Abstract

We systematically investigate various representation rings of a finite group as functors to the category of rings together with various natural transformations between them. Moreover, their species and formulae for their primitive idempotents over a splitting field are determined.

Introduction

In this article we investigate various representation rings of a finite group $G$: The Burnside ring $B(G)$, the rings of monomial representations in characteristic zero, $D(G)$, and in positive characteristic $p$, $D^p(G)$, the trivial source ring $T^p(G)$ and linear source ring $L^p(G)$ in (residual) characteristic $p$, and the classical character ring $R(G)$ and Brauer character ring $R^p(G)$ in characteristic $p$. The rings $D(G)$ and $D^p(G)$ were used for the canonical induction formulae of the character ring, the Brauer character ring, the linear source ring, and the trivial source ring (cf. [Bo98b]). We consider these representation rings as ring valued functors on the category of finite groups and use a network of natural transformations between them, cf. Diagram (1.1.a).

In Section 1 we recall the definitions of these representation rings. In order to view them as functors one has to be particularly careful in the case of linear source and trivial source modules, since usually they are defined using a complete discrete valuation ring that has enough $p$-power roots of unity for the finite group $G$. But there is no complete discrete valuation ring with residual characteristic $p$ containing all $p$-power roots of unity that can be used simultaneously.
rings the same pattern occurs: The species are indexed over already known. However, it is somewhat surprising that for all representation rings the trivial source ring that uses the canonical induction formula for the Brauer character ring.

Section 2 is devoted to the species of the above representation rings. Those for $D(G)$ and $D^p(G)$ cannot be found in the literature yet. The others are already known. However, it is somewhat surprising that for all representation rings the same pattern occurs: The species are indexed over $G$-orbits of some $G$-set, even better, these $G$-sets come from functors between the category of finite groups and the category of finite sets and the $G$-set structure comes via functoriality from the inner automorphisms of $G$. Moreover, each of the natural transformations between the representation rings we introduced is induced by a natural transformation between these set valued functors. This functorial picture is worked out in detail.

In Section 3 we give explicit formulae for the primitive idempotents of the representation rings (over a splitting field) introduced in Section 1. Using the functorial properties established in Section 2 we can derive formulae for all representation rings (over a splitting field) introduced in Section 1. Using the functorial approach allows us also to determine which idempotents vanish under restriction maps or the considered natural transformations.

Acknowledgement The author would like to thank B. Külshammer for his helpful comments on parts of the paper.

Notation Throughout this article $G$ denotes a finite group and $n := \exp(G)$ its exponent. For $g \in G$ and a set of primes $\pi$ we write $g_\pi$ for the $\pi$-part of $g$, and $G_\pi$ for the set of all $\pi$-elements of $G$, i.e., elements whose order is only divisible by primes in $\pi$. Similarly, we write $m_\pi$ for the $\pi$-part of $m \in \mathbb{N}$. If $\pi = \{p\}$ we will omit the curly brackets in the index. If $\pi$ is the set of primes different from $p$ we use the index $p'$. By $H < G$ (resp. $H \leq G$) we indicate that $H$ is a subgroup (resp. a proper subgroup) of $G$. For a prime $p$ the largest normal $p$-subgroup of $G$ is denoted by $O_p(G)$. For $H \leq G$ and $g \in G$ we set $^gH := gHg^{-1}$. If $X$ is a $G$-set and $x, x' \in X$ we write $[x]_G$ for the $G$-orbit of $x$ and $x =_G x'$ if $[x]_G = [x']_G$. The stabilizer of $x \in G$ is denoted by $N_G(x)$.

Throughout we denote by $K \subset \mathbb{C}$ the subfield obtained by adjoining all roots of unity to $\mathbb{Q}$ and by $\mathcal{O}_K$ its ring of algebraic integers. Moreover, we fix a prime $p$ and a maximal ideal $p_K$ of $\mathcal{O}_K$ containing $p$, and we denote by $\mathcal{O}$ the localization of $\mathcal{O}_K$ with respect to $\mathcal{O}_K \setminus p_K$. The maximal ideal of the local ring $\mathcal{O}$ is denoted by $p$. Then, $F := \mathcal{O}/p \cong \mathcal{O}_K/p_K$ is an algebraic closure of $\mathbb{Z}/p\mathbb{Z}$ via the natural map induced by the inclusion $\mathbb{Z} \subset \mathcal{O}_K \subset \mathcal{O}$. For $m \in \mathbb{N}$ we denote by $\zeta_m$ a primitive $m$-th root of unity and we set $K_m := \mathbb{Q}(\zeta_m), \mathcal{O}_m := K_m \cap \mathcal{O}, p_m := \mathcal{O}_m \setminus p$, and $F_m := \mathcal{O}_m/p_m$. Then, $\mathcal{O}_m$ is a discrete valuation ring with maximal ideal $p_m$, finite residue field $F_m$, and field of fractions $K_m$ such that $\bigcup_{m \in \mathbb{N}} K_m = K, \bigcup_{m \in \mathbb{N}} \mathcal{O}_m = \mathcal{O}, \bigcup_{m \in \mathbb{N}} p_m = p$, and $\bigcup_{m \in \mathbb{N}} F_m = F$. 

2
By Ri we denote the category of associative rings with identity together with ring homomorphisms preserving the identity. For \( R \in \text{Ri} \) we denote by \( R\text{mod} \) the category of finitely generated left \( R \)-modules and by \( R^\times \) the group of invertible elements of \( R \). For a commutative ring \( R \) we denote the category of \( R \)-algebras that are finitely generated as \( R \)-modules by \( R\text{alg} \). For two categories \( C \) and \( D \) and two functors \( F, G : C \to D \) we denote by \( \text{Nat}(F, G) \) the set of natural transformations between \( F \) and \( G \). Whenever this notation is used we assume implicitly that the categories \( C \) and \( D \) are small.

1 The representation rings

1.1 In this section we will give a common framework for various representation rings which we consider as contravariant functors from the category \( \text{gr} \) of finite groups to the category \( \text{Ri} \) of rings. We will obtain a commutative diagram

\[
\begin{array}{ccc}
R^{p,\text{ab}} & \overset{\rho}{\longrightarrow} & R^{\text{ab}} \\
\downarrow{\eta} & & \downarrow{\pi} \\
B & \overset{\iota}{\longrightarrow} & D^p \\
\downarrow{\beta} & & \downarrow{\beta} \\
T_p & \overset{\lambda}{\longrightarrow} & L_p \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
R^p & \overset{\delta}{\longrightarrow} & R \\
\end{array}
\]

(1.1.a)

of such functors and natural transformations between them which will be introduced in the following subsections. An exponent \( p \) always indicates that the functor depends on the prime \( p \). For each of these functors we denote the ring homomorphism associated to a group homomorphism \( f \) in \( \text{gr} \) by \( \text{res}_f \). If \( f \) is the inclusion of a subgroup \( H \) of \( G \) into \( G \), we usually write \( \text{res}_{G/H} \). For any pair of arrows in opposite directions the monomorphism is a section of the epimorphism, i.e.,

\[
\pi \circ \iota = \text{id}, \quad \rho \circ \iota = \text{id}, \quad \delta \circ \lambda = \text{id}, \quad \pi \circ \eta = \text{id}, \quad \gamma \circ \tau = \text{id}.
\]
Moreover, all possible squares commute in the sense that the composition of
two arrows equals the composition of the other two, i.e.,
\[\ell \circ \eta = \eta \circ \ell, \quad \ell \circ \pi = \pi \circ \ell, \quad \pi \circ \rho = \rho \circ \pi, \quad \beta \circ \ell = \ell \circ \beta, \quad \beta \circ \rho = \rho \circ \beta, \quad \gamma \circ \ell = \lambda \circ \gamma, \quad \gamma \circ \rho = \delta \circ \gamma, \] (1.1.b)

and sometimes the composition of three maps in a square equals the fourth,
namely in the cases
\[\rho \circ \eta \circ \ell = \eta, \quad \pi \circ \rho \circ \eta = \rho, \quad \pi \circ \ell \circ \eta = \ell, \quad \rho \circ \pi \circ \ell = \pi, \quad \rho \circ \beta \circ \ell = \beta, \quad \delta \circ \gamma \circ \ell = \gamma, \quad \gamma \circ \ell \circ \tau = \lambda. \] (1.1.c)

The groups \(B(G), D^p(G), D(G), T^p(G), L^p(G), R^p(G), \) and \(R(G)\) allow induction
maps from subgroups and all maps between these groups commute with
induction. All these statements are immediate consequences of the definitions
or well-known properties of these maps. The representation rings and maps of
the above diagram will be defined next.

1.2 The Burnside ring \(B(G)\) is the representation ring of the category \(G\text{-set}\) of
finite \(G\)-sets. For \(S \in G\text{-set}\) we write \([S]\) for its class in \(B(G)\). Recall that if
\(H\) runs through a set of representatives of conjugacy classes of subgroups of \(G,\)
then the elements \([G/H]\) form a \(\mathbb{Z}\)-basis of \(B(G)\). The disjoint union and the
cartesian product of two \(G\)-sets with diagonal action induces a ring structure on
\(B(G)\). If \(f : G \to G\) is a group homomorphism one obtains an induced functor
\(\text{res}_f : G\text{-set} \to G\text{-set}\) which in turn induces a ring homomorphism
\[\text{res}_f : B(G) \to B(G), \quad [G/H] \mapsto \sum_{g \in f(G) \cap G/H} [\hat{G}/f^{-1}(gH)],\]
that makes \(B\) into a contravariant functor \(B : \text{gr} \to \text{Ri}\).

