothendieck group.

We will abbreviate the basis elements $[S^G_{(H,\varphi)}]$ of $R^{ab}_{k+}(G)$ by $[H,\varphi]_G$ for $(H,\varphi) \in \mathcal{M}_k(G)$. If $M \in kG\text{-}\text{mon}$, then the associated element $[M] \in R^{ab}_{k+}(G)$ is given by
\[ [M] = \sum_{[H,\varphi]_G\in G\setminus\mathcal{M}_k(G)} r_{[H,\varphi]_G}(M) \cdot [H,\varphi]_G, \]
where $r_{[H,\varphi]_G}(M)$ is defined as in Corollary 1.18 (b). If $M = M_1 \oplus \ldots \oplus M_m$ is the decomposition of $M$, we may describe $[M]$ also by
\[ [M] = \sum_{M_i \in \{M_1,\ldots,M_m\}/G} \text{rk}_k M_i \cdot [H_i,\varphi_i]_G, \]
where $(H_i,\varphi_i)$ denotes the stabilizing pair of $M_i$. Corollary 1.18 (c) now implies that we have for arbitrary $M, N \in kG\text{-}\text{mon}$:
\[ M \cong N \iff [M] = [N] \in R^{ab}_{k+}(G). \]

1.23 Remark Subsection 1.22 gives now a very concrete meaning to the groups $R^{ab}_{+}(G)$, $R^{ab}_{k+}(G)$, and $L^{ab}_{O+}(G)$ of Sections III.1, III.2, and III.4. They
are just the Grothendieck groups of $\mathbb{C}G$-$\text{mon}$, $FG$-$\text{mon}$ and $O\mathbb{G}$-$\text{mon}$. If we define $R^\text{ab}_{k+}(H) := ZH(k)$, for $H \leq G$, then $R^\text{ab}_{k+}$ is a $k$-algebra restriction functor on $G$ with the usual conjugation and restriction maps. The discussion in 1.22 shows that for $H \leq G$, the construction $R^\text{ab}_{k+}(H)$ of I.22 and the definition of $R^\text{ab}_{k+}(H)$ as Grothendieck group yield isomorphic groups. Hence we may denote both groups by the same symbol $R^\text{ab}_{k+}(H)$. Note that $R^\text{ab}_{k+}$ in the notation of I.22 is a $\mathbb{Z}$-Green functor on $G$. In the sequel we will show that its structure maps come from the constructions on the category level. The constructions in Remark 1.6 for example provide $R^\text{ab}_{k+}(G)$ with more structure than just a free abelian group. In the next subsections we will give translations of these constructions to structure maps on $R^\text{ab}_{k+}(G)$. We will outline this translation in detail for the ring structure on $R^\text{ab}_{k+}(G)$ and leave the proofs for the other translations as an easy exercise to the reader. Note that the verification of some properties of these structures, as for example associativity and commutativity of multiplication, requires nasty calculations with double coset representatives (cf. [Bo89, Sect. III.2]). However, if we interpret $R^\text{ab}_{k+}(G)$ as the Grothendieck group of $kG$-$\text{mon}$, this is all obvious, since tensor products are commutative and associative. The same is true for restrictions along group homomorphism.

1.24 Multiplication. The tensor product on $kG$-$\text{mon}$ defines a multiplication on $R^\text{ab}_{k+}(G)$. This provides $R^\text{ab}_{k+}(G)$ with the structure of a commutative ring with unity $[G,1]_G$, since the tensor product in $kG$-$\text{mon}$ is commutative, associative and since taking tensor product with $S^G_{(G,1)}$ is up to a functorial isomorphism the identity functor on $kG$-$\text{mon}$. In order to give an explicit multiplication formula in $R^\text{ab}_{k+}(G)$ we have to decompose $S^G_{(H,\varphi)} \otimes S^G_{(K,\psi)}$ as a sum of $S^G_{(U,\mu)}$, $(U, \mu) \in G \setminus M_k(G)$:

$$S^G_{(H,\varphi)} \otimes S^G_{(K,\psi)} = \text{ind}^G_K(k_\varphi) \otimes \text{ind}^G_H(k_\psi)$$

1.7(b)

$$\cong \text{ind}^G_K(k_\psi \otimes \text{res}^G_K(\text{ind}^G_H(k_\varphi)))$$

1.7(c)

$$\cong \bigoplus_{s \in K \setminus G / H} \text{ind}^G_K(k_\psi \otimes \text{ind}^K_{K \cap s^*H}(\text{res}^{s^*H}_{K \cap s^*H}(k_\varphi)))$$

1.7(b)

$$\cong \bigoplus_{s \in K \setminus G / H} (\text{ind}^G_K \circ \text{ind}^K_{K \cap s^*H})(\text{res}^K_{K \cap s^*H}(k_\psi) \otimes \text{res}^{s^*H}_{K \cap s^*H}(k_\varphi))$$

1.7(b)

$$\cong \bigoplus_{s \in K \setminus G / H} S^G_{(K \cap s^*H, \psi |_{K \cap s^*H} \cdot s_\varphi |_{K \cap s^*H})}.$$

Hence we obtain the following explicit multiplication rule in $R^\text{ab}_{k+}(G)$:

$$[K, \psi]_G \cdot [H, \varphi]_G = \sum_{s \in K \setminus G / H} [K \cap s^*H, \psi \cdot s_\varphi]_G,$$

where we consider $\psi$ and $s_\varphi$ as restricted to $K \cap s^*H$. This coincides with the multiplication defined in 1.22.

1.25 Restriction. Let $f: G' \to G$ be a homomorphism of finite groups. The functor $\text{res}_f: kG$-$\text{mon} \to kG'$-$\text{mon}$ which commutes with direct sums and tensor
products induces a unitary ring homomorphism

$$\text{res}_{+f}: R_{k+}^{ab}(G) \rightarrow R_{k+}^{ab}(G'), \quad [H, \varphi]_G \mapsto \sum_{s \in f(G') \setminus G/H} [f^{-1}(sH), s\varphi \circ f]_{G'}.$$  

For an inclusion $K \leq G$ of a subgroup we also write

$$\text{res}^+_G: R_{k+}^{ab}(G) \rightarrow R_{k+}^{ab}(K), \quad [H, \varphi]_G \mapsto \sum_{s \in K \setminus G/H} [K \cap sH, s\varphi]_{K \cap sH}.$$  

This coincides with the definition of $\text{res}^+$ in I.2.2. Note that, if $f': G'' \rightarrow G'$ is another group homomorphism, then by the transitivity of the functors we have

$$\text{res}_{+f'} \circ \text{res}_{+f} = \text{res}_{+f \circ f'}: R_{k+}^{ab}(G) \rightarrow R_{k+}^{ab}(G'').$$

1.26 Induction. Let $K \leq G$, then the functor $\text{ind}^G_K: kK-\text{mon} \rightarrow kG-\text{mon}$ induces a group homomorphism

$$\text{ind}^G_K: R_{k+}^{ab}(K) \rightarrow R_{k+}^{ab}(G), \quad [H, \varphi]_K \mapsto [H, \varphi]_G.$$  

This coincides with the definition of $\text{ind}^+$ in I.2.2. Note that induction is transitive with respect to subgroup towers, since this is the case for the underlying functors. The canonical isomorphism in Proposition 1.7 (c) implies a Mackey formula for $R_{k+}^{ab}(G)$: Let $U, H \leq G$ and let $x \in R_{k+}^{ab}(H)$, then

$$\text{res}^+_U(\text{ind}^G_K(x)) = \sum_{U \setminus G/H} \text{ind}^U_{U \setminus H}(\text{res}^+_U(\text{ind}^G_K(x))),$$  

where $s x$ denotes the image of $x$ under $\text{res}^+_f$ for $f: sH \rightarrow H, \quad shs^{-1} \mapsto h, \quad h \in H$. This defines for each $s \in G$ and each subgroup $H \leq G$ a conjugation map

$$s: R_{k+}^{ab}(H) \rightarrow R_{k+}^{ab}(sH), \quad [K, \psi]_H \mapsto [sK, s\psi]_H.$$  

This coincides with the definition of $c_+$ in I.2.2. Note that these conjugation maps are transitive with respect to multiplication in $G$, and that they commute with restriction and induction. Moreover, if $s \in H$, then conjugation with $s$ on $R_{k+}^{ab}(H)$ is the identity. This shows that $R_{k+}^{ab}$ is a Mackey functor for $G$.

The canonical isomorphism in Proposition 1.7 (b) implies for $H \leq G$, $x \in R_{k+}^{ab}(G)$, $y \in R_{k+}^{ab}(H)$:

$$x \cdot \text{ind}^+_H(y) = \text{ind}^+_H((\text{res}^+_H(x)) \cdot y).$$  

This shows again that $R_{k+}^{ab}$ is a Green functor.

1.27 Bilinear form. The bilinear form $[-, -]_{G}$ on $kG-\text{mon}$ induces a bilinear form, which we still denote by $[-, -]_{G}$, on $R_{k+}^{ab}(G)$ with values in $\mathbb{Z}$. For the basis elements of $R_{k+}^{ab}(G)$ we have (cf. Proposition 1.20 (b))

$$[[K, \psi]_G, [H, \varphi]_G]_G = \gamma^G_{[K, \psi]_G, [H, \varphi]_G} = \# \{s \in K \setminus G/H \mid (K, \psi) \in \ast(H, \varphi)\}.$$
5.1. KG-MONOMIAL MODULES

Note that for $k = \mathbb{C}$ this definition of $[-,-]_G$ on $R^\text{ab}_k(G)$ coincides with the one of Section III.1. Proposition 1.21 implies that $[-,-]_G$ is non-degenerate in both arguments, i.e. for all $x \in R^\text{ab}_k(G)$ we have

$$[x,y]_G = 0 \text{ for all } y \in R^\text{ab}_k(G) \iff x = 0$$

and

$$[y,x]_G = 0 \text{ for all } y \in R^\text{ab}_k(G) \iff x = 0.$$  

Moreover Proposition 1.7 (a) implies that $\text{res}_{+}^{G}H$ is the right adjoint of $\text{ind}_{+}^{G}H$ with respect to $[-,-]$ for $H \leq G$:

$$[x,\text{res}_{+}^{G}H(y)]_H = [\text{ind}_{+}^{G}H(x),y]_G$$

for all $x \in R^\text{ab}_k(H)$, $y \in R^\text{ab}_k(G)$.

1.28 Abelianization. The image of $kG-\text{mon}_\text{ab}$ in $R^\text{ab}_k(G)$ is given by the elements $\sum_{\varphi \in \hat{G}(k)} \alpha_{\varphi} [G,\varphi]_G$ with $\alpha_{\varphi} \in \mathbb{N}_0$. The Grothendieck group $R^\text{ab}_k(G)$ of $kG-\text{mon}_\text{ab}$ can be considered as the subgroup of $R^\text{ab}_k(G)$ generated by the elements $[G,\varphi]_G$, $\varphi \in \hat{G}(k)$. This is a subring of $R^\text{ab}_k(G)$ which is isomorphic to the group ring $\mathbb{Z} \hat{G}(k)$, when we identify $[G,\varphi]_G \in R^\text{ab}_k(G)$ with $\varphi \in \mathbb{Z} \hat{G}(k)$. In this way we can consider $R^\text{ab}_k(G)$ as a $\mathbb{Z} \hat{G}(k)$-algebra. Now the functor $P : kG-\text{mon} \to kG-\text{mon}_\text{ab}$ induces the ring homomorphism

$$\pi_G : R^\text{ab}_k(G) \to \mathbb{Z} \hat{G}(k), \quad (H,\varphi) \mapsto \begin{cases} \varphi, & \text{if } H = G, \\ 0, & \text{if } H \neq G, \end{cases}$$

which coincides with the Brauer morphism $\pi^A_G : A_+(G) \to A(G)$ of Section I.3 with $A(H) := R^\text{ab}_k(H)$ for $H \leq G$.

1.29 The forgetful functor. The forgetful functor $V : kG-\text{mon} \to kG-\text{mod}^{\text{pr}}_k$ induces the map

$$b_G : R^\text{ab}_k(G) \to R_k(G), \quad [H,\varphi]_G \mapsto [\text{ind}_{H}^{G}(k\varphi)],$$

where $R_k(G)$ is the Grothendieck ring of the category $kG-\text{mod}^{\text{pr}}_k$, and the brackets denote the image of an object in the Grothendieck group. If $k$ is a field of characteristic 0, then $R_k(G)$ is the $k$-character ring (of generalized characters of $k$-representations) of $G$ and $[\text{ind}_{H}^{G}(k\varphi)]$ corresponds to the character $\text{ind}_{H}^{G}(\varphi)$. Since $V$ commutes with $\oplus$, $\otimes$, $\text{res}_f$ and $\text{ind}_H^G$, the map $b_G$ is a ring homomorphism commuting with restriction and induction. Hence, the collection $(b_H)_{H \leq G}$ forms a morphism of Green functors. Moreover, $b_G$ is a $\mathbb{Z} \hat{G}(k)$-algebra homomorphism, if we view $R_k(G)$ as a $\mathbb{Z} \hat{G}(k)$-algebra by the obvious identification of $\mathbb{Z} \hat{G}(k) = R^\text{ab}_k(G)$ (the Grothendieck ring of $kG-\text{mod}^{\text{pr}}_k$) as subring of $R_k(G)$ (the Grothendieck ring of $kG-\text{mod}^{\text{pr}}_k$), cf. Corollary 1.12 (b). Note that $(b_H)_{H \leq G}$ coincides with the induction morphism $b^{M,A}$ of Definition II.1.1 with $A(H) := R^\text{ab}_k(H)$ and $M(H) := R_k(H)$ for $H \leq G$. 


1.30 Remark  Assume that $k = \mathbb{C}$. Then the Grothendieck ring of $\mathbb{C}G\text{-mod}$ is given by the ordinary character ring $R(G)$. The Grothendieck ring of $\mathbb{C}G\text{-mod}_{ab}$ is the subring of $R(G)$ generated by $\{ \varphi \in \text{Irr}(G) \mid \varphi \in \hat{G} \}$, and is isomorphic to the group ring $\mathbb{Z}\hat{G}$, which we have already considered as the Grothendieck ring of $\mathbb{C}G\text{-mon}_{ab}$. Therefore we denote the Grothendieck ring of $\mathbb{C}G\text{-mon}_{ab}$ also by $R_{ab}(G)$.

The abelianization functor $\mathcal{P} : \mathbb{C}G\text{-mod} \to \mathbb{C}G\text{-mod}_{ab}$ (which can also be considered as the functor of taking $G'$-fixed points of a $\mathbb{C}G$-module, where $G'$ is the commutator subgroup of $G$) induces the map

$$p_G : R(G) \to \mathbb{Z}\hat{G}, \quad \text{Irr}(G) \ni \chi \mapsto \begin{cases} 
\chi, & \text{if } \chi \in \hat{G}, \\
0, & \text{if } \chi \notin \hat{G},
\end{cases}$$

which we have already encountered in Example III.1.1. Note that $p_G$ in contrast to $\pi_G$ is not multiplicative.

Using the abelianization maps on $\mathbb{C}G\text{-mon}$ and $\mathbb{C}G\text{-mod}$ together with the fact that their abelian subcategories are equivalent, gives us a possibility to form a link between $R_{ab}^+(G)$ and $R(G)$ via $R_{ab}(G)$. More precisely, we can embed both of these two Grothendieck groups into the group $\prod_{H \leq G} R_{ab}(H)$ by defining

$$r_G = (p_H \text{res}_H^G)_{H \leq G} : R(G) \to \prod_{H \leq G} R_{ab}(H), \quad \chi \mapsto \left( p_H(\text{res}_H^G(\chi)) \right)_{H \leq G}$$

and

$$\rho_G = (\pi_H \text{res}_H^+)^{H \leq G} : R_{ab}^+(G) \to \prod_{H \leq G} R_{ab}(H), \quad x \mapsto \left( \pi_H(\text{res}_H^+(x)) \right)_{H \leq G}.$$

Note that $\rho_G$ which equals the mark homomorphism of Section I.3 is a ring homomorphism, since $\pi_H$ and $\text{res}_H^+$ are ring homomorphisms for all $H \leq G$. However $r_G$ is not a ring homomorphism in general. Clearly $r_G$ is injective, since a character is uniquely determined by restriction to all cyclic subgroups, and since $p_H$ is the identity for $H$ cyclic. Also $\rho_G$ is injective as we have seen in Proposition I.3.2. This can also be seen by the fact that $\rho_G$ is induced from the functor

$$kG\text{-mon} \to \prod_{H \leq G} kH\text{-mon}_{ab}, \quad M \mapsto (\mathcal{P}(\text{res}_H^G(M)))_{H \leq G}.$$
We have defined $b_G$ as induced from the forgetful functor $V: \mathbb{C}G-\text{mon} \to \mathbb{C}G-\text{mod}$. Of course $a_G$ can not be induced by a functor $\mathbb{C}G-\text{mod} \to \mathbb{C}G-\text{mon}$, since in this case any $\mathbb{C}G$-module would be isomorphic to an image under $V$, and hence monomial. Another reason for the non-existence of such a functor is, that $a_G(\chi)$ may have negative coefficients (as examples show). Hence, the only way of lifting $a_G$ to a functor is to associate to each $\mathbb{C}G$-module $V$ a 'virtual' $\mathbb{C}G$-monomial module $M$, with the property

$$\dim \mathbb{C} M^{(H, \varphi)} = \dim \mathbb{C} V^{(H, \varphi)} \quad \text{for all } (H, \varphi) \in \mathcal{M}(G),$$

In fact, the equation $\rho_G \circ a_G = r_G$ implies the above equation, since

$$p_H \text{res}^G_H([V]) = \sum_{\varphi \in \hat{H}} \dim \mathbb{C} V^{(H, \varphi)} \varphi \in \mathbb{Z} \hat{H} \quad \text{for } V \in \mathbb{C}G-\text{mod}$$

and

$$\pi_H \text{res}^G_H([M]) = \sum_{\varphi \in \hat{H}} \dim \mathbb{C} M^{(H, \varphi)} \varphi \in \mathbb{Z} \hat{H} \quad \text{for } M \in \mathbb{C}G-\text{mon}.$$

In the next section we will enlarge the category $kG-\text{mon}$ to the homotopy category of chain complexes of $kG$-monomial modules in order to give the negative coefficients in $a_G(\chi)$ a meaning in terms of ‘real’ objects by considering Lefschetz invariants.

### 5.2 The monomial resolution

In this section we define the notion of a monomial resolution (Definition 2.1), prove the existence and uniqueness of monomial resolutions (Theorem 2.4), and show that over the complex numbers there exist always finite monomial resolutions (Theorem 2.14). Moreover, we show in Corollary 2.16 that the Lefschetz element of a finite monomial resolution of $V \in \mathbb{C}G-\text{mod}$ equals the canonical induction formula of the character of $V$, as defined in Example III.3.1.

We keep the notation of the previous section.

#### 2.1 Definition

Let $V \in kG-\text{mod}$. A $kG$-monomial resolution of $V$ is a (not necessarily finite) chain complex

$$M_* = \ldots \xrightarrow{\partial_2} M_2 \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_0} M_0$$

with $M_i \in kG-\text{mon}$ and $\partial_i \in \text{mon}_{kG}(M_{i+1}, M_i)$ for all $i \geq 0$, together with a $kG$-linear map $\varepsilon: M_0 \to V$ such that

$$\ldots \xrightarrow{\partial_2} M_2^{(H, \varphi)} \xrightarrow{\partial_1} M_1^{(H, \varphi)} \xrightarrow{\partial_0} M_0^{(H, \varphi)} \xrightarrow{\varepsilon} V^{(H, \varphi)} \longrightarrow 0 \quad (5.1)$$

is an exact sequence of $k$-modules for all $(H, \varphi) \in \mathcal{M}_k(G)$. Note that the maps $\partial_i$ in (5.1) are well defined by Proposition 1.11 (b) and also that $\varepsilon$ in (5.1) is well defined by Remark 1.9 (h). Note furthermore that (5.1) is an exact sequence of $kN_G(H, \varphi)$-modules. In particular we obtain for $(H, \varphi) = (E, 1)$ an exact sequence

$$\ldots \xrightarrow{\partial_2} M_2 \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_0} M_0 \xrightarrow{\varepsilon} V \longrightarrow 0$$
of $kG$-modules which justifies the terminology.

