A characterization of Adams operations on representation rings*

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Abstract

We show that the Adams operations on the character ring, the Brauer character ring, and the trivial source ring can be characterized as the natural endomorphisms of these representation rings if these rings are considered as functors form the category of finite groups to the category of unitary rings. In the case of the trivial source ring one has to take a vertex filtration into account.

Introduction

In this article we consider Adams operations on the character ring $R(G)$, and for a prime p, on the Brauer character ring $R^p(G)$ and the trivial source ring $T^p(G)$ of a finite group G. The latter ones where introduced by Benson in [Be84a]. We show that the Adams operations on the character ring and on the Brauer character ring can be characterized as the natural endomorphisms of these representation rings considered as functors from the category of finite groups to the category of unitary rings. In the case of the trivial source ring we characterize the Adams operations as those natural endomorphisms that preserve the vertex filtration, i.e., which map an indecomposable trivial source module with a certain vertex to a linear combination of such with smaller or equal vertex. If one views these representation rings as functors on varying finite groups, one is forced to consider Adams operations Ψ^κ for elements κ in the Prüfer ring $\hat{\mathbb{Z}}$ rather than in the integers. The main theorem for these three representation rings are stated as Theorem 1.2, 2.2, and 3.7.

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Each of these three representation rings becomes a split K-algebra after tensoring with the field extension of $\mathbb{Q}$ that is generated by the roots of unity. This allows to define K-algebra maps between their K-tensored versions using their species. Even more, as pointed out in [Bo01], their species are again indexed in a functorial way by the orbits of G-sets (elements of G, p'-elements of G, pairs (P, g) with P a p-subgroup of G and g a p'-element in $N_G(P)$) and the natural transformations between the K-tensored versions of the representation rings correspond to the natural transformation between the indexing sets of species. Each of these G-sets has a natural power operation $g \mapsto g^\kappa$ for $\kappa \in \hat{\mathbb{Z}}$ and induces Adams operations on the K-tensored representation rings. That these K-algebra maps are integral, i.e., that they map the representation rings to the representation rings, is classically known for the character ring. Using the surjectivity of the decomposition map it also follows for the Brauer character ring. Benson gave an adhoc construction which implies the integrality in the case of the trivial source ring. Alternatively, we use for the trivial source ring the canonical induction formula from [Bo98a] to express the Adams operations and show their integrality. This was earlier done for the character ring and can also be done without efforts for the Brauer character ring.

The reader might ask why we do not include the case of the linear source ring. The answer is that everything is much more complicated there. The naive analogous definition of Adams operations using the canonical induction formula for linear source modules fails to be multiplicative in general. We will investigate this case in a subsequent paper.

Notation We denote by gr the category of finite groups and by Ri the category of associative rings with identity and identity preserving ring homomorphisms. As usually, the Prüfer ring is denoted by $\hat{\mathbb{Z}}$. It is the projective limit of the rings $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{Z}$, with respect to the natural epimorphisms $\pi^n_m \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, $z + n\mathbb{Z} \mapsto z + m\mathbb{Z}$, for $n, m \in \mathbb{N}$ with m dividing n. There is a canonical ring isomorphism $\hat{\mathbb{Z}} \to \prod_p \mathbb{Z}_p$ into the product of the p-adic integers indexed over the primes p. For an element $\kappa = (\kappa_n)_{n\in\mathbb{N}}$ in $\hat{\mathbb{Z}}$ and a group element g of finite order n we set $g^\kappa := g^{\kappa_n}$.

Throughout, K denotes the field subfield of the complex numbers obtained by adjoining all roots of unity to $\mathbb{Q}$.

Throughout G denotes a finite group and we write $H \leqslant G$ (resp. $H < G$) if H is a subgroup (resp. proper subgroup) of G. For $U, H \leqslant G$ we write $U \leqslant_G H$ if U is G-conjugate to a subgroup of H, and $U =_G H$ if U is G-conjugate to H. For a set of primes π and an element $g \in G$ we set g_π for the π-part of g. Moreover, we denote the set of the π-elements of G by G_π. If π is the complement of a prime p we usually write p' instead of π. For an element x in a G-set X we write $[x]_G$ for its G-orbit.

1 The character ring

In this section we will show that the Adams operations on the character ring are characterized as the natural endomorphisms of the character ring viewed as contravariant functor from the category of finite groups to the category of rings.