1.3 We define \(R^{ab}(G)\) as the group ring of the multiplicative group \(\hat{G} := \text{Hom}(G, K^\times)\) and \(R^{p,\text{ab}}(G)\) as the group ring of the \(p\)'-part \(\hat{G}_{p'}\) over \(\mathbb{Z}\). Obviously we have natural identifications \(R^{ab}(G) \cong \mathbb{Z}\text{Hom}(G, O^\times) \cong 
\mathbb{Z}\text{Hom}(G, K^\times) \cong \mathbb{Z}\text{Hom}(G, O_n^\times)\) and \(R^{p,\text{ab}}(G) \cong \mathbb{Z}\text{Hom}(G, F^\times) \cong Z\text{Hom}(G, F_{p'}^\times)\) using the various natural ring homomorphisms between \(O, F, O_n, K,\) and \(K_n\). There is a natural inclusion \(\iota_G : R^{p,\text{ab}}(G) \to R^{ab}(G)\)
and a left inverse \(\rho_G : R^{ab}(G) \to R^{p,\text{ab}}(G)\) mapping \(\varphi \in \hat{G}\) to its \(p\)'-part, or if we interpret \(R^{p,\text{ab}}(G)\) as \(\mathbb{Z}\text{Hom}(G, F^\times)\) by mapping \(\varphi\) to its reduction mod \(p\). If
\(f : G \to G\) is a group homomorphism between finite groups, we obtain a restriction
map \(\text{Hom}(G,K^\times) \to \text{Hom}(\hat{G}, K^\times)\) which induces ring homomorphisms
\(\text{res}_f : R^{ab}(G) \to R^{ab}(G)\) and \(\text{res}_f : R^{p,\text{ab}}(G) \to R^{p,\text{ab}}(\hat{G})\) so that \(R^{ab} : \text{gr} \to \text{Ri}\) becomes a contravariant functor and \(R^{p,\text{ab}}\) a subfunctor. The notation \(R^{p,\text{ab}}(G)\) and \(R^{p,\text{ab}}(\hat{G})\) will become more meaningful after the introduction of the character
ring functor \(R\) and the Brauer character ring functor \(R^p\). It is immediate that
\(R^{ab}(G) \cong R(G^{ab})\) and \(R^{p,\text{ab}} \cong R^p(G^{ab})\), where \(G^{ab}\) denotes the commutator
factor group of \(G,\).
1.4 Next we describe $D(G)$ and $D^p(G)$ (the letter $D$ paying tribute to Dress who studied similar rings earlier, cf. [Dr71]). For a commutative ring $R$ we set

$$\mathcal{M}_R(G) := \{(H, \varphi) \mid H \leq G, \varphi \in \text{Hom}(H, R^\times)\}.$$ 

We will view $\text{Hom}(H, R^\times)$ as multiplicative group. The set $\mathcal{M}_R(G)$ is a $G$-poset, i.e., a partially ordered set (with $(I, \psi) \leq (H, \varphi)$ if and only if $I \leq H$ and $\psi = \varphi|_I$) and a $G$-set under conjugation $g(H, \varphi) := (gH, g\varphi)$ (with $g\varphi(x) := \varphi(g^{-1}xg)$ for $x \in gH$) such that the $G$-action respects the poset structure. Similarly, we define $\mathcal{M}_{R^p}(G)$ as the $G$-subposet consisting of those $(H, \varphi) \in \mathcal{M}_R(G)$ where $\varphi$ has $p'$-order. Now we define $D_R(G)$ (resp. $D^p_R(G)$) as the free abelian group on the $G$-conjugacy classes $[H, \varphi]_G$ of elements in $\mathcal{M}_R(G)$ (resp. $\mathcal{M}_{R^p}(G)$). Similar to the Burnside ring, $D_R(G)$ and $D^p_R(G)$ can be considered as representation rings of certain categories of finite $G$-equivariant line bundles (cf. [Bo97b]) but we do not need this interpretation here. $D_R(G)$ is a commutative ring under

$$[H, \varphi]_G \cdot [I, \psi]_G = \sum_{g \in H \cap G / I} [H \cap gI, \varphi|_{H \cap gI} \cdot g\varphi|_{H \cap gI}]_G,$$

see [Bo97b, 5.3]. With this definition, $D^p_R(G)$ is a subring of $D_R(G)$. If $R$ is any domain containing $\mathcal{O}_n$, then $\mathcal{M}_{\mathcal{O}_n}(G) \cong \mathcal{M}_R(G)$, $\mathcal{M}^p_{\mathcal{O}_n}(G) \cong \mathcal{M}^p_R(G)$, $D_{\mathcal{O}_n}(G) \cong D_R(G)$, and $D^p_{\mathcal{O}_n}(G) \cong D^p_R(G)$ in a canonical way, using the inclusion $\mathcal{O}_n^\times \leq R^\times$. We set $\mathcal{M}(G) := \mathcal{M}_{\mathcal{O}_n}(G)$, $\mathcal{M}^p(G) := \mathcal{M}^p_{\mathcal{O}_n}(G)$, $D(G) := D_{\mathcal{O}_n}(G)$, and $D^p(G) := D^p_{\mathcal{O}_n}(G)$, and use the above identifications whenever $R$ is a domain containing $\mathcal{O}_n$. Note also that we can identify $\mathcal{M}^p(G)$ with $\mathcal{M}_F(G)$ or $\mathcal{M}_{F_n}(G)$. There is an obvious inclusion $i_G : D^p(G) \to D(G)$.

If $f : \hat{G} \to G$ is a group homomorphism, then

$$\text{res}_f : D(G) \to D(\hat{G}), \quad [H, \varphi]_G \mapsto \sum_{g \in f(H) \cap G / H} [f^{-1}(gH), g\varphi \circ f^{-1}(gH)]_{\hat{G}}$$

is a ring homomorphism (see [Bo97b, 5.4]) which maps $D^p(G)$ to $D^p(\hat{G})$ and makes $D$ and $D^p$ into contravariant functors from gr to $\text{Ri}$. Note that the embedding

$$i_G : B(G) \to D^p(G), \quad [H, \varphi]_G \mapsto [H, 1]_G,$$

is a ring homomorphism which is natural in $G$. The injective ring homomorphisms $i_G : B(G) \to D^p(G)$ and $i_G : D^p(G) \to D(G)$ have natural left inverses

$$\pi_G : D^p(G) \to B(G), \quad [H, \varphi]_G \mapsto [G / H],$$

$$\rho_G : D(G) \to D^p(G), \quad [H, \varphi]_G \mapsto [H, \varphi]_G,$$

in the category $\text{Ri}$. Recall that $\varphi_{p'}$ denotes the $p'$-part of $\varphi$.

Note that $D^p(G)$ is canonically isomorphic to $D_F(G) = D^p_{F_n}(G)$ or $D_{F_n}(G) = D^p_{F_n}(G)$ via the isomorphism $\text{Hom}(H, \mathcal{O}_n^\times)_{p'} \to \text{Hom}(H, F_n^\times) \to \text{Hom}(H, F_n^\times)$ induced by the reduction homomorphism $\mathcal{O}_n \to F_n$ and the inclusion $F_n \to F$. 
With these identifications one can interpret $\rho_G$ also as reduction in the second argument of $[H, \varphi]_G$.

We have a ring homomorphism

$$ \eta_G: \mathcal{R}^\text{ab}(G) \to D(G), \quad \varphi \mapsto [G, \varphi]_G, $$

which is natural in $G$ and restricts to a natural transformation $\eta: \mathcal{R}^\text{p,ab} \to D^p$. Moreover, we have a left inverse of $\eta$, given by

$$ \pi_G: D(G) \to \mathcal{R}^\text{ab}(G), \quad [H, \varphi]_G \mapsto \begin{cases} \varphi, & \text{if } H = G, \\ 0, & \text{otherwise,} \end{cases} \quad (1.4.a) $$

which again restricts to a natural transformation $\pi: D^p \to \mathcal{R}^\text{p,ab}$. It is now easy to check that the first four equations in (1.1.b) and (1.1.c) hold.

1.5 Let $R$ be a commutative ring. A permutation $RG$-module is an $RG$-module which has a finite $G$-stable $R$-basis. A monomial $RG$-module is an $RG$-module $M$ which has a finite $R$-basis such that $G$ permutes the rank-one $R$-submodules spanned by these basis elements. This is the same as saying that $M$ is isomorphic to a direct sum of modules of the form $\text{Ind}_H^G(R_\varphi)$, where $(H, \varphi) \in \mathcal{M}_R(G)$ and $R_\varphi$ denotes the $RH$-module $R$ with $H$-action defined by $hr := \varphi(h)r$ for $h \in H$ and $r \in R$. We call an $RG$-module a trivial source (resp. linear source) $RG$-module if it is isomorphic to a direct summand of a permutation (resp. monomial) $RG$-module. If $R$ is a complete discrete valuation ring, these modules can be characterized via the sources of their indecomposable direct summands, cf. [Bo98a, Proposition 1.1, 1.2]. This explains the terminology. For general $R$ it might be more suitable to call these modules quasi permutation (resp. monomial) $RG$-modules. We denote by $R_G^{\text{triv}}$ (resp. $R_G^{\text{lin}}$) the categories of trivial source (resp. linear source) $RG$-modules. We assume that the Krull-Schmidt-Theorem holds in $R_G^{\text{lin}}$. Then we define $L_R(G)$ as the free abelian group on the set of isomorphism classes $[M]$ of indecomposable modules $M \in R_G^{\text{lin}}$. If $M \in R_G^{\text{lin}}$ and $M = M_1 + \cdots + M_r$ is a decomposition into indecomposable modules, we set $[M] = [M_1] + \cdots + [M_r] \in L_R(G)$. The group $L_R(G)$ has a commutative ring structure induced by the tensor product $M \otimes_R N$ of two modules $M, N \in R_G^{\text{lin}}$. Similarly, we define $T_R(G)$ as the representation ring of $R_G^{\text{triv}}$. This is a subring of $L_R(G)$. If $f: \tilde{G} \to G$ is a group homomorphism then every $M \in R_G^{\text{lin}}$ is also an object in $G$ using the $G$-action given by restriction along $f$. In fact, the restriction preserves permutation modules and monomial modules. This induces a ring homomorphism $\text{res}_f: L_R(G) \to L_R(\tilde{G})$ such that $\text{res}_f(T_R(G)) \subseteq T_R(\tilde{G})$, and $L_R$ and $T_R$ become functors form $\mathfrak{gr}$ to $\mathfrak{Ri}$.

We are mainly interested in $R = \mathcal{O}_m$ for certain multiples $m$ of $n$. Note that by [CR81, Theorem 30.18(iii)] the Krull-Schmidt-Theorem holds in $\sigma_m G^{\text{lin}}$ whenever $m$ is a multiple of $n$. Moreover, by [Fe82, Lemma I.18.7] there exists a multiple $m$ of $n$ such that for any multiple $l$ of $m$ the functor $\mathcal{O}_l \otimes_{\mathcal{O}_m} - : \sigma_m G^{\text{lin}} \to \sigma_l G^{\text{lin}}$ preserves indecomposability and induces an isomorphism $L_{\mathcal{O}_m}(G) \to L_{\mathcal{O}_l}(G)$, since every indecomposable linear source $\mathcal{O}_m G$-module must be a direct summand of some $\text{Ind}_P^G(\mathcal{O}_m)_\psi$ for some $p$-subgroup $P$.
of $G$ and some $\psi \in \text{Hom}(P, \mathcal{O}_m^\times)$. In this case we call $K_m$ a splitting field for
the linear source modules of $G$. If $K_m$ and $K_{m'}$ are two such splitting fields, then we obtain a canonical isomorphism $L_{\mathcal{O}_m}(G) \to L_{\mathcal{O}_{m'}}(G)$ via $L_{\mathcal{O}_i}(G)$ with $l = \text{lcm}(m, m')$ or any common multiple $l$ of $m$ and $m'$.