2.2 Lemma Let

$$\begin{array}{c}
P \\
M \quad \xrightarrow{f} \\
N
\end{array}$$

be a diagram, where $P, M \in kG\text{-}\text{mon}$, $N \in kG\text{-}\text{mod}$ and $f, g \in kG\text{-}\text{mod}$. Assume furthermore that

$$f(P^{(H,\varphi)}) \subseteq g(M^{(H,\varphi)}) \text{ for all } (H, \varphi) \in M_k(G).$$

Then there exists $h \in \text{mon}_{kG}(P, M)$ such that $g \circ h = f$.

Proof By Corollary 1.18 (d) we may assume that $P = \text{ind}^G_H L_{(H,\varphi)}$ for some $(H, \varphi) \in M_k(G)$, where $L_{(H,\varphi)}$ is a $\varphi$-homogeneous $kH$-monomial module, and hence also a $\varphi$-homogeneous $kH$-module. We consider the following diagram of $kH$-monomial modules and $kH$-modules:

$$\begin{array}{c}
1 \otimes_{kH} L_{(H,\varphi)} \subseteq P^{(H,\varphi)} \\
M^{(H,\varphi)} \quad \xrightarrow{f} \\
g(M^{(H,\varphi)}) \subseteq N^{(H,\varphi)},
\end{array}$$

where $f$ is well defined by the hypothesis. Since $1 \otimes_{kH} L_{(H,\varphi)} \cong L_{(H,\varphi)}$ is $k$-projective, we obtain a $k$-linear map $h: 1 \otimes_{kH} L_{(H,\varphi)} \rightarrow M^{(H,\varphi)}$ such that $g \circ h = f$ on $1 \otimes_{kH} L_{(H,\varphi)}$. We consider $h$ as a $kH$-linear map $L_{(H,\varphi)} \rightarrow M$. Since $L_{(H,\varphi)}$ is $\varphi$-homogeneous, we have $h \in \text{mon}_{kH}(L_{(H,\varphi)}; \text{res}^G_H(M))$ by Proposition 1.11 (c). By Proposition 1.7 (a), $h$ extends to a $kG$-monomial homomorphism $h: \text{ind}^G_H L_{(H,\varphi)} \rightarrow M$. Since $g \circ h$ and $f$ coincide on $L_{(H,\varphi)}$, they also coincide on $\text{ind}^G_H L_{(H,\varphi)}$ by $kG$-linearity.

2.3 Proposition Let $V, W \in kG\text{-}\text{mod}$ and $f \in \text{Hom}_{kG}(V, W)$. Assume that

$$\begin{array}{c}
M_s \quad \xrightarrow{\varepsilon} \\
\xrightarrow{f} \\
N_s \quad \xrightarrow{\delta} \\
V \quad \xrightarrow{W}
\end{array}$$

are $kG$-monomial resolutions of $V$ and $W$. Then there exists a $kG$-monomial chain map $f_s = (f_i)_{i \geq 0}: M_s \rightarrow N_s$ such that

$$\begin{array}{c}
M_s \quad \xrightarrow{\varepsilon} \\
\xrightarrow{f} \\
N_s \quad \xrightarrow{\delta} \\
V \quad \xrightarrow{W}
\end{array}$$

commutes. Moreover, $f_s$ is uniquely determined up to $kG$-monomial homotopy by the commutativity of the above diagram; i.e. if $g_s: M_s \rightarrow N_s$ is another $kG$-monomial chain map which renders the above diagram commutative, then there exist $h_i \in \text{mon}_{kG}(M_i, N_{i+1}), i \geq 0$, such that $f_0 - g_0 = \partial_0 \circ h_0$ and $f_i - g_i = h_{i-1} \circ \partial_{i-1} + \partial_i \circ h_i$ for $i \geq 1$. 
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**Proof**  First we prove the existence of \((f_i)_{i \geq 0}\) by induction on \(i\). We consider the following diagram

\[
\begin{array}{ccc}
M_0 & \xrightarrow{\varepsilon} & V \\
\downarrow & & \downarrow f \\
N_0 & \xrightarrow{\delta} & W.
\end{array}
\]

Since \(f(\varepsilon(M_0^{(H,\varphi)})) \subseteq W^{(H,\varphi)}\) for all \((H, \varphi) \in \mathcal{M}_k(G)\), and since \(\delta : N_0^{(H,\varphi)} \to W^{(H,\varphi)}\) is surjective, we can apply Lemma 2.2 and obtain \(f_0 : M_0 \to N_0\) in \(kG\)-mon such that \(\delta \circ f_0 = f \circ \varepsilon\). Now we assume that \(f_0, \ldots, f_n, n \geq 0\), are already defined in such a way that \(\delta \circ f_0 = f \circ \varepsilon\), \(\partial_{i-1} \circ f_i = f_{i-1} \circ \partial_{i-1}\) for \(1 \leq i \leq n\). Then we consider the diagram

\[
\begin{array}{ccc}
M_{n+1} & \xrightarrow{\partial_n} & M_n \\
\downarrow & & \downarrow f_n \\
N_{n+1} & \xrightarrow{\partial_n} & N_n,
\end{array}
\]

and observe that by Lemma 2.2 we obtain a morphism \(f_{n+1} \in \text{mon}_k(G)(M_{n+1}, N_{n+1})\) which makes the above diagram commutative, since \((f_n \circ \partial_n)(M_{n+1}^{(H,\varphi)}) \subseteq \partial_n(N_{n+1}^{(H,\varphi)})\) for all \((H, \varphi) \in \mathcal{M}_k(G)\). In fact, \(\partial_n(N_{n+1}^{(H,\varphi)}) = \ker(\partial_{n-1} : N_{n-1}^{(H,\varphi)} \to N_{n-1}^{(H,\varphi)})\) and \(\partial_{n-1} \circ f_n \circ \partial_n = f_{n-1} \circ \partial_{n-1} \circ \partial_n = 0\). (If \(n = 0\), we have to set \(\partial_{-1} := \varepsilon\) and \(N_{-1} := W\).)

Now suppose that we have another chain map \(g_* : M_* \to N_*\) such that \(\delta \circ g_0 = f \circ \varepsilon\). Then we define inductively \(kG\)-monomial homomorphisms \(h_i : M_i \to N_{i-1}, i \geq 0\), such that

\[
f_i - g_i = \partial_i \circ h_i + h_{i-1} \circ \partial_{i-1}, \quad i \geq 0.
\]  \((5.2)\)

Here we set \(\partial_{-1} = 0\) and \(h_{-1} = 0\). For \(i = 0\) consider the following diagram

\[
\begin{array}{ccc}
M_0 & \xrightarrow{f_0-g_0} & N_0 \\
\downarrow & & \downarrow \partial_0 \\
N_1 & \xrightarrow{\partial_0} & N_0.
\end{array}
\]

We may apply Lemma 2.2, since \((f_0 - g_0)(M_0^{(H,\varphi)}) \subseteq \partial_0(N_1^{(H,\varphi)})\) for all \((H, \varphi) \in \mathcal{M}_k(G)\). In fact, \(\partial_0(N_1^{(H,\varphi)}) = \ker(\delta : N_0^{(H,\varphi)} \to W^{(H,\varphi)})\) and \(\delta \circ (f_0 - g_0) = f \circ \varepsilon - f \circ \varepsilon = 0\). Hence, there is \(h_0 \in \text{mon}_k(G)(M_0, N_1)\) such that \(f_0 - g_0 = \partial_0 \circ h_0\).

Now let \(h_i \in \text{mon}_k(G)(M_i, N_{i+1})\) be already constructed for \(0 \leq i \leq n\), such that \((5.2)\) holds for all \(0 \leq i \leq n\). Then we consider the diagram

\[
\begin{array}{ccc}
M_{n+1} & \xrightarrow{\partial_n} & M_n \\
\downarrow f_{n+1}-g_{n+1} & & \left\uparrow h_n \\
N_{n+2} & \xrightarrow{\partial_{n+1}} & N_{n+1} \\
\downarrow \partial_{n+1} & & \downarrow \partial_n \\
N_{n+1} & \xrightarrow{\partial_n} & N_n.
\end{array}
\]

Here again we may apply Lemma 2.2 for the maps \(f_{n+1} - g_{n+1} - h_n \circ \partial_n\) and \(\partial_{n+1}\).

In fact, we have

\[
\partial_n \circ (f_{n+1} - g_{n+1} - h_n \circ \partial_n) = (f_n - g_n) \circ \partial_n - (\partial_n \circ h_n) \circ \partial_n
\]

\[
= (f_n - g_n) \circ \partial_n (f_n - g_n - h_{n-1} \circ \partial_{n-1}) \circ \partial_n
\]

\[
= 0,
\]
and hence
\[(f_{n+1} - g_{n+1} - h_n \circ \partial_n)(M^{(H,\varphi)}_{n+1}) \subseteq \ker(\partial_n : N^{(H,\varphi)}_{n+1} \to N^{(H,\varphi)}_n) = \partial_{n+1}(N^{(H,\varphi)}_{n+2}),\]
for all \((H, \varphi) \in \mathcal{M}_k(G)\). Lemma 2.2 now implies the existence of a morphism \(h_{n+1} \in \text{mon}_{kG}(M_{n+1}, N_{n+2})\) such that \(f_{n+1} - g_{n+1} = h_n \circ \partial_n + \partial_{n+1} \circ h_{n+1}.\)

2.4 Theorem  Each \(V \in kG\text{-mod}\) has a \(kG\)-monomial resolution, and any two \(kG\)-monomial resolutions \(M_s^{\varepsilon} \to V\) and \(N_s^{\delta} \to V\) of \(V\) are homotopy equivalent as chain complexes in \(kG\text{-mon}\), i.e. chain maps and homotopies are \(kG\)-monomial homomorphisms.

Proof First we prove the existence of a \(kG\)-monomial resolution \(M_s^{\varepsilon} \to V\) of \(V\). We define \(M_0\) by defining its homogeneous components \(M_0([H, \varphi]_G)\) in degree \([H, \varphi]_G \in G\setminus \mathcal{M}_k(G)\) for all \((H, \varphi) \in \mathcal{R}_k(G)\) according to Remark 1.9 (f) (or see also 1.10):

Let \(\varepsilon^{(H,\varphi)} : P_0^{(H,\varphi)} \to V^{(H,\varphi)}\) be \(k\)-linear and surjective with \(P_0^{(H,\varphi)} \in k\text{-mod}^{\text{pr}}\) (here we use the assumption that \(k\) is noetherian). We can provide \(P_0^{(H,\varphi)}\) with the structure of a \(\varphi\)-homogeneous \(kH\)-module, and \(\varepsilon^{(H,\varphi)}\) is automatically \(kH\)-linear. Now set
\[M_0([H, \varphi]_G) := \text{ind}_H^G(P_0^{(H,\varphi)}),\]
where we consider \(P_0^{(H,\varphi)} \in kH\text{-mon}_{\text{ab}}\) by Corollary 1.12 (b). We define \(\varepsilon : M_0 \to V\) by defining its restrictions to the homogeneous components:
\[\varepsilon : M_0([H, \varphi]_G) \to V, \quad g \otimes_{kH} p \mapsto g\varepsilon^{(H,\varphi)}(p),\]
for \(g \in G, \ p \in P_0^{(H,\varphi)}, \ (H, \varphi) \in \mathcal{R}_k(G).\) The map \(\varepsilon\) is obviously \(kG\)-linear and induces surjective maps \(\varepsilon : M_s^{(H,\varphi)} \to V^{(H,\varphi)}\) for all \((H, \varphi) \in \mathcal{R}_k(G)\) (we may choose \(g = 1\) in the above definition). Since \(\varepsilon\) is \(kG\)-linear, surjectivity holds also for all \((H, \varphi) \in \mathcal{M}_k(G)\).

Now assume that we have already defined
\[M_n \xrightarrow{\partial_{n-1}} M_{n-1} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_0} M_0 \xrightarrow{\varepsilon} V\]
such that
\[M_n^{(H,\varphi)} \xrightarrow{\partial_{n-1}} M_{n-1}^{(H,\varphi)} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_0} M_0^{(H,\varphi)} \xrightarrow{\varepsilon} V^{(H,\varphi)} \xrightarrow{} 0\]
is exact for all \((H, \varphi) \in \mathcal{M}_k(G)\). Then let \(M_{n+1} \in kG\text{-mon}\) be defined by the following construction for its homogeneous components in degree \([H, \varphi]_G \in G\setminus \mathcal{M}_k(G)\), \((H, \varphi) \in \mathcal{R}_k(G).\) Let \(\Omega_n^{(H,\varphi)}\) be the kernel of the \(kH\)-linear map \(\partial_{n-1} : M_n^{(H,\varphi)} \to M_{n-1}^{(H,\varphi)}\) (if \(n=0, \partial_{-1}\) is interpreted as \(\varepsilon\)), then \(\Omega_n^{(H,\varphi)}\) is a \(\varphi\)-homogeneous \(kH\)-module.

Let \(\partial_n^{(H,\varphi)} : P_{n+1}^{(H,\varphi)} \to \Omega_n^{(H,\varphi)}\) be a surjective \(kH\)-linear map with a \(\varphi\)-homogeneous module \(P_{n+1}^{(H,\varphi)} \in kH\text{-mod}^{\text{pr}}\) (as for \(P_0^{(H,\varphi)}\) it is enough to choose a surjective \(k\)-linear map from a finitely generated \(k\)-projective module to \(\Omega_n^{(H,\varphi)}\)). We consider \(P_{n+1}^{(H,\varphi)} \in kH\text{-mon}_{\text{ab}}\) by Corollary 1.12 (b) and define
\[M_{n+1}([H, \varphi]_G) := \text{ind}_H^G(P_{n+1}^{(H,\varphi)}).\]
5.2. THE MONOMIAL RESOLUTION

Since $P^{(H,\varphi)}_{n+1}$ is $\varphi$-homogeneous, $M_{n+1}([H,\varphi]_G)$ is homogeneous in degree $[H,\varphi]_G \in G\backslash M_k(G)$.

Next we define for all $(H,\varphi) \in \mathcal{R}_k(G)$

$$
\partial_n: M_{n+1}([H,\varphi]_G) \to M_n, \quad g \otimes_{kH} p \mapsto g \partial_n^{(H,\varphi)}(p),
$$

for $g \in G$ and $p \in P^{(H,\varphi)}_{n+1}$. This defines a $kG$-linear map $\partial_n : M_{n+1} \to M_n$. First note that $\partial_{n-1} \circ \partial_n = 0$, since for $g \in G$ and $p \in P^{(H,\varphi)}_{n+1}$ we have $(\partial_{n-1} \circ \partial_n)(g \otimes_{kH} p) = g(\partial_{n-1} \circ \partial_n)(1 \otimes_{kH} p) = g \partial_{n-1}(\partial_n^{(H,\varphi)}(p)) = 0$, since $\partial_n^{(H,\varphi)}(p) \in \ker \partial_{n-1}$. Next note that $\partial_n$ satisfies the condition in Proposition 1.11 (c). In fact, if $(H,\varphi) \in \mathcal{R}_k(G)$, then

$$
M_{n+1}(H,\varphi) = \bigoplus_{s \in N_G(H,\varphi)/H} s \otimes_{kH} P^{(H,\varphi)}_{n+1}
$$

and

$$
\partial_n(s \otimes_{kH} P^{(H,\varphi)}_{n+1}) = s \cdot \Omega_n^{(H,\varphi)} \subseteq s M_n^{(H,\varphi)} = M_n^{(H,\varphi)} = M^{(H,\varphi)}
$$

for $s \in N_G(H,\varphi)$. And if $(H,\varphi) \in \mathcal{M}_k(G)$ is arbitrary, then $\gamma(H,\varphi) \in \mathcal{R}_k(G)$ for some $s \in G$, and we have

$$
\partial_n(M_{n+1}(H,\varphi)) \subseteq s^{-1} \partial_n(M_{n+1}(\gamma(H,\varphi))) \subseteq s^{-1} M_n^{(H,\varphi)} = M^{(H,\varphi)}.
$$

Hence $\partial_n$ is a $kG$-monomial homomorphism by Prop. 1.11. Finally,

$$
M_{n+1}^{(H,\varphi)} \xrightarrow{\partial_n} M_n^{(H,\varphi)} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_0} M_0^{(H,\varphi)} \xrightarrow{\varepsilon} V^{(H,\varphi)} \to 0
$$

is exact for all $(H,\varphi) \in \mathcal{M}_k(G)$. In fact, we only have to show exactness at degree $n$. For $(H,\varphi) \in \mathcal{R}_k(G)$ we have

$$
\ker(M_n^{(H,\varphi)} \partial_{n-1}) = \Omega_n^{(H,\varphi)} = \partial_n(1 \otimes_{kH} P^{(H,\varphi)}_{n+1}) \subseteq \partial_n(M_{n+1}^{(H,\varphi)}).
$$

For arbitrary $(H,\varphi) \in \mathcal{M}_k(G)$ there is some $s \in G$ such that $\gamma(H,\varphi) \in \mathcal{R}_k(G)$. Then we have by $kG$-linearity and Remark 1.9 (d)

$$
\ker(M_n^{(H,\varphi)} \partial_{n-1}) = s^{-1} \ker(M_n^{\gamma(H,\varphi)} \partial_{n-1}) \subseteq s^{-1} \partial_n(M_{n+1}^{\gamma(H,\varphi)}) = \partial_n(M_{n+1}^{(H,\varphi)}).
$$

Since $\partial_{n-1} \circ \partial_n = 0$, this completes the proof of existence.

Now we prove the uniqueness part of the assertion. Let $M_* \xrightarrow{\varepsilon} V$ and $N_* \xrightarrow{\delta} V$ be two $kG$-monomial resolutions of $V$, then by Proposition 2.3 we have chain maps $f_* = (f_i)_{i \geq 0}$ and $g_* = (g_i)_{i \geq 0}$ in $kG\text{-}\text{mod}$ from $M_*$ to $N_*$ such that they extend the identity on $V$. By the uniqueness part of Proposition 2.3 we then obtain $g_* \circ f_* \sim 1_{M_*}$ and $f_* g_* \sim 1_{N_*}$, where ‘$\sim$’ means homotopic as chain maps in $kG\text{-}\text{mod}$. 

2.5 Remark Proposition 2.3 and Theorem 2.4 can be interpreted as follows: The construction of $kG$-monomial resolutions provides a functor from $kG\text{-}\text{mod}$ to the homotopy category of chain complexes in $kG\text{-}\text{mon}$, i.e. the category of chain complexes in $kG\text{-}\text{mon}$ with homotopy classes of $kG$-monomial chain maps as morphisms. This functor is a full embedding, since — in the situation of Proposition 2.3
mapping $f$ to the homotopy class of $f_*$ is an isomorphism between $\text{Hom}_G(V,W)$ and the $k$-module of homotopy classes of $kG$-monomial chain maps from $M_*$ to $N_*$. In fact, the inverse map is given by associating to a representative $f_* : M_* \to N_*$ of a homotopy class the unique map $f : V \to W$ such that

$$
\begin{array}{ccc}
M_* & \xrightarrow{\epsilon} & V \\
\downarrow f_* & & \downarrow f \\
N_* & \xrightarrow{\delta} & W
\end{array}
$$

commutes. This argument also shows that the full embedding of $kG$–mod into the homotopy category of $kG$–mon has the functor $H_0\mathcal{V}$ (taking homology in degree 0) as a left inverse.

**2.6 Lemma** Let $f : G' \to G$ be a group homomorphism, $V \in kG$–mod$^\text{pt}$ and $M \in kG$–mon. Moreover let $(H', \varphi') \in M_k(G')$.