1.1 Let $R\colon \mathsf{gr} \to \mathsf{Ri}$ denote the contravariant functor which associates to each finite group G its character ring $R(G)$ and to each homomorphism $f\colon \widetilde{G} \to G$ between finite groups the ring homomorphism $\mathrm{res}_f\colon R(G) \to R(\widetilde{G})$, $\chi \mapsto \chi \circ f$. For a finite group G and an integer k let $\Psi_G^k\colon R(G) \to R(G)$ denote the k-th Adams operation on $R(G)$ which takes a virtual character χ of G to the class function $\Psi_G^k(\chi)\colon G \to \mathbb{C}$, $g \mapsto \chi(g^k)$. See for example [CR81, §12B] for a proof that this class function is again a virtual character. Note that one could also define Ψ_G^k by using the canonical Brauer induction formula (cf. [Bo90]). This approach also shows that Ψ_G^k takes a virtual character to a virtual character. We will use this approach again for the trivial source ring later. It is immediate from the definition that Ψ_G^k is a ring homomorphism and that $\Psi_G^k \circ \Psi_G^l = \Psi_G^{kl}$ for any $k, l \in \mathbb{Z}$. Note also that Ψ_G^k depends only on the class of k module the exponent $\exp(G)$ of G. So, also $\Psi_G^{\kappa_n}\colon R(G) \to R(G)$ is defined for any $\kappa_n \in \mathbb{Z}/n\mathbb{Z}$. If we want to consider all finite groups at the same time, the appropriate ring to consider is $\hat{\mathbb{Z}}$ instead of $\mathbb{Z}$. In fact, each $\kappa = (\kappa_n) \in \hat{\mathbb{Z}}$ gives rise to a ring homomorphism

$$\Psi_G^\kappa := \Psi_G^{\kappa_{\exp(G)}}\colon R(G) \to R(G)$$

for *any* $G \in \mathsf{gr}$. One clearly has a commutative diagram

$$\begin{array}{ccc} R(\widetilde{G}) & \xrightarrow{\Psi^\kappa_{\widetilde{G}}} & R(\widetilde{G}) \\ {\scriptstyle \mathrm{res}_f}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}_f} \\ R(G) & \xrightarrow{\Psi^\kappa_{G}} & R(G) \end{array}$$

for any $f\colon \widetilde{G} \to G$ in the category gr. Thus, $\Psi^\kappa\colon R \to R$ is a natural endomorphism and we obtain a morphism

$$\hat{\mathbb{Z}} \to \mathrm{End}(R)\,, \quad \kappa \mapsto \Psi^\kappa\,, \tag{1.1.a}$$

of monoids with respect to multiplication in $\hat{\mathbb{Z}}$ and composition in $\mathrm{End}(R)$, the set of natural transformations $R \to R$ from the contravariant functor $R\colon \mathsf{gr} \to \mathsf{Ri}$ to itself.

The following theorem shows that the Adams operations on the character ring can be characterized as its natural endomorphisms.

1.2 Theorem *The map in* (1.1.a) *is an isomorphism.*

Proof Let $\kappa = (\kappa_n)$ and $\kappa' = (\kappa'_n)$ be elements in $\hat{\mathbb{Z}}$ with $\Psi^\kappa = \Psi^{\kappa'}$. Then, for a cyclic group C_n of order n, the evaluation of $\Psi^{\kappa_n}_{C_n}$ and $\Psi^{\kappa'_n}_{C_n}$ on a faithful irreducible character of C_n implies $\kappa_n = \kappa'_n$. This shows the injectivity of the map in (1.1.a).

To show the surjectivity, assume that $\Psi \in \mathrm{End}(R)$. Note that, since we assume that ring homomorphisms are unitary, we have $\Psi_1 = \mathrm{id}_{R(1)}$ for the trivial group 1. For each $n \in \mathbb{N}$, let C_n denote a cyclic group of order n and let φ be a faithful irreducible character of C_n. Then the ring $R(C_n)$ is the group ring $\mathbb{Z}\hat{C}_n$ of the multiplicative group $\hat{C}_n := \mathrm{Hom}(C_n, \mathbb{C}^\times)$. Therefore, Ψ_{C_n} takes the torsion unit φ of $\mathbb{Z}\hat{C}_n$ to an element in the torsion subgroup $\{\pm\varphi^k \mid k = 0, \dots, n-1\}$ of the unit group of $\mathbb{Z}\hat{C}_n$ (cf. [Se78, 6.2, Exercise 2(d)]). Using the functoriality of Ψ with respect to the inclusion $1 \to C_n$, we obtain $\Psi_{C_n}(\varphi) = \varphi^{\kappa_n}$ for a unique $\kappa_n \in \mathbb{Z}/n\mathbb{Z}$. This implies $\Psi_{C_n}(\varphi) = \varphi^{\kappa_n}$ for all $\varphi \in \hat{C}_n$. Now the functoriality with respect to an embedding $C_m \to C_n$ implies $\pi^n_m(\kappa_n) = \kappa_m$. Therefore, the elements κ_n, $n \in \mathbb{N}$, define an element $\kappa = (\kappa_n) \in \hat{\mathbb{Z}}$ and $\Psi_C = \Psi^\kappa_C$ for each cyclic group C. Now let G be an arbitrary finite group and let $\chi \in R(G)$. Then the restrictions of $\Psi^\kappa_G(\chi)$ and $\Psi_G(\chi)$ to all the cyclic subgroups of C of G coincide by functoriality with respect to the inclusion $C \leqslant G$. Therefore, also $\Psi^\kappa_G(\chi)$ and $\Psi_G(\chi)$ coincide and we have $\Psi_G = \Psi^\kappa_G$. □

Note that in the surjectivity part of the proof we made only use of the functoriality of Ψ with respect to injective group homomorphisms. therefore, by a similar proof we can show the following local version of the previous theorem.