Let $m$ be a multiple of $n$ such that $K_m$ is a splitting field for the linear source modules of $G$. In order to relate the ring $L_{\mathcal{O}_m}(G)$ to the linear source rings used in the literature let $\tilde{\mathcal{O}}_m$ denote a $p$-adic completion of $\mathcal{O}_m$ and $\tilde{p}_m$ the maximal ideal of $\tilde{\mathcal{O}}_m$. It is well-known that the Krull-Schmidt-Theorem holds for $\tilde{\mathcal{O}}_m$. By [CR81, Theorem 30.18], the functor $\tilde{\mathcal{O}}_m \otimes_{\mathcal{O}_m} - : \mathcal{O}_m\text{lin} \to \tilde{\mathcal{O}}_m\text{lin}$ preserves indecomposability and induces an isomorphism $L_{\mathcal{O}_m}(G) \to L_{\tilde{\mathcal{O}}_m}(G)$. From the preceding discussions it follows that again for any multiple $l$ of $m$ the functor $\tilde{\mathcal{O}}_l \otimes_{\mathcal{O}_m} - : \mathcal{O}_m\text{lin} \to \tilde{\mathcal{O}}_l\text{lin}$ preserves indecomposability and induces an isomorphism $L_{\mathcal{O}_m}(G) \to L_{\tilde{\mathcal{O}}_l}(G)$. Let $\tilde{R}$ denote the valuation ring in a maximal unramified extension $E$ of the field of fractions of $\tilde{\mathcal{O}}_m$. In other words, $E$ is isomorphic to the field obtained by adjoining an $m_p$-th root of unity and all roots of unity of $p'$-order to the field $\mathbb{Q}_p$ of $p$-adic numbers. The ring $\tilde{R}$ is then a complete discrete valuation ring whose residue field is an algebraic closure of the field of $p$ elements. Since every indecomposable linear source $\tilde{R}G$-module is already defined over $\tilde{\mathcal{O}}_l$ for some multiple $l$ of $m$, we obtain that the functor $\tilde{R} \otimes_{\tilde{\mathcal{O}}_l} - : \tilde{\mathcal{O}}_l\text{lin} \to \tilde{R}\text{lin}$ preserves indecomposability and induces an isomorphism $L_{\tilde{\mathcal{O}}_l}(G) \to L_{\tilde{R}}(G)$. So, finally, the functor $\tilde{R} \otimes_{\mathcal{O}_m} - : \mathcal{O}_m\text{lin} \to \tilde{R}\text{lin}$ preserves indecomposability and induces an isomorphism $L_{\mathcal{O}_m}(G) \to L_{\tilde{R}}(G)$ so that we can use result about $L_{\tilde{R}}(G)$ from the literature also for $L_{\mathcal{O}_m}(G)$.

Everything we established above for linear source modules holds in particular for trivial source modules. Moreover, with $\tilde{R}$ as above it is well-known that reduction modulo the radical of $\tilde{R}$ gives a functor $\tilde{R}_{\text{triv}} \to F_{\text{triv}}$ that preserves indecomposability and induces an isomorphism $T_{\tilde{R}}(G) \to T_F(G)$, cf. [Br85]. Together with the above results reinterpreted for trivial source modules one obtains that also the functors $F_m \otimes_{\mathcal{O}_m} - : \mathcal{O}_m\text{triv} \to F_{\text{triv}}$ and $F \otimes_{F_m} - : F_{\text{triv}} \to F_{\text{triv}}$ preserve indecomposability and induces isomorphisms $T_{\mathcal{O}_m}(G) \to T_{F_m}(G)$ and $T_{F_m}(G) \to T_F(G)$.

Now we define $L^p(G) := L_{\mathcal{O}_m}(G)$ and $T^p(G) := T_{\mathcal{O}_m}(G)$, where $K_m$ is a splitting field for the linear source modules of $G$. Moreover, whenever $m' \in \mathbb{N}$ is such that also $K_{m'}$ is a splitting field for the linear source modules of $G$, then we identify $L^p(G)$ and $T^p(G)$ with $L_{\mathcal{O}_{m'}}(G)$ and $T_{\mathcal{O}_{m'}}(G)$ as above. In particular, if $f : G \to G$ is a group homomorphism then $\text{res}_f : L^p(G) \to L^p(\tilde{G})$ is well-defined and restricts to a ring homomorphism $T^p(G) \to T^p(\tilde{G})$. Summarizing the above considerations we obtain functors $T^p, L^p : \mathfrak{g}_F \to \tilde{R}i$ such that $\iota : T^p \subseteq L^p$ is a natural inclusion.

Sometimes we will identify $T^p(G)$ with $T_F(G)$ or $T_{F_m}(G)$ and $L^p(G)$ with $L_{\mathcal{O}_m}(G)$ or $L_{\tilde{R}}(G)$ always using the above isomorphism.
The homomorphism

$$\beta_G: D(G) \to L(G), \quad [H, \varphi]_G \mapsto \left[\text{Ind}^G_H(\mathcal{O}_m)_\varphi\right],$$

defines a natural transformation $D \to L^p$, and its restriction to $D^p(G)$ defines a natural transformation $\beta: D^p \to T^p$, since $\mathcal{O}_m$ is a trivial source $\mathcal{O}H$-module, if $\varphi$ has $p'$-order. The maps $\beta_G: D(G) \to L^p(G)$ and $\delta_G: D^p(G) \to T^p(G)$ are surjective, since there exist induction formulae for $L^p(G)$ and $T^p(G)$ using $D(G)$ and $D^p(G)$, cf. [Bo98a]. If we use the identifications $D^p(G) \cong D_F(G)$ and $T^p(G) \cong T_F(G)$, then the map $\beta: D^p(G) \to T^p(G)$ is given by $[H, \varphi]_G \mapsto \left[\text{Ind}^G_H(F_\varphi)\right]$ and the natural inclusion $\nu_G: T^p(G) \to L^p(G)$ has the natural left inverse $\rho_G: L_{\mathcal{O}_m}(G) \to L_{F_p}(G)$ given by reduction modulo $p_m$.

It is easy to check that the fifth and sixth equation in (1.1.b) and the fifth equation in (1.1.c) hold.

1.6 We denote by $R(G)$ (resp. $R^p(G)$) the character ring (resp. the Brauer character ring) of $G$. We identify $R(G)$ (resp. $R^p(G)$) with the Grothendieck ring of $\mathcal{K}G\text{mod}$ (resp. $FG\text{mod}$). For that reason we identify $F^\times$ with the $p'$-part of the group of roots of unity of $K$ via the natural surjection $\mathcal{O} \to \mathcal{O}/p = F$. Hence, we may assume that the Brauer characters as well as the characters have their values in $\mathcal{O}_K \subset K$. Then the decomposition map $\delta_G: R(G) \to R^p(G)$ is given by restriction of a character to the $p'$-elements of $G$. $\delta_G$ is a surjective ring homomorphism which commutes with induction and restrictions res$_f$ with respect to arbitrary group homomorphisms $f$. In short, $\delta: R \to R^p$ is a natural transformation between the ring valued functors $R$ and $R^p$ (see [Se78, §14-§18] for more details). The map $\delta_G$ has a left inverse, the Brauer lift $\lambda_G: R^p(G) \to R(G)$, which extends a class function $\chi$ on the set of $p'$-elements of $G$ to the class function on $G$ that maps $g \in G$ to $\chi(g_{p'})$. The Brauer lift is natural in $G$. With Brauer’s characterization of virtual characters it is easy to see that $\lambda_G(\chi) \in R(G)$. It is also easy to see that the last two equations in (1.1.b) and the last equation in (1.1.c) hold.

Moreover, if $m \in \mathbb{N}$ is such that $K_m$ is a splitting field for the linear source modules of $G$, the functors

$$\mathcal{O}_m G\text{triv} \to F_m G\text{mod} \to F_G\text{mod}$$

and

$$\mathcal{O}_m(G)\text{lin} \to K_m G\text{mod} \to K_G\text{mod},$$

given by scalar extensions, induce natural surjective ring homomorphisms

$$\gamma_G: T^p(G) \to R^p(G) \quad \text{and} \quad \gamma_G: L^p(G) \to R(G),$$

since they commute with induction, and since every element in $R^p(G)$ and $R(G)$ is a $\mathbb{Z}$-linear combination of induced one-dimensional characters.

We still have to define the map $\tau_G: R^p(G) \to T^p(G)$. This is the only map in Diagram (1.1.a) that is not obvious. We use the canonical induction
formula $a_G: R^p(G) \to D^p(G)$ from [Bo98b, Example 9.8] and define $\tau_{\tilde{G}}$ as the composition

$$\tau_{\tilde{G}}: R^p(G) \xrightarrow{a_G} D^p(G) \xrightarrow{\beta_G} T^p.$$ 

Since the map $a_G$ commutes with restrictions (cf. [Bo98b, Theorem 10.3]), the map $\tau_{\tilde{G}}$ commutes with restrictions. Moreover, $\tau_{\tilde{G}}$ is a ring homomorphism. In fact, since elements in $T^p(G)$ are determined by their restrictions to $p$-hypoelementary subgroups (i.e., subgroups $H$ satisfying that $H/O_p(H)$ is cyclic), it suffices to show that $\text{res}_{\tilde{H}}^H \circ \tau_{\tilde{G}}$ is a ring homomorphism for each $p$-hypoelementary subgroup $H$ of $G$. But since $\tau_{\tilde{G}}$ commutes with $\text{res}_H^G$, it suffices to show that $\tau_H$ is a ring homomorphism whenever $H$ is $p$-hypoelementary. In this case, $R^p(H)$ is the free abelian group on the classes $[F_{\varphi}]$, $\varphi \in \text{Hom}(H, F^\times)$, of one-dimensional modules and one has $a_H([F_{\varphi}]) = [H, \varphi]_H$. This implies $\tau_H([F_{\varphi}]) = [F_{\varphi}] \in T^p(H)$ and $\tau_H$ is a ring homomorphism.