(a) If $\ker f \cap H' = \ker(f|_{H'}) \not\subseteq \ker \varphi'$, then

$$(\text{res}_f V)^{(H', \varphi')} = 0 \quad \text{and} \quad (\text{res}_f M)^{(H', \varphi')} = 0.$$

(b) If $\ker f \cap H' = \ker(f|_{H'}) \subseteq \ker \varphi'$, then there is a unique $\varphi \in \text{Hom}(f(H'), k^\times)$ with $\varphi \circ f = \varphi'$, and we have

$$(\text{res}_f V)^{(H', \varphi')} = V(f(H'), \varphi) \quad \text{and} \quad (\text{res}_f M)^{(H', \varphi')} = M(f(H'), \varphi).$$

**Proof** (a) Let $h' \in H'$ with $f(h') = 1$ and $\varphi'(h') \neq 1$. Then we have for all $v \in V$ (resp. $v \in M$): $h'v = f(h')v = 1v = v$. Hence, there is no $0 \neq v \in V$ (resp. no $0 \neq v \in M$) with $h'v = \varphi'(h')v$, since $V$ and $M$ are $k$-torsion free. This implies that $(\text{res}_f V)^{(H', \varphi')} = 0$ and $(\mathcal{V}(\text{res}_f M))^{(H', \varphi')} = 0$. But $(\text{res}_f M)^{(H', \varphi')} \subseteq (\mathcal{V}(\text{res}_f M))^{(H', \varphi')}.$

(b) The existence and uniqueness of $\varphi \in \text{Hom}(f(H'), k^\times)$ is clear. For $v \in V$ we have

\begin{align*}
v \in (\text{res}_f V)^{(H', \varphi')} & \iff h'v = \varphi'(h')v \text{ for all } h' \in H' \\
& \iff f(h')v = \varphi(f(h'))v \text{ for all } h' \in H' \\
& \iff hv = \varphi(h)v \text{ for all } h \in f(H') \\
& \iff v \in V(f(H'), \varphi).
\end{align*}

And if $M = M_1 \oplus \ldots \oplus M_m$ is the decomposition of $M$, then we have for each $M_i, 1 \leq i \leq m$:

\begin{align*}
M_i \subseteq (\text{res}_f M)^{(H', \varphi')} & \iff h'v = \varphi'(h')v \text{ for all } h' \in H', \ v \in M_i \\
& \iff f(h')v = \varphi(f(h'))v \text{ for all } h' \in H', \ v \in M_i \\
& \iff hv = \varphi(h)v \text{ for all } h \in f(H'), \ v \in M_i \\
& \iff M_i \subseteq M(f(H'), \varphi).
\end{align*}
2.7 Proposition  
Let $V \in kG-\text{mod}^{pr}$ and let $f: G' \to G$ be a group homomorphism. If $M_* \xrightarrow{\varepsilon} V$ is a $kG$-monomial resolution of $V$, then $\text{res}_f(M_*) \xrightarrow{\varepsilon} \text{res}_f(V)$ is a $kG'$-monomial resolution of $\text{res}_f(V)$.

Proof  
This is obvious with Lemma 2.6.

In the above proposition the hypothesis $V \in kG-\text{mod}^{pr}$ cannot be weakened to $V \in kG-\text{mod}$: Assume that $G'$ is non-trivial and $f: G' \to G$ is the trivial map. Let $1 \neq \varphi' \in \hat{G}'(k)$ such that $\{1 - \varphi'(g') \mid g' \in G'\}$ generates a proper ideal $I$ in $k$. Set $V := k/I$ with the trivial $G'$-action. Then $\ker f \cap G' = G' \not\leq \ker \varphi'$ and therefore $(\text{res}_f M_*)^{(G', \varphi')} = 0$ by Lemma 2.6. But $(\text{res}_f V)^{(G', \varphi')} = \text{res}_f V = V$.

2.8 Definition  
For a finite chain complex

$$M_* = \ldots 0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \ldots \longrightarrow M_m \longrightarrow 0 \ldots$$

in $kG-\text{mon}$ we define the Lefschetz invariant $\Lambda(M_*) \in R_{k+}^{ab}(G)$ of $M_*$ by

$$\Lambda(M_*) := \sum_{i=m}^{n} (-1)^i [M_i],$$

where $[M_i] \in R_{k+}^{ab}(G)$ was defined in 1.22.

2.9 Lemma  
Let $k$ satisfy condition $(\ast)$ and let

$$M_* = \ldots 0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \ldots \longrightarrow M_r \longrightarrow 0 \ldots,$$

$$N_* = \ldots 0 \longrightarrow N_n \longrightarrow N_{n-1} \longrightarrow \ldots \longrightarrow N_s \longrightarrow 0 \ldots$$

be finite homomotopy equivalent chain complexes in $kG-\text{mon}$. Then we have

$$\Lambda(M_*) = \Lambda(N_*).$$

Proof  
By the injectivity of $\rho_{k+}^{Rab}$ (cf. Proposition I.3.2), we only have to show that

$$\pi_H(\text{res}_H^G(\Lambda(M_*))) = \pi_H(\text{res}_H^G(\Lambda(N_*))) \in \hat{H}(k)$$

for all $H \leq G$. If we define integers $c_{(H, \varphi)}$, $d_{(H, \varphi)}$, for $(H, \varphi) \in \mathcal{M}_k(G)$, by setting

$$\pi_H(\text{res}_H^G(\Lambda(M_*))) = \sum_{\varphi \in \hat{H}(k)} c_{(H, \varphi)} \varphi \quad \text{and} \quad \pi_H(\text{res}_H^G(\Lambda(N_*))) = \sum_{\varphi \in \hat{H}(k)} d_{(H, \varphi)} \varphi,$$

it suffices to show $c_{(H, \varphi)} = d_{(H, \varphi)}$ for all $(H, \varphi) \in \mathcal{M}_k(G)$. However, by the definition of $\Lambda(M_*)$ we have

$$c_{(H, \varphi)} = \sum_{i \in \mathbb{Z}} (-1)^i \text{rk}_k(\text{res}_H^G(M_i))(H, \varphi)$$
and

\[ d_{(H,\varphi)} = \sum_{i \in \mathbb{Z}} (-1)^i \text{rk}_k (\text{res}_H^G(N_i))(H, \varphi). \]

Now we know from Remark 1.9 (g) that \( (\text{res}_H^G(M_i))(H, \varphi) = M_i^{(H,\varphi)} \) and similarly for \( N_i \), so that it suffices to show

\[ \sum_{i \in \mathbb{Z}} (-1)^i \text{rk}_k M_i^{(H,\varphi)} = \sum_{i \in \mathbb{Z}} (-1)^i \text{rk}_k N_i^{(H,\varphi)} \]

for all \((H, \varphi) \in M_k(G)\). But this last equation holds, since with \( M^* \) and \( N^* \) also the two chain complexes \( M_i^{(H,\varphi)} \) and \( N_i^{(H,\varphi)} \) of \( k \)-modules are homotopy equivalent. In fact, the original chain maps and homotopies for the homotopy equivalence of \( M^* \) and \( N^* \) induce via the functor \(-^{(H,\varphi)}\) again chain maps and homotopies (cf. Proposition 1.11 (b)).

2.10 Remark Let \( k \) be a field of characteristic \( p > 0 \) and let \( G \) be cyclic of order \( p \). Then \( M_k(G) = \{(E, 1), (G, 1)\} \). Let \( V_i, 1 \leq i \leq p \), denote the indecomposable \( kG \)-module of dimension \( i \). Assume that there is a finite \( kG \)-monomial resolution

\[ 0 \rightarrow M_n \rightarrow \cdots \rightarrow M_0 \rightarrow V \rightarrow 0. \]

Then \( \sum_{i=0}^n (-1)^i \dim_k M_i^{(H,\varphi)} = \dim_k V_i^{(H,\varphi)} \) for all \((H, \varphi) \in M_k(G)\). Hence, by Corollary 1.18 (b), the vector

\[ \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \dim_k V_i^{(G,1)} \\ \dim_k V_i^{(E,1)} \end{pmatrix} \]

is an integral linear combination of the vectors

\[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \dim_k(S^G_{(G,1)})^{(G,1)} \\ \dim_k(S^G_{(G,1)})^{(E,1)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ p \end{pmatrix} = \begin{pmatrix} \dim_k(S^G_{(E,1)})^{(G,1)} \\ \dim_k(S^G_{(E,1)})^{(E,1)} \end{pmatrix}. \]

This implies that \((i - 1)/p\) is an integer, which is only possible for \( i = 1 \).

Hence we have seen that \( V_2, \ldots, V_p \) don’t have finite monomial resolutions. \( V_1 \), the trivial module has a monomial resolution, namely

\[ 0 \rightarrow S^G_{(G,1)} \xrightarrow{\varepsilon} V_1 \rightarrow 0, \]

where \( \varepsilon \) is the identity, if we identify both \( V_1 \) and \( S^G_{(G,1)} \) with the trivial module \( k \).

2.11 Definition Let \( V \in kG\text{-mod} \). The monomial homological dimension \( \dim_{\text{mh}}(V) \) of \( V \) is defined to be \( \infty \), if there is no finite \( kG \)-monomial resolution of \( V \), and it is defined to be \( n \in \mathbb{N}_0 \), if \( n \) is minimal with respect to the existence of a \( kG \)-monomial resolution \( 0 \rightarrow M_n \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_0 \rightarrow V \rightarrow 0 \) of length \( n \).

2.12 Remark If \( V \in kG\text{-mod}_{\text{ab}}^{\text{pr}} \), then we may consider \( M_0 := V \) as a \( kG \)-monomial module by Corollary 1.12 (b), and \( 0 \rightarrow M_0 \xrightarrow{\varepsilon} V \rightarrow 0 \) with \( \varepsilon \) the
identity becomes a $kG$-monomial resolution of $V$. Hence, all $k$-projective abelian $kG$-modules have monomial homological dimension zero.

For the rest of this section let $k = \mathbb{C}$, the field of complex numbers. We omit the index $k$ in most of the notations. In this situation we are able to prove the finiteness of the monomial homological dimension and to give upper bounds. In order to formulate these upper bounds we have to introduce the following notions from [Bo89] or [Bo90]. Moreover we will prove in Corollary 2.16 that the $\mathcal{C}G$-monomial resolution can be considered as a categorical version of the canonical Brauer induction formula $a_G$ of Example III.1.1.

2.13 Definition (cf. [Bo90, 2.17–2.20]) Let $V \in \mathcal{C}G$–mod. We call $(H, \varphi) \in \mathcal{M}(G)$ admissible for $V$, if $V^{(H, \varphi)} \neq \{0\}$. Let $S(V)$ denote the set of non-zero subspaces of $V$, and let $A(V) \subseteq \mathcal{M}(G)$ denote the set of admissible pairs for $V$. We define

$$F_V: A(V) \rightarrow S(V), \quad P_V: S(V) \rightarrow A(V)$$

$$(H, \varphi) \mapsto V^{(H, \varphi)}, \quad W \mapsto \sup\{(H, \varphi) \in \mathcal{M}(G) \mid W \subseteq V^{(H, \varphi)}\}.$$ 

Note that $P_V$ is well-defined (although in general suprema don’t exist in $\mathcal{M}(G)$), since $W \subseteq V^{(E,1)} = V$, and since $W \subseteq V^{(H, \varphi)}$ and $W \subseteq V^{(H', \varphi')}$ implies that the subgroup $U := <H, H'>$ generated by $H$ and $H'$ also acts by scalar multiplication on $W$ via a unique $\mu \in \widehat{U}$ which extends $\varphi$ and $\varphi'$.

The sets $A(V)$ and $S(V)$ are $G$-posets (i.e. $G$ acts by poset automorphisms) with the poset structure inherited from $\mathcal{M}(G)$ and the inclusion in $S(V)$. The maps $F_V$ and $P_V$ form a Galois connection in the sense of [Ro64] or [Walk81], i.e. $F_V$ and $P_V$ are order reversing and

$$W \subseteq (F_V \circ P_V)(W), \quad (H, \varphi) \leq (P_V \circ F_V)(H, \varphi),$$

(5.3)

for all $W \in S(V), (H, \varphi) \in A(V)$. Moreover, $F_V$ and $P_V$ are maps of $G$-sets. The relations in (5.3) imply

$$P_V(W) = (P_V \circ F_V \circ P_V)(W) \quad \text{and} \quad F_V(H, \varphi) = (F_V \circ P_V \circ F_V)(H, \varphi)$$

(5.4)

for all $W \in S(V), (H, \varphi) \in A(V)$. For $(H, \varphi) \in A(V)$ we define the $V$-closure $\text{cl}_V(H, \varphi)$ as

$$\text{cl}_V(H, \varphi) := (P_V \circ F_V)(H, \varphi),$$

in other words, $\text{cl}_V(H, \varphi)$ is the biggest pair $(H', \varphi')$ in $A(V)$ such that $V^{(H, \varphi)} = V^{(H', \varphi')}$. A pair $(H, \varphi) \in A(V)$ is called closed for $V$, if $\text{cl}_V(H, \varphi) = (H, \varphi)$. The closure map is order preserving and has the following properties:

$$(H, \varphi) \leq \text{cl}_V(H, \varphi), \quad \text{cl}_V(\text{cl}_V(H, \varphi)) = \text{cl}_V(H, \varphi), \quad \text{cl}_V(\varphi(H, \varphi)) = \varphi(\text{cl}_V(H, \varphi))$$

(5.5)

for all $(H, \varphi) \in A(V), s \in G$. The subset $\text{Cl}(V) \subseteq A(V)$ of closed pairs forms a $G$-subposet of $\mathcal{M}(G)$ which is isomorphic to the $G$-poset $\{V^{(H, \varphi)} \mid (H, \varphi) \in A(V)\}$ via $F_V$ and $P_V$.

For $(H, \varphi) \in \text{Cl}(V)$ we define the $V$-rank $r_V(H, \varphi)$ by the maximal number $n$ such that there is a strictly ascending chain $(H, \varphi) = (H_0, \varphi_0) < \ldots < (H_n, \varphi_n)$
In particular, the pairs of $V$-rank 0 are exactly the maximal elements in $\text{Cl}(V)$. Finally we define the rank $r(\text{Cl}(V))$ of the poset $\text{Cl}(V)$ as the maximum of all ranks $r_V(H, \varphi)$, $(H, \varphi) \in \text{Cl}(V)$.

2.14 Theorem Each $V \in \mathcal{C}G\text{-mod}$ has a $\mathcal{C}G$-monomial resolution $M_* \to V \to 0$ with the following properties:

(i) $M_i(H, \varphi) = 0$ for all $(H, \varphi) \notin \text{Cl}(V)$, $i \geq 0$.

(ii) $M_i(H, \varphi) = 0$ for all $(H, \varphi) \in \text{Cl}(V)$, $i \geq 0$, with $r_V(H, \varphi) < i$.

In particular, $V$ has finite monomial homological dimension and

$$\dim_{\text{mh}} V \leq r(\text{Cl}(V)).$$

Proof We define

$$M_n \xrightarrow{\partial_{n-1}} M_{n-1} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_0} M_0 \xrightarrow{\varepsilon} V$$

by induction on $n$ with the following properties:

$(A_n)$ $M_i(H, \varphi) = 0$ for all $(H, \varphi) \notin \text{Cl}(V)$, $0 \leq i \leq n$.

$(B_n)$ $M_i(H, \varphi) = 0$ for all $(H, \varphi) \in \text{Cl}(V)$, $0 \leq i \leq n$, with $r_V(H, \varphi) < i$.

$(C_n) M_n^{(H, \varphi)} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_0} M_0^{(H, \varphi)} \xrightarrow{\varepsilon} V^{(H, \varphi)} \to 0$ is exact for all $(H, \varphi) \in \mathcal{M}(G)$.

$(D_n)$ $0 \to M_n^{(H, \varphi)} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_0} M_0^{(H, \varphi)} \xrightarrow{\varepsilon} V^{(H, \varphi)} \to 0$ is exact for all $(H, \varphi) \in \text{Cl}(V)$ with $r_V(H, \varphi) \leq n$.

Note that for $n = r(\text{Cl}(V)) + 1$ the properties $(A_n)$ and $(B_n)$ imply that $M_n = 0$, and property $(C_n)$ shows that $0 \to M_{n-1} \to \cdots \to M_0 \to V$ is a finite $kG$-monomial resolution of $V$. Thus, part (i) and (ii) of the theorem hold.

(a) First we note that it suffices to prove $(C_n)$ only for closed pairs, provided that $(A_n)$ holds:

Let $(H, \varphi) \in \mathcal{M}(G)$ be arbitrary. If $(H, \varphi) \notin A(V)$, then $(H', \varphi') \notin A(V)$ for all $(H', \varphi') \geq (H, \varphi)$. Hence, $M_i(H, \varphi) = M_i^{(H, \varphi)} = 0$ for all $i = 0, \ldots, n$, by $(A_n)$, and also $V^{(H, \varphi)} = 0$.

If $(H, \varphi) \in A(V)$, then $V^{(H, \varphi)} = V^{\text{cl}(H, \varphi)}$ by (5.4), and $M_i^{(H, \varphi)} = M_i^{\text{cl}(H, \varphi)}$ for all $0 \leq i \leq n$. In fact, $M_i^{\text{cl}(H, \varphi)} \subseteq M_i^{(H, \varphi)}$, since $(H, \varphi) \leq \text{cl}(H, \varphi)$. On the other hand, for all $(H', \varphi') \geq (H, \varphi)$ we have $M_i(H', \varphi') = 0$ unless $(H', \varphi')$ is closed (by $(A_n)$), but in this case $(H', \varphi') = \text{cl}(H', \varphi') \geq (H, \varphi)$ and therefore $M_i(H', \varphi') \subseteq M_i^{\text{cl}(H, \varphi)}$. Since $(C_n)$ holds for $\text{cl}(H, \varphi)$ by assumption, it also holds for $(H, \varphi)$.

(b) We begin the induction on $n$ by defining $M_0 \xrightarrow{\varepsilon} V$. Let $(H, \varphi) \in \mathcal{R}(G)$. If $(H, \varphi) \in \text{Cl}(V)$, then we define the $[H, \varphi]_G$-homogeneous component $M_0([H, \varphi]_G)$ as $\text{ind}_H^G(V^{(H, \varphi)})$, where we consider $V^{(H, \varphi)}$ as object in $\mathcal{C}H\text{-mon}_{\text{ab}}$ via Corollary 1.12 (b). The map $\varepsilon$ is defined on $\text{ind}_H^G(V^{(H, \varphi)})$ by $s \otimes_{CH} v \mapsto sv$ for $s \in G$, except for the case $(H, \varphi) \notin \text{Cl}(V)$, where we define $\varepsilon = 0$.

In the following, we shall denote the $C_{\text{hom}}$-monomial components of $V$ by $V_{(H, \varphi)}$ and the $C_{\text{hom}}$-monomial components of $M_i$ by $M_i^{(H, \varphi)}$.
5.2. THE MONOMIAL RESOLUTION

\( v \in V^{(H, \varphi)}. \) (This is analogous to the proof of Theorem 2.4 with \( P_0^{(H, \varphi)} := V^{(H, \varphi)}. \))

If \( (H, \varphi) \notin \text{Cl}(V) \), then we define \( M_0[[H, \varphi]]_G = 0 \) so that \( (A_0) \) is satisfied.

Condition \( (B_0) \) is empty. By part (a) it suffices to prove \( (C_0) \) only for \( (H, \varphi) \in \text{Cl}(V) \), i.e. we have to show that \( \varepsilon: M_0^{(H, \varphi)} \to V^{(H, \varphi)} \) is surjective. But this is clear from the definition of \( \varepsilon \) on \( 1 \otimes \mathbb{C} H V^{(H, \varphi)} \subseteq M_0(H, \varphi) \subseteq M_0^{(H, \varphi)} \) if \( (H, \varphi) \in \mathcal{R}(G) \). If \( (H, \varphi) \notin \mathcal{R}(G) \), then \( (C_0) \) follows from the previous case by the \( kG \)-linearity of \( \varepsilon \). In order to prove \( (D_0) \) we have to assume that \( (H, \varphi) \) is a maximal element in \( \text{Cl}(V) \), hence it is also maximal in \( A(V) \) (by the \( kG \)-linearity of \( \varepsilon \) we may assume that \( (H, \varphi) \in \mathcal{R}(G) \)). This implies that \( N_G(H, \varphi) = H \), since otherwise \( \varphi \) could be extended to a bigger subgroup \( H' \triangleright H \), and \( V^{(H', \varphi)} \neq 0 \) then implies by Clifford theory that \( V^{(H', \varphi)} \neq 0 \) for some \( (H', \varphi') > (H, \varphi) \); this contradicts the maximality of \( (H, \varphi) \) in \( A(V) \). Since \( (H, \varphi) \) is maximal in \( \text{Cl}(V) \) we see, by \( (A_0) \) and \( N_G(H, \varphi) = H \), that \( M_0^{(H, \varphi)} = M_0(H, \varphi) = 1 \otimes \mathbb{C} H V^{(H, \varphi)} \). Thus \( \varepsilon: M_0^{(H, \varphi)} \to V^{(H, \varphi)} \) is an isomorphism and \( (D_0) \) holds.