1.3 Theorem *Let G be a finite group with exponent n, and let $\big(\Psi_H \colon R(H) \to R(H)\big)_{H \leqslant G}$ be a family of ring homomorphisms satisfying*

$$\mathrm{res}_f \circ \Psi_H = \Psi_U \circ \mathrm{res}_f \tag{1.3.a}$$

for each injective homomorphism $f \colon U \to H$ between subgroups U and H of G. Then there exists an integer k, unique modulo n, such that $\Psi_H = \Psi^k_H$ for all $H \leqslant G$.

1.4 Remark Note that the assertion in Theorem 1.3 is no longer true, if we only require the functoriality (1.3.a) with respect to compositions of subgroup inclusions and conjugations by elements of G. For example, let G be the quaternion group of order 8 and let $H_0 \leqslant G$ be a cyclic subgroup of order 4. Set $\Psi_H := \mathrm{id}_{R(H)}$ for all $H \leqslant G$ different from H_0 and set $\Psi_{H_0} = \Psi^3_{H_0} \colon R(H_0) \to R(H_0)$. Then it can easily be checked that Equation (1.3.a) holds for all conjugations and inclusions.

1.5 Remark Another approach to Adams operations on the character ring uses its species, i.e., the $\mathbb{C}$-algebra maps $s \colon \mathbb{C} \otimes R(G) \to \mathbb{C}$ or what is the same, its K-species $s \colon K \otimes R(G) \to K$. Let $\mathcal{R}(G)$ denote the G-set consisting of the elements of G under the conjugation action. For each $g \in G$ there is a K-algebra map

$$s_g^{R(G)} \colon K \otimes R(G) \to K\,, \quad 1 \otimes \chi \mapsto \chi(g)\,.$$

Obviously, from character theory, $s_g^{R(G)} = s_{g'}^{R(G)}$ if and only if g and g' are conjugate under G. Since K-algebra maps are always K-linearly independent, these are all the species of $R(G)$, and their collection gives a K-algebra isomorphism

$$s^{R(G)} = (s_g^{R(G)})_{g \in \mathcal{R}(G)/G} \colon K \otimes R(G) \to \prod_{g \in \mathcal{R}(G)/G} K \;\cong\; \mathrm{Hom}_G(\mathcal{R}(G), K)\,,$$

where the last set describes the G-equivariant maps $\mathcal{R}(G) \to K$ regarding K endowed with the trivial G-action. Now, any natural ring homomorphism $\Psi \colon R(G) \to R(G)$ induces a natural map $f \colon \mathcal{R}(G)/G \to \mathcal{R}(G)/G$ by

$$s_{[g]_G}^{R(G)} \circ \Psi = s_{f([g]_G)}^{R(G)}\,. \tag{1.5.a}$$

Conversely, every natural map $f \colon \mathcal{R}(G)/G \to \mathcal{R}(G)/G$ defines a unique natural K-algebra map $\Psi \colon K \otimes R(G) \to K \otimes R(G)$ such that Equation (1.5.a) holds, cf. [Bo01, Remark 2.2]. In the case $\Psi = \Psi_G^\kappa$, $\kappa \in \hat{\mathbb{Z}}$, the corresponding map $f \colon \mathcal{R}(G)/G \to \mathcal{R}(G)/G$ is induced by the natural κ-power maps

$$\psi_G^\kappa \colon \mathcal{R}(G) \to \mathcal{R}(G)\,, \quad g \mapsto g^\kappa\,.$$

We will see that other representation rings follow the same pattern: Their species are indexed by G-orbits of a G-set which allows natural power operations which induce natural K-algebra maps of the representation ring tensored with K. Then it remains to be investigated if these K-algebra maps are integral, i.e., if the representation ring is stable under them.

2 The Brauer character ring

2.1 For $G \in \mathsf{gr}$, let $R^p(G)$ denote the Brauer character ring of G in positive characteristic p, which we identify with the Grothendieck group of the category ${}_{FG}\mathsf{mod}$ of finitely generated FG-modules for a fixed choice of an algebraically closes field F of characteristic p after identifying the group of roots of unity in F with the p'-part of the group of roots of unity of K, cf. [Se78, §18]. We will consider R^p as a contravariant functor $R^p \colon \mathsf{gr} \to \mathsf{Ri}$.