Similarly, one checks that $\gamma_{\tilde{G}} \circ \tau_{\tilde{G}}$ is the identity on $R^p(G)$. In fact, it suffices to show that $\text{res}_{\tilde{H}}^H \circ \gamma_{\tilde{G}} \circ \tau_{\tilde{G}} = \text{res}_H^G: R^p(G) \to R^p(H)$ for cyclic $p'$-subgroups $H$ of $G$. Since $\tau$ and $\gamma$ commute with restrictions, it suffices to show that $\gamma_H \circ \tau_H$ is the identity on $R^p(H)$, which is easily verified starting with $[F_{\varphi}]$ as above.

Finally, $\lambda = \gamma_{\tilde{G}} \circ \tau_{\tilde{G}}$, since it suffices to show this only for cyclic groups $H$ and for those it follows immediately by inspection of what happens to $[F_{\varphi}] \in R^p(H)$, $\varphi \in \text{Hom}(H, F^\times)$, on both sides of the equation.

## 2 The species

### 2.1 In this section we recall the species of the representation rings $X(G)$ where $X$ stands for $B$, $R^p, R^{p, ab}, D^p, D, T^p, L^p, R^p$, or $R$. For any of these choices of $X$ we will introduce a functor $X: \text{gr} \to \text{set}$ to the category of finite sets and a pairing $(-, -)_G: X(G) \times X(G) \to O_K$ for every $G \in \text{gr}$ such that for each group homomorphism $f: G \to \tilde{G}$ we have

$$\text{res}_f(\tilde{\chi}, x)_G = (\tilde{\chi}, f(x))_{\tilde{G}}, \quad (2.1.a)$$

for all $\tilde{\chi} \in X(\tilde{G})$ and $x \in X(G)$, if we denote the map $X(f): X(G) \to X(\tilde{G})$ again by $f$. Moreover in all cases, for fixed $x \in X(G)$, the map

$$s^X(x)_G: X(G) \to O_K, \chi \mapsto (\chi, x)_G,$$

will be a ring homomorphism with the property

$$s^X(x)_G \circ \text{res}_f = s^X(\tilde{G})(f(x)), \quad (2.1.b)$$
whenever \( f : G \to \tilde{G} \) is a group homomorphism, by (2.1.a). Note that, since \( \mathcal{X} \) is a functor, the conjugation maps \( s : G \to G, \ g \mapsto sg s^{-1} \), for \( s \in G \), provide \( \mathcal{X} \) with the structure of a \( G \)-set. For any \( x_1, x_2 \in \mathcal{X}(G) \) we will have

\[
s_{x_1}^{X(G)} = s_{x_2}^{X(G)} \iff x_1 = g x_2 \text{ for some } g \in G.
\] (2.1.c)

Moreover the collection \( s_{x}^{X(G)} \) of maps \( s_{x}^{X(G)} \), where \( x \) runs through a set of representatives for the \( G \)-orbits \( \mathcal{X}(G)/G \), induces an isomorphism

\[
s^{X(G)} : K \otimes X(G) \to \prod_{x \in \mathcal{X}(G)/G} K
\] (2.1.d)

of \( K \)-algebras. Therefore, the maps \( s_{x}^{X(G)} \), \( x \in \mathcal{X}(G) \), constitute all the ring homomorphisms \( X(G) \to K \). They are called the species of \( X(G) \). We may identify the last product with the \( K \)-algebra \( \text{Hom}_{G}(\mathcal{X}(G), K) \) in the obvious way, where \( \text{Hom}_{G} \) stands for \( G \)-equivariant maps and \( K \) is considered as \( G \)-set with trivial action. Then the isomorphism \( s^{X(G)} \) has the form

\[
s^{X(G)} : K \otimes X(G) \to \text{Hom}_{G}(\mathcal{X}(G), K),
\] (2.1.e)

and we have \( (s^{X(G)}(a \otimes \chi))(x) = as_{x}^{X(G)}(\chi) = a(\chi, x)_{G} \) for \( \chi \in X(G), x \in \mathcal{X}(G), \) and \( a \in K \).

The natural transformations \( \alpha : X \to Y \) between the representation ring functors in Diagram (1.1.a) will be induced by natural transformations \( \alpha^*: \mathcal{Y} \to \mathcal{X} \) after identifying \( K \otimes X(G) \) and \( K \otimes Y(G) \) under the isomorphism (2.1.e) with the \( K \)-algebras of \( G \)-equivariant functions on \( \mathcal{X}(G) \) and \( \mathcal{Y}(G) \) such that we have

\[
\alpha_{G} : \text{Hom}_{G}(\mathcal{X}(G), K) \to \text{Hom}_{G}(\mathcal{Y}(G), K), \quad f \mapsto f \circ \alpha_{G}^{*}.
\] (2.1.f)

If we denote by \( \overline{\alpha_{G}^{*}} : \mathcal{Y}(G)/G \to \mathcal{X}(G)/G \) the induced map on \( G \)-orbits, it is easy to verify that one has the following relations:

\[
\alpha_{G}^{*} \text{ injective} \iff \overline{\alpha_{G}^{*}} \text{ injective} \iff \alpha_{G} \text{ surjective}
\] (2.1.g)

and

\[
\alpha_{G}^{*} \text{ surjective} \iff \overline{\alpha_{G}^{*}} \text{ surjective} \iff \alpha_{G} \text{ surjective}.
\]

**2.2 Remark** (a) If \( X : \text{gr} \to \mathcal{K}	ext{alg} \) is a contravariant functor with the property that, for each \( G \in \text{gr} \), \( X(G) \) is isomorphic to a finite product of copies of \( K \), then one obtains a covariant functor \( \mathcal{X} : \text{gr} \to \text{set} \) by setting

\[
\mathcal{X}(G) := \mathcal{K}	ext{alg}(X(G), K)
\]

and

\[
\mathcal{X}(f) : \mathcal{K}	ext{alg}(X(G), K) \to \mathcal{K}	ext{alg}(X(\tilde{G}), K), \quad s \mapsto s \circ X(f),
\]

for any \( f : G \to \tilde{G} \) in \( \text{gr} \). Conversely, if \( \mathcal{X} : \text{gr} \to \text{set} \) is a covariant functor, then one obtains a contravariant functor \( X : \text{gr} \to \mathcal{K}	ext{alg} \) by setting

\[
X(G) := \text{set} \mathcal{X}(G), K)
\]
and 
\[ X(f) : \text{set}(\mathcal{X} (\tilde{G}), K) \to \text{set}(\mathcal{X} (G), K), \quad \varepsilon \mapsto \varepsilon \circ \mathcal{X}(f), \]

for any \( f : G \to \tilde{G} \) in gr. Obviously, \( X(G) \) is isomorphic to a finite product of copies of \( K \). It is not difficult to see that the composition of these two constructions (in both orders) yields a functor that is naturally isomorphic to the original one. In fact, if we start with \( X \), then

\[ X(G) = \bigoplus_{s \in \mathcal{K}_{\text{alg}}(X(G), K)} K \cdot e_s, \tag{2.2.a} \]

where \( e_s \) is the unique primitive idempotent of \( X(G) \) with \( s(e_s) = 1 \) and we have a natural isomorphism

\[ X(G) \cong \mathcal{K}_{\text{alg}}(\text{set}(X(G), K), K) \]

in \( \mathcal{K}_{\text{alg}} \) sending \( e_s \in X(G) \) to the characteristic map \( \varepsilon_s \) with \( \varepsilon_s(t) := \delta_{s,t} \) and sending \( \varepsilon \in \text{set}(\mathcal{K}_{\text{alg}}(X(G), K), K) \) to \( \sum_{s \in \mathcal{K}_{\text{alg}}(X(G), K)} \varepsilon(s) \cdot e_s \). Also, if we start with \( \mathcal{X} \), then we have a natural bijection

\[ \mathcal{X}(G) \cong \mathcal{K}_{\text{alg}}(\text{set}(\mathcal{X}(G), K), K) \]

sending \( x \in \mathcal{X}(G) \) to \( s \) with \( s(\varepsilon) := \varepsilon(x) \) for \( \varepsilon \in \text{set}(\mathcal{X}(G), K) \) and sending \( s \in \mathcal{K}_{\text{alg}}(\text{set}(\mathcal{X}(G), K), K) \) to the unique \( x \in \mathcal{X}(G) \) with \( s(\varepsilon_x) = 1 \), where \( \varepsilon_x : \mathcal{X}(G) \to K \) is the characteristic function on \( \{x\} \).

(b) If \( Y : \text{gr} \to \mathcal{K}_{\text{alg}} \) is a second contravariant functor with the property that \( Y(G) \) is isomorphic to a finite product of copies of \( K \) for every \( G \in \text{gr} \) and \( Y : \text{gr} \to \text{set} \) is the associated covariant functor according to part (a), then one obtains a bijection

\[ \text{Nat}(X, Y) \cong \text{Nat}(Y, \mathcal{X}) \]

between these sets of natural transformations as follows. We map \( \varphi \in \text{Nat}(X, Y) \) to \( \psi \in \text{Nat}(Y, \mathcal{X}) \) with

\[ \psi_G : Y(G) \to \mathcal{X}(G), \quad s \mapsto s \circ \varphi_G, \]

for \( G \in \text{gr} \), and we map \( \psi \in \text{Nat}(Y, \mathcal{X}) \) to \( \varphi \in \text{Nat}(X, Y) \) with

\[ \varphi_G : X(G) \to Y(G), \quad e_s \mapsto \sum_{t \in \psi^{-1}(s)} e_t, \]

where we use the notation from part (a) and Equation (2.2.a).

(c) If \( \mathcal{X} : \text{gr} \to \text{set} \) is a covariant functor then we denote by \( \overline{\mathcal{X}} : \text{gr} \to \text{set} \) the covariant functor defined by

\[ \overline{\mathcal{X}}(G) := \mathcal{X}(G)/G \]

and

\[ \overline{\mathcal{X}}(f) : \overline{\mathcal{X}}(G)/G \to \overline{\mathcal{X}}(\tilde{G})/\tilde{G}, \quad [x]_G \mapsto [(\mathcal{X}(f))(x)]_{\tilde{G}}, \]
which is well-defined. Here the $G$-set structure on $\mathcal{X}(G)$ comes by functoriality from the inner automorphisms of $G$.