(c) Next we assume that we have already constructed

\[
M_n \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_0} M_0 \xrightarrow{\varepsilon} V
\]

such that \( (A_n) \) holds. If \( n = 0 \), we interpret \( \partial_{n-1} \) as \( \varepsilon \) and \( M_{n-1} \) as \( V \).

We are going to define \( M_{n+1} \) by defining, for each \( (H, \varphi) \in \mathcal{R}(G) \), the \([H, \varphi]_G\)-homogeneous component \( M_{n+1}([H, \varphi]_G) \) and a \( CG \)-monomial homomorphism \( \partial_n: M_{n+1}([H, \varphi]_G) \to M_n \). So let \( (H, \varphi) \in \mathcal{R}(G) \).

If \( (H, \varphi) \notin \text{Cl}(V) \) we define \( M_{n+1}([H, \varphi]_G) := 0 \), hence \( (A_{n+1}) \) is satisfied.

If \( (H, \varphi) \in \text{Cl}(V) \) and \( r_V(H, \varphi) \leq n \), we also define \( M_{n+1}([H, \varphi]_G) := 0 \). Then, for this \( (H, \varphi) \), \( (B_{n+1}) \) is satisfied, and \( (D_n) \) implies \( (C_{n+1}) \) and \( (D_{n+1}) \) for this \( (H, \varphi) \). Obviously \( (B_{n+1}) \) \( (D_{n+1}) \) are also satisfied for all elements in the \( G \)-orbit \([H, \varphi]_G\) of \( (H, \varphi) \).

If \( (H, \varphi) \notin \text{Cl}(V) \) and \( r_V(H, \varphi) > n + 1 \), then we define \( M_{n+1}([H, \varphi]_G) := \text{ind}^G_H(\Omega^{(H, \varphi)}_n) \) with \( \Omega^{(H, \varphi)}_n := \ker(\partial_{n-1}: M_n^{(H, \varphi)} \to M_{n-1}^{(H, \varphi)}) \), and as in the proof of Theorem 2.4 we define \( \partial_n: M_{n+1}([H, \varphi]_G) \to M_n \) (we may take \( P_n^{(H, \varphi)} := \Omega^{(H, \varphi)}_n \) with the identity map). Then \( (C_{n+1}) \) holds for this \( (H, \varphi) \) as is proved in Theorem 2.4, and the conditions \( (B_{n+1}) \) and \( (D_{n+1}) \) are empty for \( (H, \varphi) \). By the \( kG \)-linearity of \( \partial_n \), \( (B_{n+1}) \) \( (D_{n+1}) \) are also satisfied for all elements in \([H, \varphi]_G\).

If \( (H, \varphi) \in \text{Cl}(V) \) and \( r_V(H, \varphi) = n + 1 \), we will show in Lemma 2.15 that

\[
\Omega^{(H, \varphi)}_n := \ker(M_n^{(H, \varphi)} \xrightarrow{\partial_{n-1}} M_{n-1}^{(H, \varphi)}) \cong \text{ind}^G_H(N_G(H, \varphi)(L(H, \varphi)))
\]

as \( \mathbb{C} N_G(H, \varphi) \)-module for some \( \varphi \)-homogeneous \( \mathbb{C} H \)-submodule \( L(H, \varphi) \subseteq \Omega^{(H, \varphi)}_n \). Considering \( L(H, \varphi) \) as object in \( \mathbb{C} H \text{-} \text{mon}_{ab} \) by Corollary 1.12 (b), we can define

\[ M_{n+1}([H, \varphi]_G) := \text{ind}^G_H(L(H, \varphi)), \]

and

\[ \partial_n: M_{n+1}([H, \varphi]_G) \to M_n, \quad g \otimes \mathbb{C} H v \mapsto gv, \]

for \( g \in G, \ v \in L(H, \varphi) \). The map \( \partial_n \) is a \( CG \)-monomial homomorphism by Proposition 1.7 (a), since \( \partial_n(L(H, \varphi)) \subseteq M_n^{(H, \varphi)} \) (use Proposition 1.11 (c) \( \implies \) (a)).
For this \((H, \varphi), (B_{n+1})\) is empty and \((D_{n+1})\) implies \((C_{n+1})\). We will show that \((D_{n+1})\) holds for \((H, \varphi)\). Since \((A_{n+1})\) and \((B_{n+1})\) hold for all closed pairs of rank less than \(n+1\) we have

\[
M^{(H, \varphi)}_{n+1} = \left( \text{ind}_{H}^{G}(L(H, \varphi)) \right)^{(H, \varphi)} = \bigoplus_{s \in N_G(H, \varphi)/H} s \otimes_{C_{H}} L(H, \varphi) = \text{ind}_{H}^{G}(H, \varphi)(L(H, \varphi)),
\]

and hence by (5.6), \(\partial_{n}\) induces an isomorphism \(M^{(H, \varphi)}_{n+1} \rightarrow \Omega^{(H, \varphi)}_{n+1}\). Again it is obvious that \((B_{n+1}) - (D_{n+1})\) also hold for all \(G\)-conjugates of the pair \((H, \varphi)\).

Therefore, the theorem is proved up to the following lemma.

**2.15 Lemma** Let \(M_{n} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_{0}} M_{0} \xrightarrow{\varepsilon} V\) be such that conditions \((A_{n}) - (D_{n})\) in the proof of Theorem 2.14 are satisfied. Let \((H, \varphi) \in \text{Cl}(V)\) with \(r_{V}(H, \varphi) = n + 1\), then the character \(\theta\) of the \(\mathbb{C}N_G(H, \varphi)\)-module

\[
\Omega^{(H, \varphi)}_{n} := \ker(\partial_{n-1}: M^{(H, \varphi)}_{n} \rightarrow M^{(H, \varphi)}_{n-1})
\]

is a multiple on \(\text{ind}_{H}^{G}(\varphi)\), possibly zero.

**Proof** Let \(N := N_G(H, \varphi)\), then we have the following exact sequence of \(\mathbb{C}N\)-modules

\[
0 \rightarrow \Omega^{(H, \varphi)}_{n+1} \rightarrow M^{(H, \varphi)}_{n} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_{0}} M^{(H, \varphi)}_{0} \xrightarrow{\varepsilon} V^{(H, \varphi)} \rightarrow 0.
\]

As a consequence of the semisimplicity of \(\mathbb{C}N\) we obtain the following equation in the character ring \(R(N)\):

\[
(-1)^{n+1}\theta + \sum_{i=0}^{n} (-1)^{i} \chi_{i} = \nu,
\]

where by \(\chi_{i} \in R(N)\) we denote the character of \(M_{i}^{(H, \varphi)}\), \(0 \leq i \leq n\), and by \(\nu \in R(N)\) we denote the character of \(V^{(H, \varphi)}\). On the other hand we can consider \(M_{i}^{(H, \varphi)}\) as \(kN\)-monomial module in its own right (cf. Remark 1.9 (d)), and since \(\chi_{i} = b_{N}([M_{i}^{(H, \varphi)}])\) and \(\nu = b_{N}a_{N}(\nu)\), the last equation can be written as

\[
(-1)^{n+1}\theta = b_{N}\left(a_{N}(\nu) - \sum_{i=0}^{n} (-1)^{i}[M_{i}^{(H, \varphi)}]\right).
\]  

(5.7)

First note that all the stabilizing pairs of the components of \(M_{i}^{(H, \varphi)} \in \mathbb{C}N_{\text{mon}}, 0 \leq i \leq n\), are of the form \((H' \cap N, \varphi'|_{H' \cap N})\) for some \((H', \varphi') \geq (H, \varphi)\) in \(\mathcal{M}(G)\). Hence, \([M_{i}^{(H, \varphi)}] \in R_{+}^{ab}(N)\) may have non-zero coefficients only at basis elements \([K, \psi]_{N} \in N \backslash \mathcal{M}(N)\) with \((H, \varphi) \leq (K, \psi)\), for all \(0 \leq i \leq n\). The same is true for the element \(a_{N}(\nu)\) by Theorem III.1.2 (xi), since \(\nu|_{\mathfrak{h}}\) is a multiple of \(\varphi\). This shows that we can write

\[
a_{N}(\nu) - \sum_{i=0}^{n} (-1)^{i}[M_{i}^{(H, \varphi)}] = \sum_{[H, \varphi]_{N} \leq [K, \psi]_{N} \in N \backslash \mathcal{M}(N)} \alpha_{[K, \psi]_{N}}[K, \psi]_{N} (5.8)
\]
with suitable $\alpha_{[K, \psi]} N \in \mathbb{Z}$. We will show that

$$\alpha_{[K, \psi]} N = 0 \text{ for all } [K, \psi] N > [H, \varphi] N.$$ (5.9)

This implies that the element in (5.8) is a multiple of $[H, \varphi] N$, and substitution of (5.8) in (5.7) establishes the assertion of the lemma.

In order to show (5.9) let us assume that there is $[K_0, \psi_0] N > [H, \varphi] N$ with $\alpha_{[K_0, \psi_0]} N \neq 0$ and which is maximal under this condition. This will lead us to a contradiction as follows. By the maximality of $[K_0, \psi_0] N$ and by Proposition 1.20 (b) we have

$$
\left[ [K_0, \psi_0] N, \sum_{[H, \varphi] N \leq [K, \psi] N \in N \setminus \mathcal{M}(N)} \alpha_{[K, \psi]} N [K, \psi] N \right]_N
= \sum_{[H, \varphi] N \leq [K, \psi] N \in N \setminus \mathcal{M}(N)} \alpha_{[K, \psi]} N \left[ [K_0, \psi_0] N, [K, \psi] N \right]_N
= \alpha_{[K_0, \psi_0]} N \left[ [K_0, \psi_0] N, [K_0, \psi_0] N \right]_N
= \alpha_{[K_0, \psi_0]} N (N_N(K_0, \psi_0) : K_0)
\neq 0.
$$

On the other hand we obtain

$$
\left[ [K_0, \psi_0] N, a_N(\nu) - \sum_{i=0}^n (-1)^i [M_i^{(H, \varphi)}]_N \right]_N
= [K_0, \psi_0] N, a_N(\nu)]_N - \sum_{i=0}^n (-1)^i \left[ [K_0, \psi_0] N, [M_i^{(H, \varphi)}]_N \right]_N
\text{III.1.2 (iii)} \quad \left( \text{ind}_{K_0}^N(\psi_0, \nu) \right)_N - \sum_{i=0}^n (-1)^i \left[ S_{(K_0, \psi_0)}^N, M_i^{(H, \varphi)} \right]_N
\text{1.20 (a)} \quad \left( \psi_0, \text{res}_{K_0}^N(\nu) \right)_{K_0} - \sum_{i=0}^n (-1)^i \dim_{\mathbb{C}} \left( M_i^{(H, \varphi)} \right)_{K_0, \psi_0}
= \dim_{\mathbb{C}} V(K_0, \psi_0) - \sum_{i=0}^n (-1)^i \dim_{\mathbb{C}} M_i^{(K_0, \psi_0)}.
$$

We will get the desired contradiction by showing that the last expression is zero. If $(K_0, \psi_0)$ is not admissible for the $\mathbb{C}N$-module $V^{(H, \varphi)}$, i.e. $V^{(K_0, \psi_0)} = 0$, then also $M_i^{(K_0, \psi_0)} = 0$ for all $0 \leq i \leq n$ by $(A_n)$, and the above expression is zero. If $(K_0, \psi_0)$ is admissible for $V^{(H, \varphi)}$, then we have $\text{cl}_{V} (H, \varphi) = (H, \varphi) < (K_0, \psi_0) \leq \text{cl}_{V} (K_0, \psi_0)$ which implies $r_{V} (\text{cl}(K_0, \psi_0)) < r_{V} (H, \varphi) = n + 1$. Now $(D_n)$ of the proof of Theorem 2.14 implies that the sequence

$$0 \rightarrow M_n^{cl(K_0, \psi_0)} \rightarrow \ldots \rightarrow M_0^{cl(K_0, \psi_0)} \rightarrow V^{cl(K_0, \psi_0)} \rightarrow 0$$

is exact. But $V^{(K_0, \psi_0)} = V^{cl(K_0, \psi_0)}$ by (5.4), and by an argument we already used in part (a) of the proof of Theorem 2.14 we know that $M_i^{(K_0, \psi_0)} = M_i^{cl(K_0, \psi_0)}$ for all $0 \leq i \leq n$. Hence, also in this case the above expression is zero. \qed
2.16 Corollary Let \( V \in \mathbb{C}G\text{-mod} \) with character \( \chi \), and let \( M_* \xrightarrow{\varepsilon} V \) be a finite \( \mathbb{C}G\)-monomial resolution of \( V \). Then we have
\[
a_G(\chi) = \Lambda(M_*).
\]

Proof It suffices to show that \( \rho_G(a_G(\chi)) = \rho_G(\Lambda(M_*)) \), since \( \rho_G \) is injective. So let \( H \leq G \), then the commutativity of Diagram (2.2) implies
\[
\pi_H \left( \text{res}_{+H}^G(a_G(\chi)) \right) = \rho_H(\text{res}_{H}^G(\chi)) = \sum_{\varphi \in \hat{H}} \dim \mathbb{C} V(\varphi) \cdot \varphi,
\]
and by Remark 1.9 (g) and the exactness of the complex in (5.1) we obtain
\[
\pi_H \left( \text{res}_{+H}^G(\Lambda(M_*)) \right) = \sum_{\varphi \in \hat{H}} \left( \sum_{i \geq 0} (-1)^i \dim \mathbb{C} M_i(\varphi) \right) \cdot \varphi = \sum_{\varphi \in \hat{H}} \dim \mathbb{C} V(\varphi) \cdot \varphi.
\]

2.17 Remark The last theorem shows that the construction of \( \mathbb{C}G\)-monomial resolutions provides a full embedding of \( \mathbb{C}G\text{-mod} \) into the homotopy category of finite chain complexes of \( \mathbb{C}G\)-monomial modules. The corollary to the theorem has the following interpretation: The canonical Brauer induction formula \( a_G: R(G) \rightarrow R^\text{ab}(G) \) is the shadow of the functor of taking monomial resolutions on the level of Grothendieck groups. More precisely, if \( K^b(\mathbb{C}G\text{-mon}) \) denotes the homotopy category of finite chain complexes in \( \mathbb{C}G\text{-mon} \), then we have the following commutative diagram
\[
\begin{array}{ccc}
\mathbb{C}G\text{-mod} & \longrightarrow & K^b(\mathbb{C}G\text{-mon}) \\
\downarrow & & \downarrow \Lambda \\
R(G) & \xrightarrow{a_G} & R^\text{ab}(G),
\end{array}
\]
where the vertical arrows stand for taking characters and Lefschetz invariants, and the upper horizontal arrow is the functor of taking \( \mathbb{C}G\)-monomial resolutions.

2.18 Lemma (Stopping criterion) Let \( n \geq 0 \) and let
\[
M_n \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_0} M_0 \xrightarrow{\varepsilon} V
\]
be a chain complex with \( M_n, \ldots, M_0 \in \mathbb{C}G\text{-mon} \), \( V \in \mathbb{C}G\text{-mod} \), \( \partial_{n-1}, \ldots, \partial_0 \) \( \mathbb{C}G\)-monomial homomorphisms and \( \varepsilon \in \text{Hom}_{\mathbb{C}G}(M_0, V) \) such that
\[
M_n^{(H, \varphi)} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_0} M_0^{(H, \varphi)} \xrightarrow{\varepsilon} V^{(H, \varphi)} \longrightarrow 0
\]
is exact for all \((H, \varphi) \in M(G)\). Let
\[
U := \ker \partial_{n-1} \quad \text{and} \quad U^{(H, \varphi)} := U \cap M_n^{(H, \varphi)} = \ker \left( M_n^{(H, \varphi)} \xrightarrow{\partial_{n-1}} M_{n-1}^{(H, \varphi)} \right)
\]
for all \((H, \varphi) \in M(G)\) (for \( n = 0 \) we set \( \partial_{-1} = \varepsilon \) and \( M_{-1} = V \)). Then the following are equivalent:

...
(a) The above sequence can be completed to a $CG$-monomial resolution

$$ 0 \to M_{n+1} \xrightarrow{\partial_n} M_n \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_0} M_0 \xrightarrow{\varepsilon} V \to 0 $$

of $V$.

(b) For each $(H, \varphi) \in \mathcal{M}(G)$ the character of the $\mathbb{C}N_G(H, \varphi)$-module

$$ U^{(H, \varphi)}/ \sum_{(H, \varphi) < (H', \varphi')} U^{(H', \varphi')} $$

is a multiple of $\text{ind}_{H}^{N_G(H, \varphi)}(\varphi)$.

**Proof**

(a) $\implies$ (b): Under the hypothesis (a), the map $\partial_n$ induces isomorphisms $M_{n+1} \xrightarrow{\partial_n} U$ and $M_{n+1}^{(H, \varphi)} \xrightarrow{\partial_n} U^{(H, \varphi)}$ for all $(H, \varphi) \in \mathcal{M}(G)$. Hence, there are resulting isomorphisms

$$ \overline{\partial_n}: M_{n+1}^{(H, \varphi)}/ \sum_{(H, \varphi) < (H', \varphi')} M_{n+1}^{(H', \varphi')} \xrightarrow{\sim} U^{(H, \varphi)}/ \sum_{(H, \varphi) < (H', \varphi')} U^{(H', \varphi')} $$

of $\mathbb{C}N_G(H, \varphi)$-modules. But clearly

$$ M_{n+1}^{(H, \varphi)}/ \sum_{(H, \varphi) < (H', \varphi')} M_{n+1}^{(H', \varphi')} \cong M_{n+1}(H, \varphi) $$

as $\mathbb{C}N_G(H, \varphi)$-modules, and the character of the latter module is a multiple of $\text{ind}_{H}^{N_G(H, \varphi)}(\varphi)$.

(b) $\implies$ (a): For all $(H, \varphi) \in \mathcal{R}(G)$ let $X(H, \varphi) \in \mathbb{C}N_G(H, \varphi)\text{-mod}$ be a complement of $\sum_{(H, \varphi) < (H', \varphi')} U^{(H', \varphi')} U^{(H, \varphi)}$. Then (b) implies that there is a $\varphi$-homogeneous $CH$-submodule $L_{(H, \varphi)} \subseteq X(H, \varphi)$ such that

$$ X(H, \varphi) = \bigoplus_{s \in N_G(H, \varphi)/H} sL_{(H, \varphi)}. \quad (5.10) $$

Now we extend the definition of $X(H, \varphi)$ and $L_{(H, \varphi)}$ to all $(H, \varphi) \in \mathcal{M}(G)$ by setting

$$ X(\underline{g}(H, \varphi)) := g \cdot X(H, \varphi) \quad \text{and} \quad L_{s(H, \varphi)} := g \cdot L_{(H, \varphi)} \subseteq X(\underline{g}(H, \varphi)). $$

Then (5.10) holds for all $(H, \varphi) \in \mathcal{M}(G)$ and we have decompositions

$$ U^{(H, \varphi)} = \bigoplus_{(H, \varphi) \leq (H', \varphi')} X(H', \varphi') $$

for all $(H, \varphi) \in \mathcal{M}(G)$ and in particular (with $(H, \varphi) = (E, 1)$)

$$ U = \bigoplus_{(H, \varphi) \in \mathcal{M}(G)} X(H, \varphi) = \bigoplus_{s \in N_G(H, \varphi)/H} sL_{(H, \varphi)}. $$
The last decomposition furnishes $U$ with the structure of a $CG$-monomial module. We will denote this module by $M_{n+1}$. The $(H,\varphi)$-homogeneous component of $M_{n+1}$ is given by $X(H,\varphi)$, so that

$$U^{(H,\varphi)} = \bigoplus_{(H',\varphi') \leq (H,\varphi)} X(H',\varphi').$$

Now we define $\partial_n: M_{n+1} \to M_n$ by the embedding $U \subseteq M_n$. Then $\partial_n$ is a $CG$-monomial homomorphism, since

$$M_n^{(H,\varphi)} = U^{(H,\varphi)} \subseteq M_n^{(H,\varphi)}$$

for all $(H,\varphi) \in M(G)$. Finally,

$$0 \to M_{n+1} \xrightarrow{\partial_n} M_n \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_0} M_0 \xrightarrow{\varepsilon} V \to 0$$

is a $CG$-monomial resolution of $V$, since

$$\partial_n M_{n+1}^{(H,\varphi)} = U^{(H,\varphi)} = \ker M_n^{(H,\varphi)} \xrightarrow{\partial_n} M_n^{(H,\varphi)}$$

for all $(H,\varphi) \in M(G)$. □

We complete the stopping criterion by determining the $CG$-monomial modules of monomial homological dimension zero.