We will use the notation and approach from [Bo01]. Let $\mathcal{R}^p(G) \subseteq \mathcal{R}(G)$ denote the set of p'-elements of the finite group G. Then $\mathcal{R}^p \colon \mathsf{gr} \to \mathsf{set}$ is a covariant functor. For each $g \in \mathcal{R}^p(G)$ one has a K-species

$$s_g^{R^p(G)} \colon R^p(G) \to K\,, \quad \chi \mapsto \chi(g)\,,$$

and it is well-known that $s_g^{R^p(G)} = s_{g'}^{R^p(G)}$ if and only if g and g' are conjugate. We obtain a K-algebra isomorphism

$$s^{R^p(G)} \colon K \otimes R^p(G) \to \prod_{g \in \mathcal{R}^p(G)/G} K \;\cong\; \mathrm{Hom}_G(\mathcal{R}^p(G), K)\,,$$

with $(s^{R^p(G)}(1\otimes\chi))(g)=\chi(g)$, for $\chi\in R^p(G)$ and $g\in\mathcal{R}^p(G)$.

Let

$$\hat{\mathbb{Z}}_{p'} := \lim_{n\in\mathbb{N}_{p'}} \mathbb{Z}/n\mathbb{Z} \cong \prod_{q\neq p} \mathbb{Z}_q\,,$$

where the set $\mathbb{N}_{p'}$ of natural p'-numbers is ordered by divisibility. Moreover, let $\pi\colon \hat{\mathbb{Z}}\to\hat{\mathbb{Z}}_{p'}$ denote the natural projection. For each $\kappa\in\hat{\mathbb{Z}}_{p'}$ we can define a K-algebra map

$$\Psi_G^\kappa\colon K\otimes R^p(G)\to K\otimes R^p(G)$$

by

$$s_g^{R^p(G)}\circ\Psi_G^\kappa = s_{g^\kappa}^{R^p(G)}\,,$$

for each $g\in\mathcal{R}^p(G)$. Since g^κ and $(g')^\kappa$ are conjugate whenever g and g' are, Ψ_G^κ is well-defined. Applying the above equation to $\chi\in R^p(G)$ one obtains $(\Psi_G^\kappa(\chi))(g)=\chi(g^\kappa)$. Moreover, since $\psi_G^\kappa\colon\mathcal{R}^p(G)\to\mathcal{R}^p(G)$, $g\mapsto g^\kappa$, is a natural transformation $\psi\colon\mathcal{R}^p\to\mathcal{R}^p$, also $\Psi^\kappa\colon K\otimes R^p\to K\otimes R^p$ is a natural transformation of the K-algebra valued functor $K\otimes R^p$, cf. [Bo01, Remark 2.2]. Note also that

$$s_g^{R^p(G)}\circ\Psi_G^\kappa\circ\Psi_G^\lambda = s_{g^\kappa}^{R^p(G)}\circ\Psi_G^\lambda = s_{g^{\kappa\lambda}}^{R^p(G)} = s_g^{R^p(G)}\circ\Psi_G^{\kappa\lambda}\,,$$

for $\kappa,\lambda\in\hat{\mathbb{Z}}_{p'}$ and $g\in\mathcal{R}^p(G)$. Hence, $\Psi_G^\kappa\circ\Psi_G^\lambda=\Psi_G^{\kappa\lambda}$.

There are different ways of seeing that $\Psi_G^\kappa(R^p(G))\subseteq R^p(G)$. One way is to use the decomposition map $d_G\colon R(G)\to R^p(G)$ and the Brauer lift $l_G\colon R^p(G)\to R(G)$ which are ring homomorphisms that are natural in G. They are given by $d_G(\chi):=\chi|_{\mathcal{R}^p(G)}$ and $(l_G(\chi))(g):=\chi(g_{p'})$ for $g\in G$, where $g_{p'}$ denotes the p'-part of g. Using the species of $R(G)$ and $R^p(G)$ one obtains

$$\Psi_G^\kappa\circ d_G = d_G\circ\Psi_G^{\pi(\kappa)} \quad\text{and}\quad \Psi_G^\lambda\circ l_G = l_G\circ\Psi_G^\lambda$$

for $\kappa\in\hat{\mathbb{Z}}_{p'}$ and $\lambda\in\hat{\mathbb{Z}}$. This shows that $\Psi_G^{\pi(\kappa)}=d_G\circ\Psi_G^\kappa\circ l_G$ for $\kappa\in\hat{\mathbb{Z}}$ and that $\Psi_G(R^p(G))\subseteq R^p(G)$.

Summarizing the previous considerations we obtain a map

$$\hat{\mathbb{Z}}_{p'}\to\mathrm{End}(R^p)\,,\quad \kappa\mapsto\Psi^\kappa \tag{2.1.a}$$

of monoids.