Assume that $(X, \mathcal{X})$ is as in Subsection 2.1. Tensoring with $K$ yields a contravariant functor $K \otimes X : \text{gr} \to K\text{alg}$ which takes values in $K$-algebras that are isomorphic to finite products of copies of $K$. On the other hand one obtains a covariant functor $X : \text{gr} \to \text{set}$ by the above construction. By Equations (2.1.b), (2.1.c), and (2.1.d), the functors $K \otimes X$ and $X$ correspond under the construction in part (a).

If also $(Y, \mathcal{Y})$ is as in Subsection 2.1 we obtain a diagram

$$\text{Nat}(X, Y) \to \text{Nat}(K \otimes X, K \otimes Y) \cong \text{Nat}(\mathcal{Y}, \mathcal{X}) \leftarrow \text{Nat}(Y, X).$$

The first injection is defined by tensoring with $K$, the middle isomorphism is the one from part (b), and the last map is given by sending $\psi \in \text{Nat}(Y, X)$ to $\overline{\psi} \in \text{Nat}(\mathcal{Y}, \mathcal{X})$ which is defined by

$$\overline{\psi}_G : Y(G)/G \to \mathcal{X}(G)/G, \quad [y]_G \mapsto [\psi_G(y)]_G.$$

It is easy to see that $\overline{\psi}$ is again a natural transformation. By (2.1.f), the natural transformations $\alpha \in \text{Nat}(X, Y)$ we considered in Diagram (1.1.a) come from the natural transformations $\alpha^* \in \text{Nat}(Y, X)$ through the above diagram. We do not know if the map $\text{Nat}(Y, X) \to \text{Nat}(Y, X)$ is injective. Therefore, the notation $\alpha^*$ is abusive.

What one can see from the above considerations is the following: In order to construct a natural transformation $X \to Y$, one can start with a natural transformation $Y \to X$ and try to prove that the associated natural transformation $K \otimes X \to K \otimes Y$ via the above diagram has an integral form, i.e., that it maps $X(G)$ to $Y(G)$ for each $G$. In this case we call the original natural transformation $Y \to X$ also integral. Note that while not every natural transformation in $\text{Nat}(X, Y)$ comes that way from one in $\text{Nat}(Y, X)$ it does come from one in $\text{Nat}(\mathcal{Y}, \mathcal{X})$. Also, while the functor $\mathcal{X}$ is determined by $X$ up to functorial isomorphism, the functor $\mathcal{Y}$ is not. At one point (for $X = T^p$) we will have to give two different choices for $\mathcal{X}$ in order to obtain all natural transformations in $\text{Nat}(X, Y)$ as induced from some in $\text{Nat}(Y, X)$.

2.3 We define $\mathcal{R} : \text{gr} \to \text{set}$ as the forgetful functor, $\mathcal{R}(G) := G$, and

$$(\chi, g)_G := \chi(g)$$

(2.3.a)

for $\chi \in R(G)$ and $g \in G$. We obtain the species

$$s_G^{\mathcal{R}(G)} : R(G) \to \mathcal{O}_K, \quad \chi \mapsto \chi(g),$$

for $g \in G$. Similarly, we define $\mathcal{R}^p(G) := G_{p'}$, as the set of elements of $p'$-order in $G$. This defines a subfunctor $\mathcal{R}^p \subset \mathcal{R}$. Again we define a pairing $(-, -)_G : \mathcal{R}^p(G) \times \mathcal{R}^p(G) \to \mathcal{O}_K$ by (2.3.a) for $\chi \in \mathcal{R}^p(G)$ and $g \in G_{p'}$ and obtain as species the evaluation maps

$$s_G^{\mathcal{R}^p(G)} : \mathcal{R}^p(G) \to \mathcal{O}_K, \quad \chi \mapsto \chi(g).$$
for $g \in G_{p'}$.

It is obvious that the decomposition map $\delta_G: R(G) \to R^p(G)$ is induced by
the natural transformation $\delta_G^\ast: R^p(G) \to R(G)$ that is given by the inclusion.
Moreover, the Brauer lift is induced by the natural transformation $\lambda_G^\ast: R(G) \to
R^G_1(G)$ that maps $g \in G$ to its $p'$-part $g_{p'}$.

Similarly, we define functors $\mathcal{R}^{\text{ab}}, \mathcal{R}^{p,\text{ab}}: \text{gr} \to \text{set}$ by
$\mathcal{R}^{\text{ab}}(G) := G^{\text{ab}}$ and
$\mathcal{R}^{p,\text{ab}}(G) := \langle G^{\text{ab}} \rangle_{p'}$ and pairings $(\chi, gG)_{G} \mapsto \chi(g)$ for $g \in G$ or for $g \in G$
with $gG' \in (G^{\text{ab}})_{p'}$ to obtain species $s_{gG'}^{\text{R}^{\text{ab}}(G)}: \mathcal{R}^{\text{ab}}(G) \to \mathcal{O}_K$, $\chi \mapsto \chi(g)$ and
$s_{gG'}^{p,\text{ab}}(G): \mathcal{R}^{p,\text{ab}}(G) \to \mathcal{O}_K$, $\chi \mapsto \chi(g)$. As for $R(G)$ and $R^p(G)$, the inclusion
$\iota: R^{\text{ab}} \to R^{p,\text{ab}}$ is induced by $\iota_G^\ast: \mathcal{R}^{\text{ab}}(G) \to \mathcal{R}^{p,\text{ab}}(G)$, $gG' \mapsto g_{p'}G' = (gG')_{p'}$, and the reduction map $\rho: R^{\text{ab}} \to R^{p,\text{ab}}$ is induced by the inclusion
$\rho_G^\ast: \mathcal{R}^{p,\text{ab}}(G) \to \mathcal{R}^{\text{ab}}(G)$.

It is well-known that all the assertions in Subsection 2.1 hold for $X \in \{R, R^p, R^{\text{ab}}, R^{p,\text{ab}}\}$, cf. [Sc78, §14-§18].

2.4 In the case of the Burnside ring $B(G)$ we denote by $B(G)$ the set of
subgroups of $G$. Obviously, this defines a functor $B: \text{gr} \to \text{set}$ if we define
$B(f): B(G) \to B(G)$, $H \mapsto f(H)$, for every morphism $f: G \to \tilde{G}$ in $\text{gr}$. For
$H \leq G$ and $S \in \text{Oset}$ we set $([S], H) := |S^H| \in \mathbb{Z}$, where $S^H$ denotes the
set of $H$-fixed points of $S$. This induces a map $B(G) \times B(G) \to \mathbb{Z}$ such that
$s_G^{B(G)}: B(G) \to \mathbb{Z}$ is just the mark homomorphism with respect to $G$. Clearly,
all the claims in Subsection 2.1 are satisfied, cf. [CR87, §80].

2.5 Next we determine the species of $D(G)$ and $D^p(G)$. We set
$D(G) := \{(H, hH') \mid H \leq G, h \in H\}$
and
$D^p(G) := \{(H, hH') \mid H \leq G, h \in H \text{ with } hH' \in (H^{\text{ab}})_{p'}\}$,
where $H'$ denotes the derived subgroup of $H$. Obviously, we have a surjective map
$\iota_G^* : D(G) \to D^p(G)$, $(H, hH') \mapsto (H, (hH')_{p'}) = (H, h_{p'}H')$
and an injective map
$\rho_G^* : D^p(G) \to D(G)$, $(H, hH') \mapsto (H, hH')$.

For a group homomorphism $f: G \to \tilde{G}$ in $\text{gr}$ we define
$D(f): D(G) \to D(\tilde{G})$, $(H, hH') \mapsto (f(H), f(h)f(H'))$,
and similarly $D^p(f): D^p(G) \to D^p(\tilde{G})$. With this definition, $D$ and $D^p$
are functors $\text{gr} \to \text{set}$ and $\iota_G^*$ and $\rho_G^*$ from above define natural transformations.

For $\chi \in D(G)$ and $(H, hH') \in D(G)$ we set
$(\chi, (H, hH'))_G := (s_{hH'}^{R^{ab}}(H)) \circ \pi_H \circ \text{res}_H^G(\chi)$

13
with $\pi_H: D(H) \to R^{ab}(H)$ as defined in (1.4.a). Similarly, for $\chi \in D^p(G)$ and $(H, hH') \in D^p(G)$ we set

$$(\chi, (H, hH'))_G := (s_{hH'}^{R^{ab}} \circ \pi_H \circ \text{res}_H^G)(\chi).$$

From that we obtain species

$$s^{D(G)}_{(H,hH')} : D(G) \xrightarrow{\text{res}_H^G} D(H) \xrightarrow{\pi_H} R^{ab}(H) \xrightarrow{s_{hH'}^{R^{ab}}(H)} \mathcal{O}_K$$

and

$$s^{D^p(G)}_{(H,hH')} : D^p(G) \xrightarrow{\text{res}_H^G} D^p(H) \xrightarrow{\pi_H} R^{p,ab}(H) \xrightarrow{s_{hH'}^{R^{p,ab}}(H)} \mathcal{O}_K.$$  

It follows from considering the injectivity of the map $\rho_G$ in [Bo98b, Proposition 2.4] that two maps $s^{D(G)}_{(H_1,h_1H'_1)}$ and $s^{D(G)}_{(H_2,h_2H'_2)}$ coincide if and only if $(H_1, h_1H'_1)$ and $(H_2, h_2H'_2)$ are conjugate under $G$. Moreover the surjectivity of the map $\rho_G$ in [Bo98b, Proposition 2.4] implies that the number of maps $s^{D(G)}_{(H,hH')} \in \mathbb{Z}$ is the same as the ring homomorphisms into domains are always linearly independent, by Dedekind’s lemma, we obtain that (2.1.d) is an isomorphism for $X = D$. The same arguments hold for $D^p$. Note that $D(G)$ is $R^{ab}(G)$ and $D^p(G)$ is equal to $R^{p,ab}(G)$ in the terminology of [Bo98b, Proposition 2.4].

It is now easy to see that the natural inclusion $\iota: D^p \to D$ is induced by the natural surjection $\iota^*$ from above and that the natural surjection $\rho: D \to D^p$ is induced by the natural embedding $\rho^*$ from above.