**2.19 Proposition** Let $V \in CG-\text{mod}$, then we have: $\dim_{mh} V = 0 \iff V \in CG-\text{mod}_{ab}$.

**Proof** That all abelian $CG$-modules have monomial homological dimension zero has been proved in Remark 2.12 in a more general situation. Now assume that $0 \to M \xrightarrow{\varepsilon} V \to 0$ is a $CG$-monomial resolution of $V$. Then Corollary 2.16 yields $a_G(\chi) = |M|$, where $\chi$ is the character of $V$. This implies that

$$a_G(\chi) = \sum_{[H,\varphi] \in G \setminus M(G)} \alpha_{[H,\varphi]G}(\chi)[H,\varphi]_G$$

with $\alpha_{[H,\varphi]G}(\chi) \in \mathbb{N}_0$ for all $[H,\varphi]_G \in G \setminus M(G)$. Now we have the equation

$$\sum_{(H,\varphi) \in R(G)} \alpha_{[H,\varphi]G}(\chi)(G : H) = \sum_{(H,\varphi) \in R(G)} \alpha_{[H,\varphi]G}(\chi)$$

by Theorem III.1.2 (x). Since all $\alpha_{[H,\varphi]G}(\chi)$ are nonnegative, these two equations imply $\alpha_{[H,\varphi]G}(\chi) = 0$ for $H \neq G$. Hence, $\chi = b_G a_G(\chi) = \sum_{\varphi \in G} \alpha_{[H,\varphi]G}(\chi) \cdot \varphi$, and $V$ is abelian. □

**2.20 Proposition** Let $V,W \in CG-\text{mod}$. Then

$$\dim_{mh}(V \oplus W) = \max\{\dim_{mh} V, \dim_{mh} W\}.$$

**Proof** Taking the direct sum of $CG$-monomial resolutions of $V$ and $W$ results in a $CG$-monomial resolution of $V \oplus W$. Hence

$$\dim_{mh}(V \oplus W) \leq \max\{\dim_{mh} V, \dim_{mh} W\}.$$
By symmetry we only have to show $\dim_{\text{mh}} V \leq \dim_{\text{mh}} (V \oplus W)$ in order to complete the proof. Let

$$0 \longrightarrow N_n \longrightarrow \ldots \longrightarrow N_0 \longrightarrow V \oplus W \longrightarrow 0$$

be a $\text{CG}$-monomial resolution of $V \oplus W$ with $n = \dim_{\text{mh}} (V \oplus W)$. And let

$$0 \longrightarrow M_{m} \overset{\partial_{m-1}}{\longrightarrow} \ldots \overset{\partial_0}{\longrightarrow} M_0 \overset{\varepsilon}{\longrightarrow} V \longrightarrow 0$$

be a $\text{CG}$-monomial resolution of $V$. It suffices to show that if $m > n$, then this resolution of $V$ can be shortened. By Proposition 2.19 we may assume that $n \geq 1$, hence $m \geq 2$. We apply Proposition 2.3 to the canonical inclusion $\iota: V \rightarrow V \oplus W$ and to the canonical projection $\pi: V \oplus W \rightarrow V$. This yields the following commutative diagram with $\text{CG}$-monomial homomorphisms $\iota_i, \pi_i, 0 \leq i \leq n$,

$$
\begin{array}{cccccccc}
0 & \longrightarrow & M_{m} & \overset{\partial_{m-1}}{\longrightarrow} & \ldots & \overset{\partial_0}{\longrightarrow} & M_0 & \overset{\varepsilon}{\longrightarrow} & V & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \quad & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \ldots & \longrightarrow & N_0 & \longrightarrow & V \oplus W & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \quad & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{m} & \overset{\partial_{m-1}}{\longrightarrow} & \ldots & \overset{\partial_0}{\longrightarrow} & M_0 & \overset{\varepsilon}{\longrightarrow} & V & \longrightarrow & 0 \\
\end{array}
$$

Since $\pi \iota = \text{id}_V$, the uniqueness part of Proposition 2.3 implies that $\pi \iota_i: M_{s} \rightarrow M_{s}$ and $\text{id}_{M_{s}}$ are homotopy equivalent. Hence there are $\text{CG}$-monomial homomorphisms $h_i: M_i \rightarrow M_{i+1}$, $i \geq 0$ ($M_i = 0$ for $i > m$), such that

$$\text{id}_{M_i} - \pi_i \circ \iota_i = h_{i-1} \circ \partial_{i-1} + \partial_i \circ h_i$$

for all $i \geq 0$, where we interpret $h_{-1}$ and $\partial_{-1}$ as zero. In particular, for $i = m$ we obtain

$$\text{id}_{M_m} = h_{m-1} \circ \partial_{m-1}. \quad (5.11)$$

Now we apply the stopping criterion to the chain complex

$$M_{m-2} \overset{\partial_{m-3}}{\longrightarrow} \ldots \overset{\partial_0}{\longrightarrow} M_0 \overset{\varepsilon}{\longrightarrow} V \longrightarrow 0.$$

For each $(H, \varphi) \in \mathcal{M}(G)$ let

$$U^{(H, \varphi)} := \ker \left( \partial_{m-3}: M_{m-2}^{(H, \varphi)} \rightarrow M_{m-3}^{(H, \varphi)} \right) \subseteq M_{m-2}$$

as in Lemma 2.18. If $m = 2$, we interpret $\partial_{m-3}$ as $\varepsilon$ and $M_{m-3}$ as $V$. The stopping criterion says that it suffices to prove that the character of the $\mathbb{C}N_G(H, \varphi)$-module

$$U^{(H, \varphi)} \bigg/ \sum_{(H, \varphi') \prec (H, \varphi')} U^{(H', \varphi')}$$

is a multiple of $\text{ind}_{H}^{N_G(H, \varphi)}(\varphi)$, for all $(H, \varphi) \in \mathcal{M}(G)$.

Now let $T := \ker(h_{m-1}) \subseteq M_{m-1}$, and for all $(H, \varphi) \in \mathcal{M}(G)$ let

$$T^{(H, \varphi)} := \ker \left( h_{m-1}: M_{m-1}^{(H, \varphi)} \rightarrow M_{m}^{(H, \varphi)} \right) \subseteq M_{m-1}.$$
CHAPTER 5. MONOMIAL RESOLUTIONS

Then (5.11) implies that \( M_m = \partial_{m-1}(M_m) \oplus T \) and also that

\[
M_{m-1}^{(H,\varphi)} = \partial_{m-1}(M_m^{(H,\varphi)}) \oplus T^{(H,\varphi)}
\]

for all \((H, \varphi) \in \mathcal{M}(G)\). Since \( M_* \to V \to 0 \) is a \( CG \)-monomial resolution of \( V \), we have \( U^{(H,\varphi)} = \partial_{m-2}(M_{m-1}^{(H,\varphi)}) \) for all \((H, \varphi) \in \mathcal{M}(G)\). Together with (5.12) this implies \( U^{(H,\varphi)} = \partial_{m-2}(T^{(H,\varphi)}) \). But since

\[
T^{(H,\varphi)} \cap \ker \partial_{m-2} = T^{(H,\varphi)} \cap \partial_{m-1}(M_m) \subseteq T \cap \partial_{m-1}(M_m) = 0,
\]

the \( \mathbb{C}N_G(H, \varphi) \)-linear map

\[
\partial_{m-2} : T^{(H,\varphi)} \sim \to U^{(H,\varphi)}
\]

is an isomorphism for all \((H, \varphi) \in \mathcal{M}(G)\). This induces a \( \mathbb{C}N_G(H, \varphi) \)-isomorphism

\[
T^{(H,\varphi)} / \sum_{(H, \varphi) < (H', \varphi')} T^{(H',\varphi')} \cong U^{(H,\varphi)} / \sum_{(H, \varphi) < (H', \varphi')} U^{(H',\varphi')}
\]

for each \((H, \varphi) \in \mathcal{M}(G)\), and it suffices to show that the character of the left one of these two \( \mathbb{C}N_G(H, \varphi) \)-modules is a multiple of \( \text{ind}_G^{N_G(H, \varphi)}(\varphi) \). In order to show this we note that (5.12) induces a \( \mathbb{C}N_G(H, \varphi) \)-isomorphism

\[
M_{m-1}^{(H,\varphi)} / \sum_{(H, \varphi) < (H', \varphi')} M_{m-1}^{(H',\varphi')} \cong \\
\cong \partial_{m-1}(M_m^{(H,\varphi)}) / \sum_{(H, \varphi) < (H', \varphi')} \partial_{m-1}(M_m^{(H',\varphi)}) \oplus \\
\oplus T^{(H,\varphi)} / \sum_{(H, \varphi) < (H', \varphi')} T^{(H',\varphi')}.
\]

The first one of these isomorphic modules is \( \mathbb{C}N_G(H, \varphi) \)-isomorphic to \( M_{m-1}(H, \varphi) \) whose character is a multiple of \( \text{ind}_G^{N_G(H, \varphi)}(\varphi) \). The same is true for the first summand in the second module. In fact, since \( \partial_{m-1} \) is injective, we have

\[
\partial_{m-1}(M_m^{(H,\varphi)}) / \sum_{(H, \varphi) < (H', \varphi')} \partial_{m-1}(M_m^{(H',\varphi)}) \cong \\
\cong M_m^{(H,\varphi)} / \sum_{(H, \varphi) < (H', \varphi')} M_m^{(H',\varphi')} \cong M_m(H, \varphi)
\]

which now completes the proof. \( \square \)

Repeated use of the above Proposition yields

2.21 Corollary Let \( V \in \mathbb{C}G-\text{mod} \), then the monomial homological dimension of \( V \) is the maximum of the monomial homological dimensions of the irreducible constituents of \( V \).

\( \square \)
5.3 Monomial resolutions as projective resolutions

We keep the notation of the two previous sections. In this section we are going to introduce an interpretation of the monomial resolution as a projective resolution by embedding both $kG$-$\text{mod}$ and $kG$-$\text{mon}$ into an abelian category $\mathcal{C}$ such that the objects of $kG$-$\text{mon}$ become projective objects in $\mathcal{C}$. The construction of $\mathcal{C}$ from $kG$-$\text{mon}$ is a standard method, but for the reader’s convenience we will give it in detail. Recall that $\mathcal{V}: kG$-$\text{mon} \to kG$-$\text{mod}^{\text{pr}}$ denotes the forgetful functor.

### 3.1 Definition

We define $\mathcal{C}$ as the category of contravariant $k$-functors from $kG$-$\text{mon}$ to $k$-$\text{mod}$, i.e. contravariant functors which are $k$-linear on morphisms. The morphisms in $\mathcal{C}$ are the natural transformations between the contravariant functors in $\mathcal{C}$. We define $k$-functors

$$\mathcal{I}: kG$-$\text{mod} \to \mathcal{C}, \quad V \mapsto \text{Hom}_{kG}(\mathcal{V}, V)$$

and

$$\mathcal{J}: kG$-$\text{mon} \to \mathcal{C}, \quad M \mapsto \text{mon}_{kG}(\mathcal{V}, M).$$

For $V, W \in kG$-$\text{mod}$ and $f \in \text{Hom}_{kG}(V, W)$ we define

$$\mathcal{I}(f) := \text{Hom}_{kG}(\mathcal{V}, f): \text{Hom}_{kG}(\mathcal{V}, V) \to \text{Hom}_{kG}(\mathcal{V}, W),$$

and for $M, N \in kG$-$\text{mon}$ and $g \in \text{mon}_{kG}(M, N)$ we define

$$\mathcal{J}(g) := \text{mon}_{kG}(\mathcal{V}, g): \text{mon}_{kG}(\mathcal{V}, M) \to \text{mon}_{kG}(\mathcal{V}, N).$$

### 3.2 Proposition

The category $\mathcal{C}$ is abelian, and $\mathcal{I}$ and $\mathcal{J}$ are full embeddings, i.e. injective on objects and bijective on morphisms.

**Proof** Clearly $\mathcal{C}$ is an abelian category by pointwise constructions in $k$-$\text{mod}$, since $k$-$\text{mod}$ is abelian. For example, if $\varphi: \mathcal{F} \to \mathcal{G}$ is a morphism in $\mathcal{C}$, then $\ker(\varphi) \in \mathcal{C}$ is defined as the functor $M \mapsto \ker(\varphi_M) \subseteq \mathcal{F}(M), M \in kG$-$\text{mon}$.

Clearly $\mathcal{J}$ is a full embedding by Yoneda’s Lemma. In order to see that $\mathcal{I}$ is a full embedding we have to work harder. First we define a $k$-functor

$$\mathcal{T}: \mathcal{C} \to kG$-$\text{mod}$$

by the following construction. For $\mathcal{F} \in \mathcal{C}$ set $\mathcal{T}(\mathcal{F}) := \mathcal{F}(S_{(E, 1)}^{G})$ which is a $k$-module by definition. We endow $V := \mathcal{F}(S_{(E, 1)}^{G})$ with the structure of a $kG$-module: Let $g \in G$, then we define left multiplication on $V$ by $g$ as

$$\mathcal{F}(\hat{g}): \mathcal{F}(S_{(E, 1)}^{G}) \to \mathcal{F}(S_{(E, 1)}^{G}),$$

where $\hat{g} \in \text{mon}_{kG}(S_{(E, 1)}^{G}, S_{(E, 1)}^{G})$ is defined by

$$\hat{g}: S_{(E, 1)}^{G} \to S_{(E, 1)}^{G}, \quad s \otimes_{kE} \alpha \mapsto sg \otimes_{kE} \alpha,$$
for $s \in G$ and $\alpha \in k$.

On morphisms we define $\mathcal{T}$ as follows. Let $\mathcal{F}, \mathcal{G} \in \mathcal{C}$ and let $\varphi: \mathcal{F} \to \mathcal{G}$ be a natural transformation. Then

$$
\mathcal{T}(\varphi) := \varphi_{S^G_{(E,1)}}, \quad \mathcal{F}(S^G_{(E,1)}), \to \mathcal{G}(S^G_{(E,1)}).
$$

It is easy to check that $\mathcal{T}(\varphi)$ is $kG$-linear with respect to the $G$-action defined above.

The functor $\mathcal{T} \circ \mathcal{I}: kG-\text{mod} \to kG-\text{mod}$ is naturally equivalent to the identity functor via the following isomorphism

$$
\zeta_V: \mathcal{T}\mathcal{I}(V) = \text{Hom}_{kG}(\mathcal{V}(S^G_{(E,1)}),V) \sim \to V, \quad f \mapsto f(1 \otimes_{kE} 1),
$$

for $V \in kG-\text{mod}$, since the following diagram

$$
\begin{array}{ccc}
\mathcal{T}(\mathcal{I}(V)) & \mathcal{T}(\mathcal{I}(f)) & \mathcal{T}(\mathcal{I}(W)) \\
\zeta_V & \zeta_V \\
V & W
\end{array}
$$

commutes. This shows that $V$ can be recovered from $\mathcal{I}(V)$.

It is easy to check that for $V, W \in kG-\text{mod}$ and $f \in \text{Hom}_{kG}(V,W)$ the following diagram is commutative

$$
\begin{array}{ccc}
\mathcal{T}\mathcal{I}(V) & \mathcal{T}\mathcal{I}(f) & \mathcal{T}\mathcal{I}(W) \\
\zeta_V & \zeta_W \\
V & W
\end{array}
$$

This shows that $\mathcal{I}$ is injective on morphisms.

Finally we show that $\mathcal{I}$ is surjective on morphisms. Let $V, W \in kG-\text{mod}$ and let $\varphi: \mathcal{I}(V) \to \mathcal{I}(W)$ be a natural transformation. We define $f: V \to W$ by the commutative diagram

$$
\begin{array}{ccc}
\mathcal{T}\mathcal{I}(V) & \mathcal{T}\mathcal{I}(\varphi) & \mathcal{T}\mathcal{I}(W) \\
\zeta_V & \zeta_W \\
V & W
\end{array}
$$

Then $f$ is $kG$-linear, since the other three maps in the above diagram are $kG$-linear. We will show that

$$
\varphi_M = \mathcal{I}(f)(M): \quad \text{Hom}_{kG}(\mathcal{V}(M), V) \to \text{Hom}_{kG}(\mathcal{V}(M), W)
$$

for all $M \in kG-\text{mon}$. Let $h \in \text{Hom}_{kG}(\mathcal{V}(M), V)$ and $m \in M$, then we have to show that

$$
(\varphi_M(h))(m) = (fh)(m).
$$

There is a unique $g \in \text{mon}_{kG}(S^G_{(E,1)}, M)$ with $g(1 \otimes_{kE} 1) = m$, and the commutativity
of the following diagram

\[
\begin{array}{ccc}
\mathcal{I}(V)(M) = \text{Hom}_{kG}(\mathcal{V}(M), V) & \xrightarrow{\phi_M} & \text{Hom}_{kG}(\mathcal{V}(M), W) = \mathcal{I}(W)(M) \\
\text{Hom}_{kG}(\mathcal{V}(g), V) & \downarrow & \downarrow \text{Hom}_{kG}(\mathcal{V}(g), W) \\
\mathcal{I}(V)(S^G_{(E,1)}) = \text{Hom}_{kG}(\mathcal{V}(S^G_{(E,1)}), V) & \xrightarrow{\phi_{g^G_{(E,1)}}} & \text{Hom}_{kG}(\mathcal{V}(S^G_{(E,1)}), W) = \mathcal{I}(W)(S^G_{(E,1)})
\end{array}
\]

\[
\begin{array}{c}
\zeta_V \\
\downarrow \\
V \\
\xrightarrow{f} \\
W
\end{array}
\]

implies

\[
(\varphi_M(h))(m) = (\varphi_M(h))(\mathcal{V}(g)(1 \otimes_{kE} 1)) = (\varphi_M(h) \circ \mathcal{V}(g))(1 \otimes_{kE} 1) = (\zeta_W \circ \text{Hom}_{kG}(\mathcal{V}(g), W) \circ \varphi_M)(h) = (f \circ \zeta_V \circ \text{Hom}_{kG}(\mathcal{V}(g), V))(h) = (f \circ \zeta_V)(h \circ \mathcal{V}(g)) = f((h\mathcal{V}(g))(1 \otimes_{kE} 1)) = f(h(m)).
\]

\[\square\]

### 3.3 Proposition

The contravariant functor \(\mathcal{J}(M) \in \mathcal{C}\) is projective for \(M \in kG\text{-}\text{mon}\).

**Proof** Let \(\mathcal{F} := \text{mon}_{kG}(-, M) \in \mathcal{C}\) and let \(\mathcal{G}, \mathcal{H} \in \mathcal{C}\) be arbitrary. Assume that we have a diagram

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\psi} & \mathcal{H} \\
\downarrow & & \downarrow \\
\mathcal{G} & & \mathcal{H}
\end{array}
\]

(5.13)

in \(\mathcal{C}\), where \(\psi\) is an epimorphism, i.e. \(\psi_N : \mathcal{G}(N) \to \mathcal{H}(N)\) is surjective for all \(N \in kG\text{-}\text{mon}\).

We will define \(\mu \in \mathcal{C}(\mathcal{F}, \mathcal{G})\) with \(\psi \circ \mu = \varphi\). First we evaluate the above diagram at \(M\). Since \(\mathcal{F}(M) = \text{mon}_{kG}(M, M)\) is \(k\)-projective (cf. Proposition 1.20 (c)), we can complete diagram (5.13) evaluated at \(M\) by a \(k\)-linear map \(\mu_M : \mathcal{F}(M) \to \mathcal{G}(M)\).