The following theorem is proved in the same way as Theorem 1.2.

2.2 Theorem *The map in* (2.1.a) *is an isomorphism.*

2.3 Remark One can also use the canonical induction formula for the Brauer character ring from [Bo98b, Example 9.8]. This is a map $a_G\colon R^p(G)\to R_+^{p,\mathrm{ab}}(G)$ where $R_+^{p,\mathrm{ab}}(G)$ is the free abelian group on the set of G-orbits $[H,\varphi]_G$ of pairs (H,φ) with $H\leqslant G$ and $\varphi\in\mathrm{Hom}(H,F^\times)$. It has the property that $b_G\circ$

$a_G = \mathrm{id}_{R^p(G)}$, where $b_G\colon R_+^{p,\mathrm{ab}}(G) \to R^p(G)$ maps $[H,\varphi]_G$ to $\mathrm{ind}_H^G(\varphi)$ and φ is considered as a Brauer character by restriction to $\mathcal{R}^p(G)$. One can show that

$$\Psi_G^\kappa = b_G \circ \Psi_{+,G}^\kappa \circ a_G \,,$$

for $\kappa \in \hat{\mathbb{Z}}_{p'}$, where $\Psi_{+,G}^\kappa\colon R_+^{p,\mathrm{ab}}(G) \to R_+^{p,\mathrm{ab}}(G)$ maps $[H,\varphi]_G$ to $[H,\varphi^\kappa]_G$. In fact each of the four maps occurring in the above equation commutes with restrictions, and elements in $R^p(G)$ are uniquely determined by their restrictions to cyclic p'-subgroups. Thus it suffices to verify the above equation in this case, which is easily done, since $a_G(\varphi) = [G,\varphi]_G$ (by [Bo98b, Proposition 6.12]) for $\varphi \in \mathrm{Hom}(G, F^\times)$ interpreted as irreducible Brauer character.

3 The trivial source ring

3.1 The trivial source ring $T^p(G)$ of a finite group G for the prime p is the free abelian group on the set $[M]$ of isomorphism classes of indecomposable FG-modules that are isomorphic to direct summands of permutation FG-modules, where F is an algebraically closed field of characteristic p. As in 2.1 we fix an identification of the roots of unity of F with the roots of unity of K of p'-order.

Recall from [Bo01] that $K \otimes T^p(G)$ is semisimple and that one has an isomorphism

$$s^{T^p(G)}\colon K \otimes T^p(G) \to \prod_{(P,g)\in\mathcal{T}^p(G)/G} K \;\cong\; \mathrm{Hom}_G(\mathcal{T}^p(G), K)\,.$$

Here $\mathcal{T}^p(G)$ denotes the set of pairs (P,g) where P is a p-subgroup of G and $g \in N_G(P)$ is a p'-element in the normalizer of P (in [Bo01], $\mathcal{T}^p(G)$ was denoted by $\tilde{\mathcal{T}}^p(G)$). Obviously $\mathcal{T}^p(G)$ is a G-set via conjugation and forms even a functor $\mathcal{T}^p\colon \mathsf{gr} \to \mathsf{set}$. The K-species of T^p are given by $s_{(P,g)}^{T^p(G)}\colon T^p \to K$, for $(P,g) \in \mathcal{T}^p(G)$, with

$$s_{(P,g)}^{T^p(G)} = s_g^{R^p(H)} \circ \gamma_H \circ q_H^T \circ \mathrm{res}_H^G$$

where H is the *p-hypoelementary* subgroup of G (i.e., $H/O_p(H)$ is cyclic) generated by P and g, q_H^T projects on the span of the classes $[M]$ of indecomposable trivial source FH-modules M with vertex P and vanishes on the classes of indecomposable modules with smaller vertex, $\gamma_H\colon T^p(H) \to R^p(H)$ is the canonical map induced by taking the Brauer character of a trivial source FG-module, and $s_g^{R^p(H)}$ is the species on the Brauer character ring which evaluates at g. The species $s_{(P_1,g_1)}^{T^p(G)}$ and $s_{(P_2,g_2)}^{T^p(G)}$ are equal if and only if (P_1,g_1) and (P_2,g_2) are G-conjugate.