The natural embedding $\eta: R^{ab} \to D$ and the natural surjection $\pi: D \to R^{ab}$ are induced by the natural surjection $\eta_G^*: D(G) \to R^{ab}(G)$, $(H, hH') \mapsto hG'$ and the natural embedding $\pi_G^*: R^{ab}(G) \to D(G)$, $gG' \mapsto (G, gG')$. Similarly, the natural embedding $\eta: R^{p,ab} \to D^p$ and the natural surjection $\pi: D^p \to R^{p,ab}$ are induced by the natural surjection $\eta_G^*: D^p(G) \to R^{p,ab}(G)$, $(H, hH') \mapsto hG'$ and the natural embedding $\pi_G^*: R^{p,ab}(G) \to D^p(G)$, $gG' \mapsto (G, gG')$.

The natural transformations between $D^p$ and $B$ are induced by the natural transformations $\iota_G^*: D^p(G) \to B(G)$, $(H, hH') \mapsto H$ and $\pi_G^*: B(G) \to D^p(G)$, $H \mapsto (H, 1 \cdot H')$.

**2.6 Proposition** The Equation (2.1.a) holds for $(X, \mathcal{A}) = (D, D)$ and $(X, \mathcal{A}) = (D^p, D^p)$.

**Proof** Let $f: G \to \tilde{G}$ be in gr and let $(H, hH') \in D(G)$. We have to show that

$$s^{R^{ab}}_{hH'} \circ \pi_H \circ \text{res}_H^G \circ \text{res}_f = s^{R^{ab}(f(H))}_{(f(H), f(hH'))} \circ \pi_{f(H)} \circ \text{res}_{f(H)}^G$$

as maps from $D(\tilde{G})$ to $\mathcal{O}_K$. Since $D$ is a functor, one has $\text{res}_H^G \circ \text{res}_f = \text{res}_{f \circ H \to f(hH')} \circ \text{res}_{f(H)}^G$. Also, in the right hand side of the above equation we can use

$$s^{R^{ab}(f(H))}_{(f(H), f(hH'))} = s_{f(H), f(hH')} \circ \text{res}_f: H \to f(H),$$
Finally we recall from [Bo98a, §2] the species of $L^p(G)$ and $T^p(G)$. We set
$$L^p(G) := \{(H,hO_p(H))' \mid H \leq G, \langle hO_p(H) \rangle = H/O_p(H)\}$$
and
$$T^p(G) := \{(H,hO_p(H)) \mid H \leq G, \langle hO_p(H) \rangle = H/O_p(H)\}.$$ 

Note that the second condition implies that $H/O_p(H)$ is cyclic, i.e., that $H$ is $p$-hypoelementary. We obtain functors $L^p, T^p : \text{gr} \to \text{set}$ if, for $f : G \to G$ in $\text{gr}$, $(H,hO_p(H))' \in L^p(G)$, and $(H,hO_p(H)) \in T^p(G)$, we set
$$f(H,hO_p(H))' = (f(H), f(h)O_p(f(H)))'$$
and
$$f(H,hO_p(H)) = (f(H), f(h)O_p(f(H))).$$

Let $m$ be a multiple of $n$ such that $K_m$ is a splitting field for the linear source modules of $G$. We identify $L^p(G) = L_m(G)$ and $L_{\text{lin}}(G)$ as explained in Subsection 1.5. We denote by $L'(G) \subseteq L^p(G)$ the subgroup generated by the isomorphism classes $[V]$ of indecomposable $V \in \tilde{O}_{m, \text{lin}}$ having the Sylow subgroups of $G$ as vertices, and we denote by $L''(G) \subseteq L^p(G)$ the ideal generated as group by elements $[V]$, where $V$ runs through the remaining indecomposable modules in $\tilde{O}_{m, \text{lin}}$. Similarly, we define the subgroup $T'(G)$ and the ideal $T''(G)$ of $T^p(G)$. We denote by $q_\ell^G : L^p(G) \to L'(G)$ and $q_r^G : T^p(G) \to T'(G)$ the projection corresponding to the decompositions $L^p(G) = L'(G) \oplus L''(G)$ and $T^p(G) = T'(G) \oplus T''(G)$. Then $q_{\ell r}^G$ is the restriction of $q_{\ell r}^G$. If $G$ is $p$-hypoelementary then $L'(G)$ and $T'(G)$ are subrings of $L^p(G)$ and $T^p(G)$, and the maps $q_\ell^G$ and $q_r^G$ are ring homomorphisms. Now we set
$$([V], (H,hO_p(H))')_G := (s^{R(H)}_h \circ \gamma_H \circ q^L_H \circ \text{res}^G_H)([V])$$
and
$$([W], (H,hO_p(H)))_G := (s^{R(H)}_h \circ \gamma_H \circ q^T_H \circ \text{res}^G_H)([W])$$
for $V \in \tilde{O}_{m, \text{lin}}$ and $W \in \tilde{O}_{m, \text{triv}}$, $(H,hO_p(H))' \in L^p(G)$, and $(H,hO_p(H)) \in T^p(G)$. This is well-defined and induces maps $L^p(G) \times L^p(G) \to \mathcal{O}_K$ and $T^p(G) \times T^p(G) \to \mathcal{O}_K$ by linear extension. As species we obtain
$$s^G_{(H,hO_p(H))'} : L^p(G) \xrightarrow{\text{res}^G_H} L^p(H) \xrightarrow{q^L_H} L'(H) \xrightarrow{\gamma_H} R(H) \xrightarrow{s^{R(H)}_h} \mathcal{O}_K$$
such that all the assertions in Subsection 2.1 hold.

It is easy to see that the natural embedding \( \iota : T^p \to L^p \) is induced by the natural surjection

\[
\iota^* : \mathcal{L}^p(G) \to T^p(G), \quad (H, hO_p(H)) \mapsto (H, hO_p(H)),
\]

but there is no dual map \( \rho_G^* : T^p(G) \to \mathcal{L}^p(G) \) for \( \rho_G \). The only candidate that comes to one’s mind which maps \( (H, hO_p(H)) \) to \( (H, (h'p)O_p(H')) \) is not well-defined. For that reason we have to introduce a second functor \( \mathcal{T}^p : \mathfrak{g} \to \mathfrak{set} \) by

\[
\mathcal{T}^p(G) := \{(Q, g) \mid Q \leq G \text{ a } p\text{-subgroup, } g \in N_G(Q)\}_{p^r}.
\]

For a group homomorphism \( f : G \to \tilde{G} \) we set \( \mathcal{T}^p(f)(Q, g) := (f(Q), f(g)) \). It is well-known and not difficult to see that the map

\[
\kappa^*_G : \mathcal{T}^p(G) \to T^p(G), \quad (Q, g) \mapsto (\langle Q, g \rangle, gQ),
\]

which is natural in \( G \) induces an isomorphism between the \( G \)-orbits of \( \mathcal{T}^p(G) \) and \( T^p(G) \). Again we have a pairing

\[
(\cdot, \cdot)_G : T^p(G) \times \mathcal{T}^p(G) \to \mathcal{O}_K, \quad (\chi, x) \mapsto (s_g \circ \gamma_H \circ q_H^* \circ \text{res}_H^G)(\chi),
\]

with \( H := \langle Q, g \rangle \), and obtain an associated species

\[
S^p(\mathcal{T}^p(G)) : T^p(G) \xrightarrow{\text{res}_H^G} T^p(H) \xrightarrow{q_H^T} T^q(H) \xrightarrow{\gamma_H} R^p(H) \xrightarrow{s_g} \mathcal{O}_K
\]

such that their collection induces an isomorphism \( K \otimes T^p(H) \to \text{Hom}_G(T^p(G), K) \). We have \( S^{\mathcal{T}^p(G)} = S^{T^p(G)} \) so that the natural surjection \( \kappa^* \) induces the identity on \( T(G) \). Now we are able to give a map

\[
\rho_G^* : \mathcal{T}^p(G) \to \mathcal{L}^p(G), \quad (Q, g) \mapsto (\langle Q, g \rangle, gQ'),
\]

that induces \( \rho_G : L^p(G) \to T^p(G) \).

The composition of \( \beta : D(G) \to L^p(G) \) with \( S^{L^p(G)}_{(H,hO_p(H))} \) equals \( S^{D(G)}_{(H,hH')} \), since we have a commutative diagram

\[
\begin{array}{ccc}
D(G) & \xrightarrow{\text{res}_H^G} & D(H) & \xrightarrow{\pi_H} & R^{ab}(H) & \xrightarrow{s_{hH'}} \mathcal{O}_K \\
\downarrow{\beta_G} & & \downarrow{\beta_G} & & \downarrow{\gamma_H} & & \downarrow{s_{hH'}} \\
L^p(G) & \xrightarrow{\text{res}_H^G} & L^p(H) & \xrightarrow{q_H^T} & L^q(H) & \xrightarrow{\gamma_H} & R(H)
\end{array}
\]
for $H \leq G$ $p$-hypoelementary and $h \in H$ with $\langle hO_p(H) \rangle = H/O_p(H)$. In fact, for $[U, \varphi]_H \in D(H)$ we have

\[(s^U_H \circ \gamma_H \circ q^L_H([\text{Ind}_U^H(\tilde{\varphi}_m)]) = 0, \quad \text{if } U < H,\]

since $hO_p(H)$ generates $H/O_p(H)$, and if $U = H$, then both maps applied to $[U, \varphi]_H$ result in $\varphi(h)$. Similarly, one can see that the composition of $\beta: D^p(G) \to L^p(G)$ with $s^{U(G)}_{(H,hO_p(H))}$ equals $s^{U(G)}_{(H,h_pH')} \in T^p(G)$. Therefore, the natural surjections $\beta: D^p \to T^p$ and $\beta: D \to L^p$ are induced by the natural embedding

$$\beta^*_G: T^p(G) \to D^p(G), \quad (H,hO_p(H)) \mapsto (H,h_pH'),$$

and the natural transformation

$$\beta^*_G: L^p(G) \to D(G), \quad (H,hO_p(H)) \mapsto (H,hH').$$

Note that the last map is not injective in general, but that it must be injective on $G$-orbits by (2.1.g). One can also use the map

$$\beta^*_G: \bar{T}^p(G) \to D^p(G), \quad (Q,g) \mapsto (H,gH'),$$

with $H := \langle Q,g \rangle$, in order to induce the map $\beta: D^p \to T^p$.