Now let \(N \in kG\text{-}\text{mon}\) be arbitrary, and let \(f \in \mathcal{F}(N) = kG\text{-}\text{mon}(N, M)\), then we define

\[\mu_N(f) := \mathcal{G}(f)(\mu_M(\text{id}_M)) \in \mathcal{G}(N)\].

The map \(\mu_N\) is \(k\)-linear, since \(\mathcal{G}\) is a \(k\)-functor. The collection \(\mu = (\mu_N), N \in kG\text{-}\text{mon}\), is a natural transformation from \(\mathcal{F}\) to \(\mathcal{G}\), since the following diagram

\[
\begin{array}{ccc}
\mathcal{F}(N') & \xrightarrow{\mathcal{F}(g)} & \mathcal{F}(N) \\
\downarrow \mu_{N'} & & \downarrow \mu_N \\
\mathcal{G}(N') & \xrightarrow{\mathcal{G}(g)} & \mathcal{G}(N)
\end{array}
\]
commutes for all \( N, N' \in kG-\text{mon} \), \( g \in \text{mon}_{kG}(N, N') \). In fact, let \( h \in \mathcal{F}(N') = \text{mon}_{kG}(N', M) \), then

\[
\mu_N(\mathcal{F}(g)(h)) = \mu_N(h \circ g) = \mathcal{G}(h \circ g)(\mu_M(\text{id}_M)) = \mathcal{G}(g)(\mathcal{G}(h)(\mu_M(\text{id}_M))) = \mathcal{G}(g)(\mu_{N'}(h)).
\]

\[\square\]

3.4 Remark  
In general, \( \mathcal{C} \) has more projective objects than \( \mathcal{J}(M) \), \( M \in kG-\text{mon} \). In fact, let \( k = \mathbb{C} \), \( M := S^G_{(H, \varphi)} \), for some \((H, \varphi) \in \mathcal{M}_k(G)\), then \( \mathcal{C}(\mathcal{J}(M), \mathcal{J}(M)) \cong \text{mon}_{kG}(M, M) \) by Yoneda’s Lemma, and the latter \( k \)-algebra is isomorphic to the group algebra \( k[N_G(H, \varphi)/H] \) by the map

\[
k[N_G(H, \varphi)/H] \xrightarrow{\sim} \text{mon}_{kG}(S^G_{(H, \varphi)}, S^G_{(H, \varphi)}), \quad n \mapsto (g \otimes_{kH} 1 \mapsto gn \otimes_{kH} 1).
\]

Assume that \( H \neq N_G(H, \varphi) \), then \( k[N_G(H, \varphi)/H] \) may have non-trivial idempotents, and therefore \( \mathcal{J}(M) \) is decomposable in \( \mathcal{C} \). But \( M \) is not decomposable in \( kG-\text{mon} \). Hence, each proper summand of \( \mathcal{J}(M) \) is projective in \( \mathcal{C} \) and not of the form \( \mathcal{J}(N) \) for some \( N \in kG-\text{mon} \), since \( \mathcal{J} \) is a full embedding.

3.5 Lemma  
For each \( M \in kG-\text{mon} \) and each \( V \in kG-\text{mod} \) there is a \( k \)-isomorphism

\[
\kappa_{M, V} : \text{Hom}_{kG}(\mathcal{V}(M), V) \xrightarrow{\sim} \mathcal{C}(\mathcal{J}(M), \mathcal{I}(V))
\]

\[
f \mapsto \left( \begin{array}{ccc}
\text{mon}_{kG}(N, M) & \rightarrow & \text{Hom}_{kG}(\mathcal{V}(N), V) \\
h & \mapsto & f \circ \mathcal{V}(h)
\end{array} \right)_{N \in kG-\text{mon}}
\]

whose inverse maps \( \varphi \) to \( \varphi_M(\text{id}_M) \). Moreover, \( \kappa_{M, V} \) is functorial in \( M \) and \( V \), i.e. \( \kappa \) is a functorial equivalence

\[
\text{Hom}_{kG}(\mathcal{V}(M), V) \xrightarrow{\sim} \mathcal{C}(\mathcal{J}(M), \mathcal{I}(V))
\]

of functors from \( kG-\text{mon}^{\text{op}} \times kG-\text{mod} \) to \( k-\text{mod} \).

Proof  
The proof is an easy verification and is left to the reader. \[\square\]

3.6 Proposition  
Let

\[
\ldots \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_0} M_0 \xrightarrow{\varepsilon} V \rightarrow 0
\]

be a chain complex with \( M_i \in kG-\text{mon} \) and \( \partial_i \in \text{mon}_{kG}(M_{i+1}, M_i) \) for \( i \geq 0 \), \( V \in kG-\text{mod} \) and \( \varepsilon \in \text{Hom}_{kG}(\mathcal{V}(M_0), V) \). Then the following are equivalent:

(a) \( M \xrightarrow{\varepsilon} V \rightarrow 0 \) is a monomial resolution of \( V \), i.e.

\[
\ldots \xrightarrow{\partial_1} M_1^{(H, \varphi)} \xrightarrow{\partial_0} M_0^{(H, \varphi)} \xrightarrow{\varepsilon} V^{(H, \varphi)} \rightarrow 0
\]

is exact for all \((H, \varphi) \in \mathcal{M}_k(G)\).
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(b) The sequence

\[ \cdots \to J(C_1) \to J(C_0) \to J(C_0) \to J(C_0) \to \mathcal{I}(V) \to 0 \]

is exact in \( \mathcal{C} \).

Proof (a) \( \implies \) (b): We have to show that for all \( N \in kG-\text{mon} \) the following sequence in \( k-\text{mod} \) is exact:

\[ \cdots \to \text{mon}_kG(N, M_1) \to \text{mon}_kG(N, M_0) \to \text{Hom}_kG(V(N), M) \to 0. \]

Clearly this sequence is a chain complex. First we prove exactness at \( \text{mon}_kG(N, M_i) \) for \( i > 0 \). Let \( f \in \text{mon}_kG(N, M_i) \) such that \( \partial_{i-1} \circ f = 0 \). Consider the diagram

\[ \begin{array}{ccc}
N & \to & M_i \\
\downarrow f & & \downarrow \partial_{i-1} \\
M_{i+1} & \to & M_{i-1}.
\end{array} \]

By the hypothesis in (a) we have

\[ f(N^{(H, \varphi)}) \subseteq \ker \left( \partial_{i-1} : M_i^{(H, \varphi)} \to M_{i-1}^{(H, \varphi)} \right) = \partial_i(M_{i+1}^{(H, \varphi)}). \]

Hence, we may apply Lemma 2.2, and we obtain \( g \in \text{mon}_kG(N, M_{i+1}) \) with \( f = \partial_i \circ g \).

Exactness at \( \text{mon}_kG(N, M_0) \) and at \( \text{Hom}_kG(V(N), V) \) is proved in a similar way using (a) and Lemma 2.2.

(b) \( \implies \) (a): Let \( (H, \varphi) \in \mathcal{M}_k(G) \). We apply the exact sequence of functors in (b) to \( S_{(H, \varphi)}^G \). This yields an exact sequence of \( k \)-modules

\[ \cdots \to \text{mon}_kG(S_{(H, \varphi)}^G, M_1) \to \text{mon}_kG(S_{(H, \varphi)}^G, M_0) \to \text{Hom}_kG(V(S_{(H, \varphi)}^G), V) \to 0. \]

By Frobenius reciprocity (Proposition 1.7 (a)) we have functorial isomorphisms (note that \( V(S_{(H, \varphi)}^G) = \text{ind}_H^G(k \varphi) \))

\[ \text{mon}_kG(S_{(H, \varphi)}^G, M_i) \cong \text{mon}_kH(k \varphi, \text{res}_H^G(M_i)) \]

and

\[ \text{Hom}_kG(VS_{(H, \varphi)}^G, V) \cong \text{Hom}_kH(k \varphi, \text{res}_H^G(V)). \]

Hence we get an exact sequence of \( k \)-modules

\[ \cdots \to \text{mon}_kH(k \varphi, \text{res}_H^G(M_1)) \to \text{mon}_kH(k \varphi, \text{res}_H^G(M_0)) \to \text{Hom}_kH(k \varphi, \text{res}_H^G(V)) \to 0. \]

Since \( k \varphi \in kH-\text{mon} \) is homogeneous in degree \( (H, \varphi) \in \mathcal{M}_k(H) \) with respect to the \( \mathcal{M}_k(H) \)-grading of Remark 1.9 (c), Proposition 1.11 (c) yields

\[ \text{mon}_kH(k \varphi, \text{res}_H^G(M_i)) = \text{Hom}_kH(k \varphi, \text{res}_H^G(M_i))^{(H, \varphi)} = \text{Hom}_k(k, M_i^{(H, \varphi)}) \cong M_i^{(H, \varphi)} \]
by Remark 1.9 (g). Using these identifications one obtains exactly the exact sequence of part (a).

3.7 Remark  We may interpret the last proposition in the following way: let $V \in kG{-}\text{mod}$, then the map

$$
(M_\ast \xrightarrow{\varepsilon} V) \mapsto \left(\mathcal{J}(M_\ast) \xrightarrow{\kappa_{M_\ast}, V(\varepsilon)} \mathcal{I}(V)\right)
$$

is a bijection between the monomial resolutions of $V$, and the projective resolutions of $\mathcal{I}(V)$ in $\mathcal{C}$ consisting of objects from the subcategory $\mathcal{J}(kG{-}\text{mon})$. In fact Proposition 3.2 and Lemma 3.5 together with the last proposition imply the bijectivity of the above correspondence.

Finally we give alternative descriptions of the category $\mathcal{C}$ by introducing equivalences from $\mathcal{C}$ to two other categories without proving the details.

3.8  Let

$$S := \bigoplus_{(H, \varphi) \in R_k(G)} S^G_{(H, \varphi)} \in kG{-}\text{mon},$$

and let

$$E := \text{mon}_{kG}(S, S).$$

Then $E$ is a $k$-algebra, free over $k$ by Proposition 1.20 (a), and $\mathcal{C}$ is equivalent to $\text{mod}{-}E$, the category of finitely generated right $E$-modules. An equivalence is given by the following functor

$$\Phi: \mathcal{C} \rightarrow \text{mod}{-}E, \quad \mathcal{F} \mapsto \mathcal{F}(S).$$

If $\mathcal{F} \in \mathcal{C}$, then $\mathcal{F}(S)$ is a finitely generated $k$-module by definition, and the right $E$-module structure is given by the canonical $E$-action

$$\mathcal{F}(S) \otimes E \rightarrow \mathcal{F}(S), \quad v \otimes f \mapsto \mathcal{F}(f)(v).$$

If $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism in $\mathcal{C}$, i.e. a natural transformation from $\mathcal{F}$ to $\mathcal{G}$, then we define

$$\Phi(\varphi) := \varphi_S: \mathcal{F}(S) \rightarrow \mathcal{G}(S).$$

In the other direction we define the functor

$$\Psi: \text{mod}{-}E \rightarrow \mathcal{C}, \quad X \mapsto \text{Hom}_E(\text{mon}_{kG}(S, -), X),$$

where $\text{mon}_{kG}(S, M)$ is a right $E$-module for $M \in kG{-}\text{mon}$ by composition of $kG$-monomial homomorphisms. If $f: X \rightarrow Y$ is a morphism in $\text{mod}{-}E$, then we define

$$\Psi(f): \text{Hom}_E(\text{mon}_{kG}(S, -), X) \rightarrow \text{Hom}_E(\text{mon}_{kG}(S, -), Y)$$

by composition with $f$ from the left.
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Next we define natural transformations
\[ \eta : \text{Id}_{\text{mod-}E} \to \Phi \circ \Psi \quad \text{and} \quad \varepsilon : \text{Id}_E \to \Psi \circ \Phi. \]

For \( X \in \text{mod-}E \) we have an obvious isomorphism of \( E \)-modules which is natural in \( X \):
\[ \eta_X : X \sim \text{Hom}_E(E, X) = \Phi(\Psi(X)). \]

For \( F \in \mathcal{C} \) we define
\[ \varepsilon_F : F \to \text{Hom}(\text{mon}_{kG}(S), \mathcal{F}(S)) = \Psi(\Phi(F)) \]
which is supposed to be a natural transformation) at \( M \in kG-\text{mon} \) by
\[ \varepsilon_{F,M} : F(M) \to \text{Hom}_E(\text{mon}_{kG}(S, M), \mathcal{F}(S)), \quad v \mapsto (f \mapsto F(f)(v)). \quad (5.14) \]

We leave it to the reader to show that \( \varepsilon_F \) is well-defined and natural in \( \mathcal{F} \). In order to prove that (5.14) is an isomorphism we may assume that \( M = S_{(H, \varphi)}^G \) for some \( (H, \varphi) \in \mathcal{R}_k(G) \), since \( \mathcal{F} \) is a k-functor and each \( M \in kG-\text{mon} \) is a direct summand of a direct sum of these objects (cf. Proposition 1.17 (c)). For \( (H, \varphi) \in \mathcal{R}_k(G) \) let \( p_{(H, \varphi)} : S \to S_{(H, \varphi)}^G \) and \( i_{(H, \varphi)} : S_{(H, \varphi)}^G \to S \) be the canonical projection and inclusion. Then it is easy to verify that
\[ \mathcal{F}(S_{(H, \varphi)}^G) \sim \text{Hom}_E(\text{mon}_{kG}(S, S_{(H, \varphi)}^G), \mathcal{F}(S)) \]
\[ v \mapsto (f \mapsto F(f)(v)) \]
defines inverse isomorphisms.

We observe that \( S_{(H, \varphi)}^G \) is mapped under the embedding \( \Phi \circ J \) to \( e_{(H, \varphi)}E \), where \( e_{(H, \varphi)} = i_{(H, \varphi)} \circ p_{(H, \varphi)} \) is an idempotent in \( E \). The idempotents \( e_{(H, \varphi)} \), \( (H, \varphi) \in \mathcal{R}_k(G) \) are mutually orthogonal and their sum is 1.

Next we give an explicit description of \( E \) in terms of a \( k \)-basis and a multiplication rule for this basis. By Proposition 1.20 (b), \( \text{mon}_{kG}(S_{(K, \psi)}^G, S_{(H, \varphi)}^G) \) has a distinguished \( k \)-basis \( \{ f_s \mid s \in G/H \text{ with } (K, \psi) \leq \ast(H, \varphi) \} \). For each subgroup \( H \leq G \) let \( T_H \subseteq G \) be a fixed set of representatives for \( G/H \). Then \( E \) has a basis which is described by triples
\[ [(H, \varphi), s, (K, \psi)] \]
with \( (K, \psi), (H, \varphi) \in \mathcal{R}_k(G) \), \( s \in T_H \), such that \( (K, \psi) \leq \ast(H, \varphi) \). Such a triple corresponds to the \( kG \)-monomial homomorphism \( f_s : S_{(K, \psi)}^G \to S_{(H, \varphi)}^G \), \( g \otimes_K \alpha \mapsto gs \otimes_H \alpha \) for \( g \in G, \alpha \in k \), which was described in Proposition 1.20 (b). Hence, we obtain the following multiplication rule:
\[ [(H, \varphi), s, (K, \psi)] \cdot [(U, \mu), t, (L, \lambda)] = \]
\[ \begin{cases} 0, & \text{if } (K, \psi) \neq (U, \mu), \\ \varphi(u^{-1}ts) \cdot [(H, \varphi), u, (L, \lambda)], & \text{if } (K, \psi) = (U, \mu), \end{cases} \]
where \( u \in T_H \) with \( tsH = uH \).

Since the chains in \( M_k(G) \) have bounded length, this shows that \( E \) has a nilpotent ideal generated by
\[
\{[(H, \varphi), s, (K, \psi)] \mid (H, \varphi) < s(K, \psi)\}
\]
whose quotient is isomorphic to
\[
\prod_{(H, \varphi) \in \mathcal{R}_k(G)} \text{mon}_kG(S_{(H, \varphi)}^G, S_{(H, \varphi)}^G) \cong \prod_{(H, \varphi) \in \mathcal{R}_k(G)} k[N_G(H, \varphi)/H].
\]
So, if \( k \) is a field of characteristic zero, then this ideal is exactly the radical of the \( k \)-algebra \( E \).

3.9 Let \( \mathcal{D} \) be the following subcategory of the category of \( G \)-equivariant sheaves on the \( G \)-poset \( M_k(G) \). The objects of \( \mathcal{D} \) are triples \((A, r, c)\), where

- \( A = \{A_{(H, \varphi)} \mid (H, \varphi) \in M_k(G)\} \) is a family of \( k \)-modules, called stalks,

- \( r = \{r_{(K, \psi)}^{(H, \varphi)} : A_{(H, \varphi)} \to A_{(K, \psi)} \mid (K, \psi) \leq (H, \varphi) \text{ in } M_k(G)\} \) is a family of \( k \)-linear maps, called restrictions, and

- \( c = \{c_{(H, \varphi)}^g : A_{(H, \varphi)} \to A_{g(H, \varphi)} \mid g \in G, (H, \varphi) \in M_k(G)\} \) is a family of \( k \)-linear maps, called conjugations,

which are subject to the following conditions

- (i) \( r_{(H, \varphi)}^{(H, \varphi)} = \text{id}_{A_{(H, \varphi)}} \) and \( r_{(K, \psi)}^{(K, \psi)} \circ r_{(U, \mu)}^{(H, \varphi)} = r_{(U, \mu)}^{(H, \varphi)} \) for all \((U, \mu) \leq (K, \psi) \leq (H, \varphi)\) in \( M_k(G) \).

- (ii) \( c_{(H, \varphi)}^g \circ c_{(H, \varphi)}^{g'} = c_{(H, \varphi)}^{gg'} \) for all \( g, g' \in G \), \((H, \varphi) \in M_k(G)\), and \( c_{(H, \varphi)}^h \) is the scalar multiplication by \( \varphi(h) \) for all \((H, \varphi) \in M_k(G), h \in H\).

- (iii) Restrictions and conjugations commute, i.e. \( c_{(K, \psi)}^g \circ r_{(K, \psi)}^{(H, \varphi)} = r_{(K, \psi)}^{(H, \varphi)} \circ c_{(K, \psi)}^g \) for all \( g \in G, (K, \psi) \leq (H, \varphi) \in M_k(G) \).

Let \((A, r, c)\) and \((B, s, d)\) be objects in \( \mathcal{D} \). A morphism \( f \) from \((A, r, c)\) to \((B, s, d)\) is a family
\[
\{f_{(H, \varphi)} : A_{(H, \varphi)} \to B_{(H, \varphi)} \mid (H, \varphi) \in M_k(G)\}
\]
of \( k \)-linear maps such that

- (iv) \( s_{(K, \psi)}^g \circ f_{(K, \psi)}^{(H, \varphi)} = f_{(H, \varphi)}^{(H, \varphi)} \circ f_{(H, \varphi)} \) for all \((K, \psi) \leq (H, \varphi) \) in \( M_k(G) \).

- (v) \( s_{(H, \varphi)}^g \circ c_{(H, \varphi)}^g = c_{(H, \varphi)}^g \circ f_{(H, \varphi)} \) for all \( g \in G, (H, \varphi) \in M_k(G) \).