3.2 According to [Bo01, 2.2] we may define for $\kappa \in \hat{\mathbb{Z}}_{p'}$ a natural K-algebra map $\Psi_G^\kappa\colon K \otimes T^p(G) \to K \otimes T^p(G)$ by

$$\Psi_G^\kappa \circ s_{(P,g)}^{T^p(G)} = s_{(P,g^\kappa)}^{T^p(G)}\,,$$

for all $(P,g) \in \mathcal{T}^p(G)$, since the map

$$\psi_G^\kappa \colon \mathcal{T}^p(G) \to \mathcal{T}^p(G)\,, \quad (P,g) \mapsto (P, g^\kappa)\,,$$

defines a natural transformation from $\mathcal{T}^p$ to $\mathcal{T}^p$. Obviously, we have again

$$\Psi_G^\kappa \circ \Psi_G^\lambda = \Psi_G^{\kappa\lambda}\,,$$

for all $\kappa, \lambda \in \hat{\mathbb{Z}}_{p'}$. At this point it is not clear if $\Psi_G^\kappa(T^p(G)) \subseteq T^p(G)$ for all finite groups G. In order to verify this we will use the canonical induction formula for the trivial source ring (cf. [Bo98a]) which is a map $a_G \colon T^p(G) \to T_+^{p,\mathrm{ab}}(G)$, where $T_+^{p,\mathrm{ab}}(G)$ is the free abelian group on G-orbits $[H,\varphi]_G$ of pairs (H,φ) with $H \leqslant G$ and $\varphi \in \mathrm{Hom}(H, F^\times)$. In fact, $T_+^{p,\mathrm{ab}}(G) = R_+^{p,\mathrm{ab}}(G)$. After some preparations we will show in Proposition 3.5 that

$$\Psi_G^\kappa = b_G \circ \Psi_{+,G}^\kappa \circ a_G\,, \tag{3.2.a}$$

where $\Psi_{+,G}^\kappa \colon T_+^{p,\mathrm{ab}}(G) \to T_+^{p,\mathrm{ab}}(G)$ maps $[H,\varphi]_G$ to $[H,\varphi^\kappa]_G$ and $b_G \colon T_+^{p,\mathrm{ab}}(G) \to T^p(G)$ maps $[H,\varphi]_G$ to $\mathrm{ind}([F_\varphi])$. Here and in the sequel we denote by F_φ the one-dimensional FG-module F with G-action given by φ.

3.3 Proposition *One has $\Psi_G^\kappa([F_\varphi]) = [F_{\varphi^\kappa}]$ for every $\kappa \in \hat{\mathbb{Z}}_{p'}$, $G \in \mathsf{gr}$, and $\varphi \in \mathrm{Hom}(G, F^\times)$.*

Proof Using species we only have to show that

$$(s_{(P,g)}^{T^p(G)} \circ \Psi_G^\kappa)([F_\varphi]) = s_{(P,g)}^{T^p(G)}([F_{\varphi^\kappa}])$$

for all $(P,g) \in \mathcal{T}^p(G)$. By definition of Ψ_G^κ it suffices to prove that $s_{(P,g^\kappa)}^{T^p(G)}([F_\varphi]) = s_{(P,g)}^{T^p(G)}([F_{\varphi^\kappa}])$. But this follows immediately from the definition of the species. ∎

The ring $T^p(G)$ is filtered by the partially ordered set of p-subgroups of G as follows. For a p-subgroup Q of G we write $T^{p,Q}(G)$ for the span of the isomorphism classes $[M]$ of indecomposable trivial source FG-modules M which have a vertex contained in Q. Thus, $Q_1 \leqslant_G Q_2$ implies $T^{p,Q_1}(G) \subseteq T^{p,Q_2}(G)$. We call this filtration and also the one induced on $K \otimes T^p(G)$ the *vertex filtration*. For $(Q,g) \in \mathcal{T}^p(G)$ we denote the unique primitive idempotent of $K \otimes T^p(G)$ on which $s_{(Q,g)}$ does not vanish by $e_{(Q,g)}$. Thus,

$$s_{(P,h)}^{T^p(G)}(e_{(Q,g)}) = \begin{cases} 1, & \text{if } [P,h]_G = [Q,g]_G, \\ 0, & \text{if } [P,h]_G \neq [Q,g]_G, \end{cases}$$

for all $(P,h), (Q,g) \in \mathcal{T}^p(G)$.

3.4 Proposition (a) *Let $Q \leqslant G$ be a p-subgroup. The idempotents $e_{(P,g)}$ with $P \leqslant Q$ span the K-subspace $K \otimes T^{p,Q}(G)$ of $K \otimes T^p(G)$.*

(b) *Let $\kappa \in \hat{\mathbb{Z}}_{p'}$. The map $\Psi_G^\kappa \colon K \otimes T^p(G) \to K \otimes T^p(G)$ respects the vertex filtration, i.e., $\Psi_G^\kappa(K \otimes T^{p,Q}(G)) \subseteq K \otimes T^{p,Q}(G)$ for each p-subgroup Q of G.*