Next it is easy to see that the natural surjections $\gamma: T^p \to R^p$ and $\gamma: L^p \to R$ are induced by the natural embeddings

$$\gamma^*_G: R^p(G) \to T^p(G), \quad g \mapsto (g),$$

and

$$\gamma^*_G: R(G) \to L^p(G), \quad g \mapsto (g).$$

Again, one can also use $\bar{T}^p$ and define

$$\gamma^*_G: R^p(G) \to \bar{T}^p(G), \quad g \mapsto (1),$$

in order to induce the map $\gamma: T^p \to R^p$.

Finally, the map $\tau_G: T^p(G) \to R^p(G)$ is induced by

$$\tau^*_G: \bar{T}^p(G) \to R^p(G), \quad (Q,g) \mapsto g,$$

which defines a natural surjection $\tau^*: \bar{T}^p \to R^p$. In this case there is no possible map from $T^p(G)$ to $R^p(G)$ that induces $\tau_G$.

**2.8 Proposition** The Equation (2.1.a) holds for $(X, \mathcal{X}) \in \{(T^p, \mathcal{T}^p), (T^p, \mathcal{T}^p), (L^p, \mathcal{L}^p)\}$.

**Proof** We first prove the assertion for $(L^p, \mathcal{L}^p)$. Let $f: G \to \tilde{G}$ be in $\text{gr}$ and let $(H,hO_p(H)) \in D^p(G)$. We have to show that

$$s^R_H \circ \gamma_H \circ q^L_H \circ \text{res}^G_H \circ \text{res}_f = s^R_{f(H)} \circ \gamma_{f(H)} \circ q^L_{f(H)} \circ \text{res}^G_{f(H)}.$$

17
We transform the left hand side using \( \text{res}\overline{\gamma}_H \circ \text{res}_f = \text{res}_f : H \to f(H) \circ \text{res}\overline{\gamma}_H \) and the right hand side using \( s^R_{f(H)} = s^R_{h} \circ \text{res}_f : H \to f(H) \) from Equation (2.1.b) for \((R, \mathcal{R})\) so that it suffices to show that

\[
\gamma_H \circ q^L_H \circ \text{res}_f = \text{res}_f \circ \gamma_f \circ q^L_H,
\]

for a surjective map \( f : H \to J \) in \( \text{gr} \). Since \( \gamma \) is a natural transformation we have \( \text{res}_f \circ \gamma_f = \gamma_H \circ \text{res}_f \) and it suffices to show that \( q^L_H \circ \text{res}_f = \text{res}_f \circ q^L_H \). Now let \( V \in \mathcal{O}_m \) be indecomposable, where \( m \) is large enough such that \( K_m \) is a splitting field for the linear source modules of \( G \). Since \( H \) is \( p \)-hypo-elementary, also \( J \) is, and \( f(O_p(H)) = O_p(J) \). If \( V \) has vertex \( O_p(J) \), then \( q^L_H(V) = [V] \) and \( q^L_H(\text{res}_f([V])) = \text{res}_f([V]) \), since \( \text{res}_f([V]) \) has vertex \( O_p(H) \), and we are done. If \( V \) has vertex smaller than \( O_p(J) \), then \( \text{res}_f([V]) \) has vertex smaller than \( O_p(H) \) and both sides of the equation we have to prove evaluate to 0 on \([V]\). The proof for \((X, \mathcal{X}) = (T^p, \overline{T}^p)\) and \((X, \mathcal{X}) = (T^p, \overline{T}^p)\) is similar.

2.9 We summarize the preceding constructions in the diagram

![Diagram](2.9.a)

of functors \( \text{gr} \to \text{set} \) and natural transformations between them. If \( \alpha^* : \mathcal{Y} \to \mathcal{X} \) is an arrow in the above diagram then the associated natural transformation in Diagram (1.1.a) is induced by the rule in (2.1.f), or in terms of species by the rule that the composition

\[
X(G) \xrightarrow{\alpha_G} Y(G) \xrightarrow{s^G_{\gamma}} O_X
\]

equals the species \( s^X_{\gamma(G)} \) for all \( y \in \mathcal{Y}(G) \). One has the corresponding equations as in Diagram (1.1.a) and additionally that \( \beta^* \circ \kappa^* = \beta^* \) and \( \kappa^* \circ \gamma^* = \gamma^* \).
Moreover, the arrow \( i^* : \mathcal{L}^p \to \mathcal{T}^p \) is missing to avoid an overloading of the diagram and one has \( i^* \circ \rho^* = \kappa^* \).

## 3 Idempotent formulae

### 3.1 Each of the representation rings \( X(G) \), \( X \in \{ R^{p,ab}, R^{ab}, B, D, T^p, L^p, R^p, R \} \) in Diagram (1.1.a) becomes a commutative split semisimple \( K \)-algebra after tensoring with \( K \), the components indexed by the \( G \)-orbits \( X(G) \) of the associated \( G \)-set \( X(G) \) introduced in Section 2. Thus, for every \( x \in X(G) \) there exists a primitive idempotent \( e_x^X(G) \) of \( K \otimes X(G) \) with the property

\[
s_{x'}^{X(G)}(e_x^X(G)) = \begin{cases} 1, & \text{if } x =_G x', \\ 0, & \text{if } x \not=_G x', \end{cases}
\]

for \( x, x' \in X(G) \), or, if we denote by \( e_x^X(G) \) the image of \( e_x^X(G) \) under the isomorphism \( K \otimes X(G) \to \text{Hom}_G(X(G), K) \) from (2.1.e), then we have

\[
e_x^X(G)(x') = \begin{cases} 1, & \text{if } x =_G x', \\ 0, & \text{if } x \not=_G x'. \end{cases}
\]

Obviously, we have, for \( x, x' \in X(G) \),

\[
e_x^X(G) = e_x^X(G) \iff \varepsilon_x^X(G) = \varepsilon_x^X(G) \iff x =_G x'.
\]

### 3.2 In this section we will give explicit formulae for the idempotents \( e_x^X(G) \) for all \( X(G) \) and all \( x \in X(G) \) starting from those of \( K \otimes R^{ab}(G) \) by applying general abstract results. Some of these formulae are well-known, some are new. The following equation, where \( \alpha : X \to Y \) is one of the arrows in Diagram (1.1.a) and \( \alpha^* : Y \to X \) is the associated arrow in Diagram (2.9.a), will be very useful. One has

\[
\alpha_G(e_x^X(G)) = \sum_{y \in Y(G)/G \atop \alpha_G(y) =_G x} e_y^Y(G), \quad (3.2.a)
\]

for every \( x \in X(G) \). In fact, for \( y' \in Y(G) \), we have

\[
(\alpha_G(e_x^X(G)))(y') = \varepsilon_x^X(G)(\alpha_G^*(y')) = \begin{cases} 1, & \text{if } \alpha_G^*(y') =_G x, \\ 0, & \text{if } \alpha_G^*(y') \not=_G x, \end{cases}
\]

\[
= \sum_{y \in Y(G)/G \atop \alpha_G(y) =_G x} e_y^Y(G)(y').
\]

Note that if \( \alpha_G \) is injective Equation (3.2.a) simplifies to

\[
\alpha_G(e_{\alpha_G^*(y)}^X(G)) = e_y^Y(G), \quad (3.2.b)
\]
for each \( y \in Y(G) \). In this case we can therefore derive a formula for \( e^Y_{\alpha_G}(y) \) from a formula of \( e^X_{\alpha_G}(y) \). A glance at Diagram (1.1.a) shows that one can go from \( D(G) \) to any other representation ring \( Y(G) \) by surjective maps so that formulae for the idempotents of \( K \otimes D(G) \) will enable us to derive formulae for the idempotents of all the other representation rings using Equation (3.2.b), since the hypothesis to apply this equation is satisfied by (2.1.g).

Also note that Equation (3.2.a) implies that the idempotents \( e^X_{\alpha_G}(y) \) which lie in the kernel of \( \alpha_G \) are precisely those with \( x \notin \text{im}(\alpha_G^*) \). This knowledge has important applications, since it provides relations in \( Y(G) \) that were used for example in [Bo97a] and [We87].

3.3 Through our approach in Section 2 we also obtain general formulae for the behaviour of idempotents under restriction maps. If \( f : G \to \tilde{G} \) is a morphism in \( \text{gr} \) then Equation (2.1.b) implies

\[
s^X_{\alpha_G}(\text{res}_f(e^X_{\tilde{G}})) = s^X_f(e^X_{\tilde{G}}) = \begin{cases} 1, & \text{if } f(x) = \tilde{G} \tilde{x}, \\ 0, & \text{if } f(x) \notin \tilde{G} \tilde{x}. \end{cases}
\]

for \( x \in X(G) \) and \( \tilde{x} \in X(\tilde{G}) \), where we wrote \( f(x) \) instead of \( (X(f))(x) \). This implies

\[
\text{res}_f(e^X_{\tilde{G}}) = \sum_{x \in X(G)/G} \sum_{f(x) = \tilde{g} \tilde{x}} e^X_{\tilde{G}} = \sum_{x \in X(G)} \frac{1}{[G : N_G(x)]} e^X_{\tilde{G}}
\]

so that for each \( \tilde{x} \in X(\tilde{G}) \) we have

\[
\text{res}_f(e^X_{\tilde{G}}) = \sum_{\tilde{g} \in f(G)/G} \sum_{\tilde{x} \in X(\tilde{G})} \frac{[f(G) : N_{f(G)}(\tilde{g})]}{[G : N_G(x)]} e^X_{\tilde{G}},
\]

by considering the decomposition of the \( \tilde{G} \)-orbit of \( \tilde{x} \) into \( f(G) \)-orbits. In particular, if \( [\tilde{x}]_{\tilde{G}} \cap \text{im}(f) = \emptyset \), then \( \text{res}_f(e^X_{\tilde{G}}) = 0 \).