We define a \( k \)-functor \( \Sigma : \mathcal{C} \to \mathcal{D} \) by \( \Sigma(F) := (A, r, c) \), with
\[
\begin{align*}
A_{(H, \varphi)} &:= F(S_{(H, \varphi)}^G), \\
r_{(H, \varphi)}^{(H, \varphi)} &:= F(S_{(H, \varphi)}^{(H, \varphi)}), \\
c_{(H, \varphi)}^g &:= F(S_{(H, \varphi)}^g),
\end{align*}
\]
where
\[ p^{(H,\varphi)}_{(K,\psi)}: S_{(K,\psi)}^G \rightarrow S_{(H,\varphi)}^G, \quad s \otimes_{kK} \alpha \mapsto s \otimes_{kH} \alpha, \]
and
\[ \hat{g}_{(H,\varphi)}: S_{(H,\varphi)}^G \rightarrow S_{(H,\varphi)}^G, \quad s \otimes_{kH} \alpha \mapsto sg \otimes_{kH} \alpha, \]
for \( s \in G, \alpha \in k. \)

For a morphism \( \mu: \mathcal{F} \rightarrow \mathcal{G} \) in \( \mathcal{C} \) we define \( \Sigma(\mu): \Sigma(\mathcal{F}) \rightarrow \Sigma(\mathcal{G}) \) by
\[ \Sigma(\mu)_{(H,\varphi)} := \mu_{S_{(H,\varphi)}^G}: \mathcal{F}(S_{(H,\varphi)}^G) \rightarrow \mathcal{G}(S_{(H,\varphi)}^G). \]

We also define a functor \( \Pi: \mathcal{D} \rightarrow \mathcal{C} \) by
\[ \Pi(A, r, c) := \mathcal{D}(\Sigma \sigma J -, (A, r, c)) \]
on objects and the obvious definition on morphisms, and leave it to the reader to show that \( \Sigma \) and \( \Pi \) are inverse equivalences.
Chapter 6
Class Group Relations

In this chapter we recall a theorem of Yoshida which gives an equivalence between the category of cohomological Mackey functors and a functor category. The interpretation of a cohomological Mackey functor \( M \) as a functor in the usual sense allows to deduce from each relation between the permutation modules \( k[G/H], \ H \leq G \), the same relation for the \( k \)-modules \( M(H), \ H \leq G \). Since the class groups of intermediate fields of a Galois extension of number fields form a cohomological Mackey functor (we prove this very carefully, although it must be obvious for everyone familiar with Dedekind domains), we obtain relations between these class groups. Moreover, we determine explicitly all relations which can be obtained in this way by using the idempotents in the Burnside ring. In can be deduced that the class group of the top field is uniquely determined by all other fields, if the Galois group is not hypo-elementary and its reduced Euler characteristic does not vanish. Interestingly the last condition is a topological property of the simplicial complex associated to the poset of all non-trivial proper subgroups. We do not know any group theoretical translation of this property.

6.1 Cohomological Mackey functors

Throughout this section \( k \) denotes a commutative ring and \( G \) denotes a finite group. By \( kG\text{-per} \) we denote the full subcategory of \( kG\text{-mod} \) consisting of \( kG \)-permutation modules, i.e. finitely generated \( k \)-free \( kG \)-modules with a \( G \)-stable \( k \)-basis. In other words, each object in \( kG\text{-per} \) is isomorphic to a \( kG \)-module \( kS \), where \( S \) is a finite \( G \)-set and \( kS \) denotes the free \( k \)-module with basis \( S \). Let \( \text{Funct}_k(kG\text{-per}, k\text{-Mod}) \) be the category whose objects are \( k \)-functors from \( kG\text{-per} \) to \( k\text{-Mod} \), i.e. functors which are \( k \)-linear on morphisms, and whose morphisms are the natural transformations between such functors. The following theorem is due to Yoshida (cf. [Yo83b, Thm. 4.3] or [Ta89, Sect. 1]).

1.1 Theorem

There is an equivalence of categories

\[
\Psi: k\text{-Mack}^c(G) \to \text{Funct}_k(kG\text{-per}, k\text{-Mod})
\]

with the property that for \( M \in k\text{-Mack}^c(G) \) and \( H \leq G \) we have

\[
(\Psi(M))(k[G/H]) \cong M(H).
\] (6.1)
In what follows we don’t need the fact that $\Psi$ is an equivalence, but only that there exists a functor $\Psi$ such that (6.1) holds.

1.2 Corollary  Let $H_1, \ldots, H_m, U_1, \ldots, U_n \leq G$ be (not necessarily distinct) subgroups such that

$$\bigoplus_{i=1}^{m} k[G/H_i] \cong \bigoplus_{j=1}^{n} k[G/U_j]$$

in $kG\text{-per}$. Then we have

$$\bigoplus_{i=1}^{m} M(H_i) \cong \bigoplus_{j=1}^{n} M(U_j)$$

in $k\text{-Mod}$ for every cohomological $k$-Mackey functor $M$ on $G$.

Proof  This is obvious from Theorem 1.1, since the functor $\Psi(M)$ maps isomorphic objects to isomorphic objects.

Let $\text{Per}_k(G)$ denote the Grothendieck ring of the category $kG\text{-per}$ with respect to direct sums. For $V \in kG\text{-per}$ we denote by $[V]$ the associated element in $\text{Per}_k(G)$. With the usual conjugation, restriction, and induction maps, the rings $\text{Per}_k(H), H \leq G$, form a $\mathbb{Z}$-Green functor $\text{Per}_k$ on $G$. The functor $k-: G\text{-set} \to kG\text{-per}, \ S \mapsto kS,$

induces a surjective ring homomorphism

$$i_G: \Omega(G) \to \text{Per}_k(G).$$

The ring homomorphisms $i_H, H \leq G$, form the unique morphism $i: \Omega \to \text{Per}_k$ of $\mathbb{Z}$-Green functors on $G$ (cf. Proposition I.1.4). We will determine the kernel of $i_G: \mathbb{Q} \otimes \Omega \to \mathbb{Q} \otimes \text{Per}_k(G)$ for $k = \mathbb{Z}_p$, the $p$-adic completion of $\mathbb{Z}$ for a prime $p$.

1.3 Proposition  Let $p$ be a prime. The kernel of

$$i_G: \mathbb{Q} \otimes \Omega(G) \to \mathbb{Q} \otimes \text{Per}_{\mathbb{Z}_p}(G)$$

is free on the idempotents $e_H^{(G)}$, where $H$ runs through a set of representatives for the conjugacy classes of subgroups of $G$ which are not $p$-hypo-elementary (i.e. $H/O_p(H)$ is not cyclic).

Proof  First we observe that the set $\mathcal{C}(\mathbb{Q} \otimes \text{Per}_{\mathbb{Z}_p})$ of coprimordial subgroups for $\mathbb{Q} \otimes \text{Per}_{\mathbb{Z}_p}$ is the set of $p$-hypo-elementary subgroups. In fact, by Conlon’s induction theorem (cf. [CR87, 80.51]) and Proposition I.6.2 we know that $\mathcal{C}(\mathbb{Q} \otimes \text{Per}_{\mathbb{Z}_p})$ is contained in the set of $p$-hypo-elementary subgroups of $G$. Conversely, we can show that each $p$-hypo-elementary subgroup $H \leq G$ is coprimordial for $\mathbb{Q} \otimes \text{Per}_{\mathbb{Z}_p} \subseteq \mathbb{Q} \otimes \mathbb{T}_{\mathbb{Z}_p}$ by the same proof as in Proposition III.4.14 for the trivial source ring. Note that in Section III.4 we assumed that the complete discrete
6.2. APPLICATION TO CLASS GROUPS

valuation ring \( O \) is big enough. But Proposition III.4.14 also holds for \( O = \mathbb{Z}_p \). Now the proposition follows from Proposition I.6.4.

Recall the explicit formula

\[
e^{(G)}_H = \frac{1}{|N_G(H)|} \sum_{U \leq H} |U| \mu(U, H)[G/U]
\]

from Remark I.3.3 or (A.2) in Appendix A, where \( \mu \) denotes the Möbius function of the poset of subgroups of \( G \).

Let \( K_0(ab) \) denote the Grothendieck group of the category \( ab \) of finite abelian groups with respect to direct sums. For \( A \in ab \) let \([A]\) denote the associated element in \( K_0(ab) \). Note that two finite abelian groups \( A \) and \( B \) are isomorphic if and only if \([A] = [B]\) in \( K_0(ab) \), since the category \( ab \) has the cancellation property \((A \oplus C \cong B \oplus C \implies A \cong B \) for \( A, B, C \in ab \)). For \( A \in ab \) and a prime \( p \) let \( A_p \) denote the Sylow \( p \)-subgroup of \( A \).

We call a \( k \)-Mackey functor \( M \) on \( G \) finite, if \( M(H) \) is finite for all \( H \leq G \).

We call a finite group hypo-elementary, if it is \( p \)-hypo-elementary for some prime \( p \).

**1.4 Corollary** Let \( M \) be a finite cohomological \( \mathbb{Z} \)-Mackey functor on \( G \).

(a) Let \( p \) be a prime and \( H \leq G \) a subgroup which is not \( p \)-hypo-elementary, then we have the relation

\[
\sum_{U \leq H} |U| \mu(U, H)[M(U)_p] = 0
\]

in \( K_0(ab) \).

(b) Let \( H \leq G \) be a subgroup which is not hypo-elementary, then we have the relation

\[
\sum_{U \leq H} |U| \mu(U, H)[M(U)] = 0
\]

in \( K_0(ab) \).

**Proof** (a) Note that \( M_p(H) := M(H)_p \) for \( H \leq G \) defines a cohomological \( \mathbb{Z}_p \)-Mackey functor on \( G \). Hence, the result follows from Proposition 1.3 and Corollary 1.2, since two \( \mathbb{Z}_p \)G-permutation modules are isomorphic if and only if their associated elements in \( Per_{\mathbb{Z}_p}(G) \) are equal.

(b) This follows from (a), since \( A = \oplus_p A_p \) for any finite abelian group \( A \), where \( p \) runs over the set of all primes.

**6.2 Application to class groups**

In this section we are going to apply the results of the previous section to the particular finite cohomological \( \mathbb{Z} \)-Mackey functor formed by the class groups of the intermediate fields of a Galois extension of number fields.

For a number field \( K \) let \( \mathcal{O}_K \) denote its ring of integers and \( I_K \) the multiplicative group of fractional ideals of \( K \), i.e. finitely generated non-trivial \( \mathcal{O}_K \)-submodules of \( K \). The group \( I_K \) is free on the set of non-trivial prime ideals of \( \mathcal{O}_K \). Let \( P_K \leq I_K \)
be the subgroup of principal fractional ideals, i.e. non-trivial \( \mathcal{O}_K \)-submodules of \( K \) which are generated by one element. The factor group \( \mathcal{C}(L) := I_K/P_K \) is finite and is called the ideal class group \( \mathcal{C}L \) which are generated by one element. The factor group \( U\mathcal{C}L \) is a subgroup of \( U\mathcal{G} \).

Now let \( L/K \) be a finite Galois extension of number fields, and denote by \( G \) the Galois group of \( L/K \). For \( H \leq G \), let \( L^H \) denote the fixed field of \( H \) in \( L/K \). For \( U \leq H \leq G \) and \( g \in G \) we have homomorphisms

\[
\begin{align*}
  c_{g,H} : I_{L^H} \to I_{L^gH}, & \quad \varphi \mapsto g\varphi, \\
  \text{res}^H_U : I_{L^H} \to I_{L^U}, & \quad \varphi \mapsto \varphi \cdot \mathcal{O}_{L^U},
\end{align*}
\]

where \( 0 \neq \varphi \) is a prime ideal of \( \mathcal{O}_{L^H} \), and

\[
\begin{align*}
  \text{ind}^H_U : I_{L^U} \to I_{L^H}, & \quad \mathcal{P} \mapsto N_{L^U/L^H}(\mathcal{P}) = \varphi_{L^U/L^H}(\mathcal{P}),
\end{align*}
\]

where \( 0 \neq \mathcal{P} \) is a prime ideal of \( \mathcal{O}_{L^U} \), \( \varphi := \mathcal{O}_{L^H} \cap \mathcal{P} \), and \( f_{L^U/L^H}(\mathcal{P}) \) denotes the residue degree of \( \mathcal{P} \) over \( L^H \). Note that these three maps take principal fractional ideals to principal fractional ideals. Hence, we also obtain group homomorphisms

\[
\begin{align*}
  c_{g,H} : \mathcal{C}(L^H) \to \mathcal{C}(L^gH), \\
  \text{res}^H_U : \mathcal{C}(L^H) \to \mathcal{C}(L^U), \\
  \text{ind}^H_U : \mathcal{C}(L^U) \to \mathcal{C}(L^H),
\end{align*}
\]

for \( U \leq H \leq G \) and \( g \in G \).

2.1 Lemma  
Keep the above notation.

(i) The map \( \text{res}^H_U : I_{L^H} \to I_{L^U} \) is injective for \( U \leq H \leq G \).

(ii) We have \( \text{ind}^I_U(\text{res}^I_U(\varphi)) = \varphi^{\text{ind}^I_U} \) for each non-zero prime ideal \( \varphi \) of \( \mathcal{O}_K \).

(iii) We have \( \text{res}^I_U(\text{ind}^I_U(\mathcal{P})) = \prod_{g \in G/H} \text{res}^H_U(\varphi(g\mathcal{P})) \) for \( H \leq G \) and each non-zero prime ideal \( \mathcal{P} \) of \( \mathcal{O}_{L^H} \).

(iv) We have

\[
\text{res}^I_U(\text{ind}^I_U(\mathcal{P})) = \sum_{\varphi \in U \cap g\mathcal{P}} \text{ind}_{U \cap g\mathcal{P}}(\text{res}^H_U(\varphi(g\mathcal{P})))
\]

for \( U, H \leq G \) and each non-zero prime ideal \( \mathcal{P} \) of \( \mathcal{O}_{L^H} \).

Proof  
(i) The extensions of two different prime ideals have no prime ideal in common.

(ii) If \( \varphi \mathcal{O}_L = (\mathcal{P}_1 \cdots \mathcal{P}_r)^e \) with prime ideals \( \mathcal{P}_1, \ldots, \mathcal{P}_r \) of \( \mathcal{O}_L \) and \( e \) the ramification index of each \( \mathcal{P}_i \) over \( K \), then \( N_{L/K}(\mathcal{P}_i) = \varphi^f \), where \( f \) is the residue degree of \( \mathcal{P}_i \) over \( K \), for \( i = 1, \ldots, r \). Now the result follows from the well-known equation \( [L : K] = r \cdot e \cdot f \).

(iii) We first prove the result in the special case where \( H = 1 \). Let \( \varphi := \mathcal{O}_K \cap \mathcal{P} \), and \( \varphi \cdot \mathcal{O}_L = (\mathcal{P}_1 \cdots \mathcal{P}_r)^e \) with prime ideals \( \mathcal{P}_1, \ldots, \mathcal{P}_r \) of \( \mathcal{O}_L \), where \( e \) is the ramification index of \( \mathcal{P}_i \) over \( K \) for \( i = 1, \ldots, r \). Then

\[
\text{res}^I_U(\text{ind}^I_U(\mathcal{P})) = \text{res}^I_U(\varphi^f) = (\mathcal{P}_1 \cdots \mathcal{P}_r)^{ef} = \prod_{g \in G} g\mathcal{P},
\]
where \( f \) denotes the residue degree of \( \mathcal{P} \) over \( K \). Now, if \( H \) is arbitrary, then

\[
\text{res}^G_{1}(\text{ind}^G_{H}(\mathcal{P}))^{[H]} = (\text{res}^G_{1} \circ \text{ind}^G_{H})(\text{ind}^H_{1}(\text{res}^H_{1}(\mathcal{P})))
\]

\[
= (\text{res}^G_{1} \circ \text{ind}^G_{H})(\text{res}^H_{1}(\mathcal{P}))
\]

\[
= \prod_{g \in G} g(\text{res}^H_{1}(\mathcal{P}))
\]

\[
= \left( \prod_{g \in G/H} g(\text{res}^H_{1}(\mathcal{P})) \right)^{[H]}.
\]

Since \( I_L \) is torsion free, the result follows.

(iv) Since \( \text{res}^U_{1} \) is injective by (i), it suffices to show

\[
(\text{res}^G_{1} \circ \text{ind}^G_{H})(\mathcal{P}) = \sum_{g \in U \setminus G/H} (\text{res}^U_{1} \circ \text{ind}^U_{U \cap gH})(\text{res}^{gH}_{U \cap gH}(g\mathcal{P})).
\]

By (iii) we have

\[
(\text{res}^G_{1} \circ \text{ind}^G_{H})(\mathcal{P}) = \sum_{g \in G/H} \text{res}^{gH}_{1}(g\mathcal{P}),
\]

and for \( s \in G \),

\[
(\text{res}^U_{1} \circ \text{ind}^U_{U \cap sH})(\text{res}^{sH}_{U \cap sH}(s\mathcal{P})) = \sum_{t \in U \setminus U \cap sH} t\text{res}^{U \cap sH}_{U \cap sH}(s\mathcal{P})
\]

\[
= \sum_{t \in U \setminus U \cap sH} \text{res}^{tsH}_{1}(ts\mathcal{P}).
\]

Now the result follows, since if \( s \) runs through a set of representatives for the double cosets \( U \setminus G/H \) and for each \( s \), \( t \) runs through a set of representatives for \( U/U \cap sH \), then \( ts \) runs through a set of representatives for \( G/H \).

2.2 Corollary Keep the above notations. The groups \( I_{L,H} \) and \( C\mathcal{L}(L^H) \), \( H \leq G \), form a finite cohomological \( \mathbb{Z} \)-Mackey functor on \( G \).

Proof The Mackey axiom and cohomologicality axiom for \( I_{L,H} \), \( H \leq G \), follow from Lemma 2.1 (iv) and (i). The other axioms in Definition I.1.1 are clearly satisfied. Since the axioms hold for \( I_{L,H} \), they also hold for \( C\mathcal{L}(L^H) \), \( H \leq G \).

2.3 Corollary We keep the above notations.

(a) Let \( p \) be a prime. If \( H \leq G \) is not \( p \)-hypo-elementary, then we have a relation

\[
\sum_{U \leq H} |U| \mu(U, H) [C\mathcal{L}(L^U)]_p = 0
\]

in \( K_0(ab) \). In particular, \( C\mathcal{L}(L^H)_p \) is uniquely determined by the groups \( C\mathcal{L}(L^U)_p \), \( 1 \leq U < H \), through this relation.

(b) If \( H \leq G \) is not hypo-elementary, then we have a relation

\[
\sum_{U \leq H} |U| \mu(U, H) [C\mathcal{L}(L^U)] = 0
\]
in $K_0(ab)$. In particular $\mathcal{CL}(L^H)$ is uniquely determined by the group $\mathcal{CL}(L^U)$, $1 \leq U < H$, through this relation.

(c) If $H \leq G$ is not hypo-elementary and $\mu(1,H) \neq 0$, then the class group $\mathcal{CL}(L)$ is uniquely determined by the class groups $\mathcal{CL}(L^U)$, $1 < U \leq H$, through the relation in (b).

**Proof** Part (a) and (b) follow immediately from Corollary 1.4, and part (c) follows from part (b).

2.4 Example We keep the above notation.

(a) Let $G$ be isomorphic to $C_2 \times C_2$, where $C_n$ denotes the cyclic group of order $n \in \mathbb{N}$. Let $L_1, L_2, L_3$ denote the three intermediate fields different from $L$ and $K$. Since $G$ is not $p$-hypo-elementary for odd primes $p$, we obtain from Corollary 2.3 (b):

$$2[\mathcal{CL}(L)]_p - 2[\mathcal{CL}(L_1)]_p - 2[\mathcal{CL}(L_2)]_p - 2[\mathcal{CL}(L_3)]_p + 4[\mathcal{CL}(K)]_p = 0$$

in $K_0(ab)$ for each odd prime $p$, or

$$\mathcal{CL}(K)_{\text{odd}} \oplus \mathcal{CL}(K)_{\text{odd}} \oplus \mathcal{CL}(L)_{\text{odd}} \cong \mathcal{CL}(L_1)_{\text{odd}} \oplus \mathcal{CL}(L_2)_{\text{odd}} \oplus \mathcal{CL}(L_3)_{\text{odd}}$$

as abelian groups, where $A_{\text{odd}}$ denotes the odd part of $A$ for $A \in ab$.

(b) Let $G$ be the symmetric group $S_3$ on three letters. Let $H \leq G$ be the Sylow 3-subgroup and let $U$ be a Sylow 2-subgroup. $G$ is not $p$-hypo-elementary for all primes different from 3. The idempotent $e^{(G)}_G$ is given by

$$e^{(G)}_G = \frac{1}{2}([G/1] - 2[G/U] - [G/H] + 2[G/G]).$$

Hence, we obtain an isomorphism

$$\mathcal{CL}(K)_{3'} \oplus \mathcal{CL}(K)_{3'} \oplus \mathcal{CL}(L)_{3'} \cong \mathcal{CL}(L^U)_{3'} \oplus \mathcal{CL}(L_U)_{3'} \oplus \mathcal{CL}(L^H)_{3'}$$

of abelian groups.