Proof (a) By [Bo01, 3.6] the idempotent $e_{(P,g)}$ is a K-linear combination of elements of the form $\mathrm{ind}_H^G([F_\varphi])$ with H a p-hypoelementary subgroup of G such that $O_p(H) \leqslant_G P$. Thus, $e_{(P,g)} \in K \otimes T^{p,Q}(G)$. On the other hand one obtains via the Green correspondence (cf. [Br85]) that the number of (isomorphism classes of) indecomposable trivial source FG-modules with vertex Q is equal to the number of $N_G(Q)$-conjugacy classes of p'-elements in $N_G(Q)/Q$. But it is easy to see that the map $N_G(Q)_{p'} \to (N_G(Q)/Q)_{p'}$, $g \mapsto gQ$, induces a bijection between the $N_G(Q)$-conjugacy classes so that the number of indecomposable trivial source FG-modules with vertex Q is equal to the number of G-conjugacy classes of pairs $(P,g) \in \mathcal{T}^p(G)$ with $P =_G Q$. An easy induction argument on the order of Q now shows that the elements $e_{(P,g)}$ with $P \leqslant_G Q$ span $K \otimes T^{p,Q}(G)$.

(b) By definition of Ψ_G^κ we have

$$\big(s_{(Q,g)}^{T^p(G)} \circ \Psi_G^\kappa\big)(e_{(P,h)}) = s_{(Q,g^\kappa)}^{T^p(G)}(e_{(P,h)}) = \delta_{[P,h]_G,[Q,g]_G}\,,$$

and therefore,

$$\Psi_G^\kappa(e_{(P,h)}) = \sum_{[Q,g]_G \in (\overline{\psi}_G^\kappa)^{-1}([P,h]_G)} e_{(Q,g)}\,,$$

where $\overline{\psi}_G^\kappa \colon \mathcal{T}^p(G)/G \to \mathcal{T}^p(G)/G$ denotes the induced map on G-orbits, cf. also [Bo01, Equation (3.2.a)]. Together with part (a) this implies the claim, since $\psi_G^\kappa(Q,g) = (Q,g^\kappa)$. □

3.5 Proposition *The Equation* (3.2.a) *holds. In particular,* $\Psi_G^\kappa(T^p(G)) \subseteq T^p(G)$ *for every finite group* G *and every* $\kappa \in \hat{\mathbb{Z}}_{p'}$.

Proof We know from [BK00, Proposition 5.1] that elements in $T^p(G)$ are uniquely determined by their values after applying $q_H^T \circ \mathrm{res}_H^G$ for any p-hypoelementary subgroup G. Note that if M is an indecomposable trivial source FH-module for a p-hypoelementary H, then there are two cases. Either $M \cong F_\varphi$ for some $\varphi \in \mathrm{Hom}(H, F^\times)$ and then $q_H^T([M]) = [M]$, or M is not of this form and then $q_H^T([M]) = 0$. Since all four maps occurring in Equation (3.2.a) commute with restrictions, it suffices to show that

$$q_H^T \circ \Psi_H^\kappa = q_H^T \circ b_H \circ \Psi_{+,H}^\kappa \circ a_H$$

for any p-hypoelementary group H. So let H be p-hypoelementary and let $P := O_p(H)$. If M is an indecomposable trivial source FH-module of the form F_φ then $a_H([M]) = [H,\varphi]_H$ by [Bo98b, Proposition 6.12], and the right hand side of the above equation is equal to $[F_{\varphi^\kappa}]$. On the other hand, by Proposition 3.3, also the left hand side maps $[M]$ to $[F_{\varphi^\kappa}]$. If M is not of the form F_φ, then M has vertex Q for some $Q < P$ and by Proposition 3.4, we obtain $q_H^T(\Psi_H^\kappa([M])) = 0$. On the other hand, [Bo98a, Theorem 4.3(viii)] implies that $a_H([M])$ is a linear combination of elements $[U,\varphi]_H$ with $O_p(U) \leqslant_H Q$. Thus,

$(b_H \circ \Psi^\kappa_{+,H} \circ a_H)([M])$ lies again in $T^{p,Q}(H)$ so that the right hand side of the above equation applied to $[M]$ also vanishes. □

3.6 Summarizing the above considerations we obtain a monoid homomorphism

$$\hat{\mathbb{Z}}_{p'} \to \mathrm{End}^{\mathrm{f}}(T^p)\,, \quad \kappa \mapsto \Psi^\kappa\,, \tag{3.6.a}$$

where $\mathrm{End}^{\mathrm{f}}(T^p)$ denotes the set of natural endomorphisms $T^p \to T^p$ that preserve the vertex filtration.

3.7 Theorem *The map in* (3.6.a) *is an isomorphism.*

Proof If $\kappa \neq \kappa'$ in $\hat{\mathbb{Z}}_{p'}$, then $\kappa_n \neq \kappa'_n$ in $\mathbb{Z}/n\mathbb{Z}$ for some p'-number $n \in \mathbb{N}$. Let C_n be a cyclic group of order n and let $\varphi \in \mathrm{Hom}(C_n, F^\times)$ be faithful. Then $\Psi^\kappa_{C_n}(\varphi) = \varphi^{\kappa_n} \neq \varphi^{\kappa'_n} = \Psi^{\kappa'}_{C_n}(\varphi)$ and we have $\Psi^\kappa \neq \Psi^{\kappa'}$.