(c) The smallest non hypo-elementary group is the dihedral group $D_{12}$ of order 12. Moreover $\mu(1,D_{12}) = -6 \neq 0$ so that part (c) of Corollary 2.3 can be applied. The same holds for $G = S_4$, the symmetric group on 4 letters, where we have $\mu(1,S_4) = -12$. But there are also minimal non-hypo-elementary groups $G$ with $\mu(1,G) = 0$, so that part (c) of Corollary 2.3 cannot be applied as for example the group $SL_2(F_3)$.

The results we obtained by using the Mackey functor structure should be compared with the following classical results. Let $h_K$ denote the *class number* of $K$, i.e. the order of $\mathcal{CL}(K)$.

2.5 Theorem (Dirichlet 1842, [Di42]) Let $1 < d \in \mathbb{N}$ be square-free, and let $L = \mathbb{Q}(\sqrt{d}, \sqrt{-d})$. Then

$$2h_L = Q \cdot h_{\mathbb{Q}(\sqrt{d})} \cdot h_{\mathbb{Q}(\sqrt{-d})},$$

where $Q \in \{1, 2\}$ depends on $d$. 
2.6 Theorem (Brauer 1951, [Bra51]) Let $G$ be a finite group and let

$$\sum_{H \leq G} a_H \text{ind}_H^G(1) = 0$$

be a relation in the character ring $R(G)$ with integers $a_H$, $H \leq G$. Then the product

$$\prod_{H \leq G} L_{L}^{a_H}$$

assumes only finitely many values, as $L$ runs over all Galois extensions of $\mathbb{Q}$ with Galois group $G$.

2.7 Remark (a) Note that Theorem 2.6 can be considered as a generalization of Theorem 2.5. Both theorems make use of number theoretic results. Brauer for instance uses the theory of Artin $L$-functions and the analytic class number formula. In contrast to this we only used very little knowledge from number theory to verify that there is a Mackey functor structure. The rest followed from the general theory of Mackey functors. Note also that the character relations used in Theorem 2.6 are given by the idempotents $e^{(G)}_H \in \mathbb{Q} \otimes \Omega(G)$ for non-cyclic subgroups $H \leq G$. This can be proved similar to Proposition 1.3. Hence Brauer obtains more relations than we do in Corollary 2.3. But the relations are only exact up to a factor which may assume finitely many values. This finiteness property cannot be obtained by our approach. By this finiteness property Dirichlet obtains for the group $G = C_2 \times C_2$ even for the prime 2 a relation up to the factor $Q$, whereas we can’t say anything about the 2-part of the class numbers involved, cf. Example 2.4 (a). But note that in general, when the range of the factor in Brauer’s theorem is unknown we obtain for almost all primes exact relations, even for the class groups (not only the class numbers), and we are not forced to assume $K = \mathbb{Q}$.

(b) There have been improvements of Brauer’s result. Walter proved in [Walt79, Thm. 2.1] that each relation

$$\sum_{H \leq G} a_H \text{ind}_H^G(1) = \sum_{H \leq G} b_H \text{ind}_H^G(1)$$

in the character ring $R(G)$ with $a_H, b_H \in \mathbb{N}_0$, induces a relation

$$\bigoplus_{H \leq G} CL(L)^{a_H} \cong \bigoplus_{H \leq G} CL(L)^{b_H},$$

where $\pi$ is the set of all primes which do not divide $|G|$. This again is covered by our results, since each relation in the character ring comes from the idempotents $e^{(G)}_H$, $H \leq G$ non-cyclic, and since each non-cyclic $H \leq G$ is non-p-hypo-elementary if $p$ does not divide $|G|$.

(c) Roggenkamp and Scott have an approach in [RS82] which is very similar to the one we outlined here. What they call ‘Hecke actions’ is basically a functor in $\text{Funct}(kG-\text{per}, k-\text{Mod})$. They showed that Picard groups (in particular class groups) give rise to a Hecke action. In particular they have a result which is analogous to Corollary 1.2. But they didn’t determine the possible relations in $kG-\text{per}$ explicitly.
(d) Assume that $G$ is not hypo-elementary. Then the class group $\mathcal{CL}(K)$ of the bottom field is uniquely determined by the class groups of the bigger fields. In contrast to this, we need the additional information $\mu(1,G) \neq 0$ for $\mathcal{CL}(L)$ being uniquely determined by the class groups of smaller fields. This rises the general question: What group theoretic properties of a finite group $G$ imply that the reduced Euler characteristic

$$\bar{\chi}(G) := -1 + \sum_{1 < H_0 < \cdots < H_n < G} (-1)^n = \mu(1,G)$$

of the poset $\{H \mid 1 < H < G\}$ does not vanish. We only have a negative statement: If $G$ has a non-trivial Frattini subgroup, then $\mu(1,G) = 0$, since the poset is contractible with respect to the Frattini subgroup.
Appendix A

The Burnside Ring

For the reader’s convenience we fix the notation and recall some of the basic facts about the Burnside ring; see [CR87, §80] for more details.

The Burnside ring \( \Omega(G) \) of a finite group \( G \) is the Grothendieck ring of the category \( G\text{-set} \) of finite left \( G \)-sets. For any \( S \in G\text{-set} \) we denote by \([S] \in \Omega(G)\) its image in the Burnside ring. Each finite \( G \)-set \( S \) is a disjoint union \( S = S_1 \cup \ldots \cup S_r \) of transitive \( G \)-sets \( S_1, \ldots, S_r \) which are uniquely determined (namely as the \( G \)-orbits of \( S \)), and each transitive \( G \)-set is isomorphic to the set of cosets \( G/H \) for some \( H \leq G \) with the obvious left \( G \)-action. For \( K, H \leq G \) one knows that \( G/H \cong G/K \) in \( G\text{-set} \), if and only if \( H \) and \( K \) are \( G \)-conjugate. Therefore the elements \([G/H] \in \Omega(G)\), where \( H \) runs through a set \( \mathcal{R}_G \) of representatives of the conjugacy classes of subgroups of \( G \), form a \( \mathbb{Z} \)-basis of \( \Omega(G) \). Two \( G \)-sets \( S \) and \( S' \) are isomorphic if and only if \([S] = [S'] \in \Omega(G)\).

The direct product \( S \times S' \) of two finite \( G \)-sets \( S \) and \( S' \) is again a \( G \)-set by the diagonal action \( g(s, s') := (gs, gs') \) for \( g \in G \), \( s \in S \), \( s' \in S' \). This induces a multiplication \([S] \cdot [S'] := [S \times S']\) on \( \Omega(G) \) which furnishes \( \Omega(G) \) with the structure of a commutative ring with unity \([G/G] \). The multiplication is given explicitly by

\[
[G/K] \cdot [G/H] = \sum_{g \in K \backslash G/H} [G/K \cap gH]
\]

for \( K, H \leq G \).

The ring structure is most efficiently studied via the ring homomorphisms

\[
\phi_H^{(G)} : \Omega(G) \to \mathbb{Z}, \quad [S] \mapsto |S^H|, \quad H \leq G,
\]

where for \( S \in G\text{-set} \), \(|S^H|\) denotes the cardinality of the set \( S^H \) of \( H \)-fixed points of \( S \). For \( G \)-conjugate subgroups \( K \) and \( H \) of \( G \) one has \( \phi_K^{(G)} = \phi_H^{(G)} \), and the collection of these ring homomorphisms forms a ring homomorphism, the \textit{mark homomorphism}

\[
\rho_G : \Omega(G) \to \left( \prod_{H \leq G} \mathbb{Z} \right)^G \cong \prod_{H \in \mathcal{R}_G} \mathbb{Z},
\]

whose image is contained in the set of \( G \)-fixed points of \( \prod_{H \leq G} \mathbb{Z} \), where \( G \) acts by permuting the components according to the conjugation action on the index set of subgroups of \( G \). The mark homomorphism \( \rho_G \) is injective (i.e. two finite \( G \)-sets \( S \)
and \(S'\) are isomorphic, if and only if \(|S^H| = |S'^H|\) for all \(H \leq G\). Therefore, by counting ranks, the \(\mathbb{Q}\)-algebra homomorphism

\[
\mathbb{Q} \otimes \rho_G : \mathbb{Q} \otimes \Omega(G) \rightarrow \mathbb{Q} \otimes \left( \prod_{H \leq G} \mathbb{Z} \right)^G \cong \left( \prod_{H \leq G} \mathbb{Q} \right)^G \cong \prod_{H \in \mathcal{R}_G} \mathbb{Q}
\]

is an isomorphism. This shows that \(\mathbb{Q} \otimes \Omega(G)\) is semisimple with primitive idempotents \(e^{(G)}_H \in \mathbb{Q} \otimes \Omega(G)\) defined by

\[
\phi^{(G)}_K(e^{(G)}_H) = \begin{cases} 1, & \text{if } K \text{ and } H \text{ are } G\text{-conjugate}, \\ 0, & \text{otherwise}, \end{cases} \tag{A.1}
\]

for \(K, H \leq G\). Obviously \(e^{(G)}_H = e^{(G)}_K\) for \(G\)-conjugate subgroups \(K\) and \(H\) of \(G\), and the set \(\{e^{(G)}_H \mid H \in \mathcal{R}_G\}\) is a set of primitive mutually orthogonal idempotents which sum up to 1 in \(\mathbb{Q} \otimes \Omega(G)\). This set of idempotents forms a \(\mathbb{Q}\)-basis of \(\mathbb{Q} \otimes \Omega(G)\) and there is an explicit formula (see [Glu81]) which expresses \(e^{(G)}_H\) for \(H \leq G\) in terms of the basis \([G/K], K \in \mathcal{R}_G\), namely

\[
e^{(G)}_H = \frac{1}{|N_G(H)|} \sum_{K \leq H \leq G} |K| \mu(K, H) [G/K], \tag{A.2}
\]

where \(\mu\) denotes the Möbius function on the poset of subgroups of \(G\) ordered by inclusion, c.f. Appendix B. From Equation (1) one can easily derive the explicit formula

\[
(Q \otimes \rho_G)^{-1} : \left( \prod_{H \leq G} \mathbb{Q} \right)^G \rightarrow \mathbb{Q} \otimes \Omega(G),
\]

\[
(x_H)_{H \leq G} \mapsto \frac{1}{|G|} \sum_{K \leq H \leq G} |K| \mu(K, H) x_H [G/K]. \tag{A.3}
\]

For \(g \in G\) and \(H \leq G\) we have conjugation, restriction and induction functors \(c_{g, H} : H\text{-\set} \rightarrow gH\text{-\set} , \text{res}_H^G : G\text{-\set} \rightarrow H\text{-\set} , \text{ind}_H^G : H\text{-\set} \rightarrow G\text{-\set} \) which induce maps

\[
c_{g, H} : \Omega(H) \rightarrow \Omega(gH), \quad [H/K] \mapsto [gH/\mathcal{C}K],
\]

\[
\text{res}_H^G : \Omega(G) \rightarrow \Omega(H), \quad [G/U] \mapsto \sum_{g \in H \cap U} [H/H \cap gU],
\]

\[
\text{ind}_H^G : \Omega(H) \rightarrow \Omega(G), \quad [H/K] \mapsto [G/K],
\]

for \(K \leq H\) and \(U \leq G\), called conjugation, restriction, and induction maps. Conjugation and restriction maps are ring homomorphisms, whereas induction is just a group homomorphism.

For \(K, H \leq G\), the element \(\text{res}_H^G(e^{(G)}_K)\) is an idempotent in \(\mathbb{Q} \otimes \Omega(H)\), since \(\text{res}_H^G\) is a ring homomorphism. Since

\[
\phi^{(H)}_U(\text{res}_H^G(x)) = \phi^{(G)}_U(x)
\]
for arbitrary \( U \leq H \leq G, \ x \in \Omega(G) \), we have

\[
\text{res}_H^G(e_K^{(G)}) = \sum_{K' \in \mathcal{R}_H, \ K' = G} e_{K'}^{(H)} \quad \text{(A.4)}
\]

where \( \mathcal{R}_H \) is a set of representatives for the \( H \)-conjugacy classes of subgroups of \( H \).

For a commutative ring \( k \) we can form the \( k \)-algebra \( k \otimes \Omega(G) \), whose elements we write as \( k \)-linear combinations of the basis \( \{ G/H \} \), \( H \in \mathcal{R}_G \), i.e. we omit the tensor product for elements in \( k \otimes \Omega(G) \) as already done in Equation (2). The maps \( \phi^{(G)}_H \), \( \rho_G \), \( c_{g,H} \), \( \text{res}_H^G \), \( \text{ind}_H^G \) for \( g \in G, \ H \leq G \), have \( k \)-tensored versions which we denote by the same symbols. If \( |G| \) is invertible in \( k \), then \( \rho_G : k \otimes \Omega(G) \to (\prod_{H \leq G} k)^G \) is invertible and the formulae (A.2) and (A.3) remain valid.
Appendix B

Posets and Möbius Inversion

The explicit induction formulae in Chapter 2 involve alternating sums over sets of chains in posets, which can be expressed in terms of the Möbius function of the considered poset. This appendix collects the relevant notions and results about Möbius inversion and also fixes the notation. For further reference see for example [Ro64].

B.1 Definition A partially ordered set (or for short, a poset) \((X, \leq)\) is a set \(X\) together with a partial order \(\leq\) on \(X\), i.e. a relation \(\leq\) on \(X\) satisfying the following axioms:

(P1) (Reflexivity) \(x \leq x\) for \(x \in X\).

(P2) (Antisymmetry) If \(x, y \in X, x \leq y, y \leq x\), then \(x = y\).

(P3) (Transitivity) If \(x, y, z \in X, x \leq y, y \leq z\), then \(x \leq z\).

For \(x, y \in X\) we will write \(x < y\) if \(x \leq y\) and \(x \neq y\). We will mostly write just \(X\) instead of \((X, \leq)\). The zeta function (or incidence function) on \(X\), \(\zeta_X: X \times X \to \mathbb{Z}\), is defined by

\[
\zeta_X(x, y) := \begin{cases} 
1, & \text{if } x \leq y, \\
0, & \text{otherwise},
\end{cases}
\]

for \(x, y \in X\). If \(X\) is finite, and if we enumerate the elements of \(X = \{x^1, \ldots, x^r\}\) in such a way that \(x^i \leq x^j\) implies \(i \leq j\) for \(i, j \in \{1, \ldots, r\}\) (this is always possible by first enumerating the minimal elements of \(X\), then the minimal elements of the remaining elements, and so on), then the incidence matrix of \(X\),

\[
Z_X := (\zeta_X(x^i, y^j))_{i,j} = (\zeta_X(x, y))_{x,y \in X},
\]

is an upper triangular matrix whose diagonal entries are all equal to 1. Hence, \(Z_X\) is invertible over \(\mathbb{Z}\), and the entries of the Möbius matrix \(M_X\) of \(X\), where

\[
M_X := Z_X^{-1} = (\mu_X(x, y))_{x,y \in X},
\]

define the Möbius function \(\mu_X: X \times X \to \mathbb{Z}\).

A chain of length \(n \in \mathbb{N}_0\) in \(X\) is a sequence \((x_0, \ldots, x_n) \in X^{n+1}\) with \(x_{i-1} < x_i\) for \(i = 1, \ldots, n\). We will use the notation \((x_0 < \ldots < x_n)\) for such a chain.
length of the chain \( \sigma = (x_0 < \ldots < x_n) \) will be denoted by \( |\sigma| \). For \( x, y \in X \) and \( n \in \mathbb{N}_0 \) we denote by \( \Gamma^n_X(x, y) \) (resp. \( \Gamma^n_X \), resp. \( \Gamma(X) \)) the set of all chains of length \( n \) in \( X \) with minimal element \( x \) and maximal element \( y \) (resp. the set of all chains of length \( n \), resp. the set of all chains in \( X \)). The cardinality of \( \Gamma^n_X(x, y) \) (resp. \( \Gamma^n_X \)) will be denoted by \( \gamma^n_X (x, y) \) (resp. \( \gamma^n_X \)).

Let \( I \) denote the identity matrix. An induction argument shows that, for \( x, y \in X \) and \( n \in \mathbb{N}_0 \), the \((x, y)\)-entry of \((Z_X - I)^n\) is equal to \( \gamma^n_X (x, y) \). Since \( Z_X - I \) is nilpotent, we can form the matrix \( \sum_{n \geq 0} (-1)^n (Z_X - I)^n \), which is equal to \( M_X = Z_X^{-1} \) by the usual property of geometric series. Hence we obtain an explicit formula for the Möbius function:

\[
\mu_X(x, y) = \sum_{n \geq 0} (-1)^n \gamma^n_X(x, y),
\]

for \( x, y \in X \). In particular, \( \mu_X(x, y) = 0 \) unless \( x \leq y \), and \( \mu_X(x, x) = 1 \) for \( x, y \in X \).

**B.2 Proposition** Let \( A \) be an (additively written) abelian group, \( X \) a finite poset and \( f, g : X \to A \) arbitrary functions.

(i) \( g(y) = \sum_{x \leq y} f(x) \) for all \( y \in X \), if and only if, \( f(y) = \sum_{x \leq y} \mu_X(x, y) g(x) \) for all \( y \in X \).

(ii) \( g(x) = \sum_{y \leq x} f(y) \) for all \( x \in X \), if and only if, \( f(x) = \sum_{y \leq x} \mu_X(x, y) g(y) \) for all \( x \in X \).

**Proof** (i) We define row vectors \( (f(x))_{x \in X} \) and \( (g(x))_{x \in X} \). Then the first equations can be written as one matrix equation,

\[
(g(y))_{y \in X} = (f(x))_{x \in X} \cdot Z_X,
\]

and the second equations can be written as

\[
(f(y))_{y \in X} = (g(x))_{x \in X} \cdot M_X.
\]

Since \( M_X = Z_X^{-1} \), the two equations are equivalent.

(ii) This follows similarly to part (i) by additionally transposing \( Z_X \) and \( M_X \), or what amounts to the same, by considering the ‘opposite’ poset of \( X \), where all the relations are reversed.

**B.3 Corollary** Let \( A \) be an abelian group, \( X \) a finite poset and \( f : X \to A \) a function.

(i) For \( z \in X \) we have

\[
f(z) = \sum_{x \leq y \leq z} \mu_X(y, z) f(x)
\]

and

\[
f(z) = \sum_{x \leq y \leq z} \mu_X(x, y) f(x).
\]

(ii) For \( x \in X \) we have

\[
f(x) = \sum_{x \leq y \leq z} \mu_X(x, y) f(z)
\]
and
\[ f(x) = \sum_{x \leq y \leq z} \mu_X(y, z)f(z). \]

**Proof** (i) When we substitute the expression for \( g \) in Proposition B.2 (i) in the expression for \( f \), and the other way round, we obtain exactly the two equations of part (i).

(ii) is proved as (i) using Proposition B.2 (ii). \( \square \)

**B.3** Let \( G \) be a finite group. A (left) \( G \)-**poset** \( X \) is a poset \((X, \leq)\) together with a left \( G \)-action on \( X \) such that the action of any fixed group element \( g \in G \) is a poset automorphism, i.e. if \( x, y \in X \) with \( x \leq y \) then \( g\!x \leq g\!y \). The set \( G\!\setminus X \) of \( G \)-orbits is then again a poset, called the **orbit poset**, by the following definition:

\[ [x]_G \leq [y]_G : \iff \exists g \in G : x \leq g\!y \]

for \( x, y \in X \), where \([x]_G \) denotes the \( G \)-orbit in \( X \) containing \( x \).

If \( X \) is a \( G \)-poset, then \( \Gamma_X \) and \( \Gamma^n_X \), for \( n \in \mathbb{N}_0 \), are \( G \)-sets by the obvious action on chains. Note that there is a canonical map

\[ G\!\setminus \Gamma^n_X(x, y) \to \Gamma^n_G\!\setminus X([x]_G, [y]_G), \quad ([x_0 < \ldots < x_n])_G \mapsto ([x_0]_G < \ldots < [x_n]_G), \]

which is obviously surjective, but in general not injective.
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