Now let $\Psi \in \mathrm{End}^{\mathrm{f}}(T^p)$. As in the proof of Theorem 1.2 one can show that there exists $\kappa \in \hat{\mathbb{Z}}_{p'}$ such that whenever $n \in \mathbb{N}$ is a p'-number and C_n a cyclic group of order n, then $\Psi_{C_n}([F_\varphi]) = [F_{\varphi^\kappa}]$ for every $\varphi \in \mathrm{Hom}(C_n, F^\times)$. In fact, one has again $T^p(C_n) \cong \mathbb{Z}\mathrm{Hom}(C_n, F^\times)$. If $G \in \mathsf{gr}$ is arbitrary and $\varphi \in \mathrm{Hom}(G, F^\times)$ we still have $\Psi_G([F_\varphi]) = [F_{\varphi^\kappa}]$ by functoriality with respect to the natural epimorphism $f\colon G \to G/\ker(\varphi)$. We claim that $\Psi = \Psi^\kappa$. Again by [BK00, Proposition 5.1] it suffices to show that, for each $G \in \mathsf{gr}$, we have

$$q^T_H \circ \mathrm{res}^G_H \circ \Psi_G = q^T_H \circ \mathrm{res}^G_H \circ \Psi^\kappa_G\,,$$

for all p-hypoelementary subgroups H of G. Since Ψ and Ψ^κ are natural, it suffices to show that $q^T_H \circ \Psi_H = q^T_H \circ \Psi^\kappa_H$ for all p-hypoelementary groups H. But in this case, Ψ_H and Ψ^κ_H coincide on $[F_\varphi]$ for any $\varphi \in \mathrm{Hom}(H, F^\times)$ by Proposition 3.3 and $q^T_H \circ \Psi_H$ and $q^T_H \circ \Psi^\kappa_H$ vanish on indecomposable trivial source FH-modules with vertex smaller than $O_p(H)$ by Proposition 3.4(b). This concludes the proof of the theorem. □

3.8 Remark (a) Adams operations Ψ^k_G for the Green ring and an integer k where introduced by Benson in [Be84a] (see also [Be84b, 2.16]). It is not difficult to see that they map the trivial source ring to itself and coincide with Ψ^κ_G when $\kappa_n = k + n\mathbb{Z}$ with $n = \exp(G)_{p'}$, by [Be84b, 2.16.9].

(b) The map

$$\Psi_G\colon T^p(G) \to T^p(G)\,, \quad [M] \mapsto \dim_F(M) \cdot [F]\,,$$

defines an element $\Psi \in \mathrm{End}(T^p)$ that is not vertex preserving. In fact, the trivial FG-module F has as vertices the Sylow p-subgroups of G.

(c) In [Bo01] we considered the natural transformations $\gamma\colon T^p \to R^p$ and $\tau\colon R^p \to T^p$ between the trivial source ring and the Brauer character ring which satisfy $\gamma \circ \tau = \mathrm{id}_{R^p}$. Using species they translate to natural transformations

$\gamma^*\colon \mathcal{R}^p \to \mathcal{T}^p$ and $\tau^*\colon \mathcal{T}^p \to \mathcal{R}^p$ given by $\gamma^*_G(g) = (1, \langle g \rangle)$ and $\tau^*_G(P,g) = g$. Since the Adams operations on R^p and T^p where induced by the natural transformations $\psi^\kappa_G\colon \mathcal{R}^p(G) \to \mathcal{R}^p(G)$, $g \mapsto g^\kappa$, and $\psi^\kappa_G\colon \mathcal{T}^p(G) \to \mathcal{T}^p(G)$, $(P,g) \mapsto (P, g^\kappa)$, and since they commute with τ^*_G and γ^*_G we obtain commutative diagrams

$$\begin{array}{ccc} T^p(G) & \xrightarrow{\Psi^\kappa_G} & T^p(G) \\ {\scriptstyle\gamma_G}\downarrow & & \downarrow{\scriptstyle\gamma_G} \\ R^p(G) & \xrightarrow{\Psi^\kappa_G} & R^p(G) \end{array} \qquad\qquad \begin{array}{ccc} R^p(G) & \xrightarrow{\Psi^\kappa_G} & R^p(G) \\ {\scriptstyle\tau_G}\downarrow & & \downarrow{\scriptstyle\tau_G} \\ T^p(G) & \xrightarrow{\Psi^\kappa_G} & T^p(G) \end{array}$$

for every $G \in \mathsf{gr}$ and $\kappa \in \hat{\mathbb{Z}}_{p'}$.

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