3.4 First we recall and derive formulae for \( e^X_{\alpha_G}(x) \) for \( X \in \{ R, R^{ab}, R^p, R^{p,ab} \} \) and \( x \in X(G) \). It is well-known and follows from the orthogonality relation that

\[
e^R_{\alpha_G}(x) = \frac{1}{|G|} \sum_{\chi \in \text{Irr}(G)} \chi(g^{-1}) \chi \in K \otimes R(G),
\]

where \( \text{Irr}(G) \) denotes the set of irreducible characters of \( G \), and that consequently

\[
e^{R^{p,ab}}_{\alpha_G}(x) = \frac{1}{|G^{p,ab}|} \sum_{\varphi \in G} \varphi(g^{-1}) \varphi \in K \otimes R^{p,ab}(G),
\]

20
for \( g \in G \), with \( \hat{G} = \text{Hom}(G, K^*) \). Using Equation (3.2.b) for the natural
inclusions \( \rho^* : R^\rho \to R \) and \( \rho^\ast : R^{ab} \to R^{ab} \) we immediately obtain
\[
\epsilon^{R^\rho(G)}_g = \delta(\epsilon^{R(G)}_g) = \frac{1}{|G|} \sum_{\chi \in \text{Irr}(G)} \chi(g^{-1}) \chi|_{G_{p'}} \\
= \frac{1}{|G|} \sum_{\chi \in \text{Irr}(G)} \chi(g^{-1}) \sum_{\vartheta \in \text{IBr}(G)} d_{\vartheta \chi} \cdot \vartheta \\
= \frac{1}{|G|} \sum_{\vartheta \in \text{IBr}(G)} \left( \sum_{\chi \in \text{Irr}(G)} d_{\vartheta \chi}(g^{-1}) \right) \vartheta = \frac{1}{|G|} \sum_{\vartheta \in \text{IBr}} \eta_{\vartheta}(g^{-1}) \vartheta ,
\]
for \( g \in G_{p'} \). Here, \( \text{IBr}(G) \) denotes the set of irreducible Brauer characters
of \( G \), \( d_{\vartheta \chi} \) the decomposition number of \( \chi \) and \( \vartheta \), and \( \eta_{\vartheta} \) the character of the projective cover of the simple \( FG \)-module with Brauer character \( \vartheta \). Using
the Cartan invariants \( c_{\beta \vartheta} \) we can also write
\[
\epsilon^{R^\rho(G)}_g = \frac{1}{|G|} \sum_{\vartheta \in \text{IBr}(G)} \left( \sum_{\beta \in \text{IBr}(G)} c_{\beta \vartheta} \beta(g^{-1}) \right) \vartheta .
\]
Similarly, for \( g \in G \) with \( gG' \in (G^{ab})_{p'} \) we obtain
\[
\epsilon^{R^{ab}(G)}_{gG'} = \rho_G \left( \frac{1}{|G^{ab}|} \sum_{\varphi \in \hat{G}} \varphi(g^{-1}) \varphi \right) = \frac{1}{|G^{ab}|} \sum_{\varphi \in \hat{G}} \varphi(g^{-1}) \varphi_{p'} \\
= \frac{1}{|G^{ab}|_{p'}} \sum_{\varphi \in \hat{G}_{p'}} \varphi(g^{-1}) \varphi .
\]
We identify \( \hat{G} \) also with \( \text{Hom}(G, O^{\times}_K) \) and \( \text{Hom}(G, O^\times) \) and the \( p' \)-part \( \hat{G}_{p'} \) with \( \text{Hom}(G, F^\times) \) via the inclusions \( O_K \subset O \subset K \) and the canonical surjection \( O \to F \).

3.5 Next we derive a formula for \( e^{D(G)}_{(H,hH')} \in K \otimes D(G) \) for each \( (H,hH') \in D(G) \) using the formula for \( e^{R^{ab}(G)}_{gG'} \). We use [Bo98b, Proposition 2.4] which gives an explicit inverse of the isomorphism
\[
\left( \pi_H \circ \text{res} \right)_{H \leq G} : K \otimes D(G) \to \left( \prod_{H \leq G} K \otimes R^{ab}(H) \right)^G ,
\]
where \( G \) acts on the above product by natural conjugation. Combining this
isomorphism with the isomorphism \( K \otimes R^{ab}(H) \to \prod_{hH' \in H^{ab}} K \) in the \( H \)-
component gives us the collection of species of \( D(G) \).

More precisely, assume that for each \( U \leq G \) we have an element \( a_U \in K \otimes R^{ab}(U) \) such that this family is fixed under \( G \) in the sense that \( \rho(a_U) = a_{gUg^{-1}} \),
for every \( U \leq G \) and \( g \in G \), where \( \rho(-) \) is defined as \( \text{res}_f \) for \( f : gUg^{-1} \to U, v \mapsto g^{-1}vg \). Then [Bo98b, Proposition 2.4] asserts that
\[
\left( \pi_U \circ \text{res} \right) \left( \frac{1}{|G|} \sum_{H_0 \leq \ldots \leq H_n} (-1)^n |H_0| \left[ H_0, \text{res}_{H_0}^{H_n}(a_{H_n}) \right]_G \right) = a_U \quad (3.5.a)
\]
for each $U \subseteq G$. In the above equation, $[U, \sum_{\varphi \in U} \alpha \varphi_{\varphi}]_G$ is defined as
$\sum_{\varphi \in U} \alpha_{\varphi} [U, \varphi]_G \in K \otimes D(G)$ for any coefficients $\alpha_{\varphi} \in K$. Now we fix
$(H, hH') \in D(G)$ and define a family $a_U$, $U \subseteq G$, of $G$-fixed elements
$a_U \in K \otimes R^{ab}(U)$ as follows. If $U$ is not conjugate to $H$ we set $a_U := 0.$ For $U = H$ we set
$$a_H := \sum_{r \in N_G(H)/N_G(H,hH')} \left( e_{hH'}^{R^{ab}(H)} \right),$$
and for arbitrary $g \in G$ we set $a_{gHg^{-1}} := ga_H$. Then we obtain $s_{hH'}^{R^{ab}(H)}(a_H) = 1$ and $s_{hH'}^{R^{ab}(H)}(a_H) = 0$ unless $hH'$ and $hH'$ are conjugate under $N_G(H)$. If we denote the argument of $\pi_U \circ \res^G_H$ in Equation (3.5.a) by $\sigma_G((a_U)_{U \subseteq G})$, then
$$\sigma_G((a_U)_{U \subseteq G}) = \frac{1}{|G|} \sum_{g \in G} \sum_{r \in N_G(H)/N_G(H,hH')} \sum_{H_0 \cdots H_n = gHg^{-1}} (-1)^n [H_0, \res_{H_0}^{gHg^{-1}} (gr) e_{hH'}^{R^{ab}(H)}]_G$$
and
$$s_{(H,hH')} \sigma_G((a_U)_{U \subseteq G}) = \begin{cases} 1, & \text{if } (H, hH') = G(H, hH'), \\ 0, & \text{if } (H, hH') \neq G(H, hH'), \end{cases}$$
for every $(H, hH') \in D(G)$. Thus $\sigma_G((a_U)_{U \subseteq G}) = e_{(H,hH')}^{D(G)}$. We can simplify this formula using
$$[H_0, \res_{H_0}^{gHg^{-1}} (gr) e_{hH'}^{R^{ab}(H)}]_G = [r^{-1} g^{-1} H_0 r g r, \res_{H_0}^{r^{-1} g^{-1} H_0 r g r} (e_{hH'}^{R^{ab}(H)})]_G$$
and changing the summation over chains ending in $grHr^{-1}g^{-1} = gHg^{-1}$ into the summation over chains ending in $H$ by conjugation with $r^{-1}g^{-1}$. Then we obtain
$$e_{(H,hH')}^{D(G)} = \frac{1}{|N_G(H, hH')|} \sum_{H_0 \cdots H_n = H} (-1)^n [H_0, \res_{H_0}^{H^{ab}(H)} e_{hH'}^{R^{ab}(H)}]_G.$$
By the last statement in Subsection 3.3, we have $\res_{H_0}^{H^{ab}(H)}(e_{hH'}^{R^{ab}(H)}) = 0$ unless $H_0 \cap hH' \neq \emptyset$. Therefore, it suffices to sum over chains starting in subgroups $H_0$ with $hH' \in H_0H'/H'$. Together with the explicit formula for $e_{hH'}^{R^{ab}(H)}$ from Subsection 3.4 we obtain
$$e_{(H,hH')}^{D(G)} = \frac{1}{|N_G(H, hH')|} \sum_{H_0 \cdots H_n = H} (-1)^n [H_0, \res_{H_0}^{H^{ab}(H)} \sum_{\varphi \in \hat{H}} \varphi(h^{-1}) [H_0, \varphi]_G].$$
Now Equation (3.2.b) applied to $\rho: D \to D^p$, for $(H, h'H) \in D^p(G)$, i.e., $h'H \in (H^{ab})_{p'}$, we obtain
$$e_{(H,hH')}^{D^p(G)} = \rho_G(e_{(H,hH')}^{D(G)}).$$

Next we derive formulae for the idempotents of \( K \otimes L^p(G) \) and \( K \otimes T^p(G) \). Equation (3.2.b) applied to \( \beta : D \rightarrow L^p \) yields, for \((H, hO_p(H)) \in L^p(G)\),

\[
e_{(L^p(G), hO_p(H))}^D(H, H H') \beta_G(e_{(H, hO_p(H))}^D(G)) = \frac{1}{|N_G(H, hH')|} \sum_{H_0 < \cdots < H_n = H} (-1)^n \sum_{\varphi \in \hat{H}} \varphi(h^{-1}) [\text{Ind}_{H_0}^G(O_m)|_{\varphi|_{H_0}}],
\]

where \( m \in \mathbb{N} \) is chosen as a multiple of the exponent \( n \) of \( G \) such that \( K_m \) is a splitting field for the linear sources modules of \( G \), cf. Subsection 1.5. Note that in the above summation, since \( H \) is \( p \)-hypo-elementary and \( (hO_p(H)) = H/O_p(H) \), the condition \( H_0 \cap hH' \neq \emptyset \) implies that \( H_0 \) contains a subgroup of order \( |H|_{p'} \).

Similarly, from Equation (3.2.b) applied to \( \beta : D^p \rightarrow T^p \) we obtain, for \((H, hO_p(H)) \in T^p(G)\),

\[
e_{(T^p(G), hO_p(H))}^D(H, H H') \beta_G(e_{(H, hO_p(H))}^D(G)) = \frac{1}{|N_G(H, H H')|} \sum_{H_0 < \cdots < H_n = H} (-1)^n \sum_{\varphi \in \hat{H}} \varphi(h^{-1}) [\text{Ind}_{H_0}^G(O_m)|_{\varphi|_{H_0}}].
\]

Note that, since \( h_{p'} H' \) is a \( p' \)-element in \( H_{p'} \), the condition \( H_0 \cap h_{p'} H' \neq \emptyset \) is now equivalent to the condition that \( |H|_{p'} \) divides \( |H_0| \) and we sum exactly over those chains \( H_0 < \cdots < H_n = H \) with the property that \( H_0 \) contains a Hall \( p' \)-subgroup of \( H \).

It should be noted that in contrast to the idempotent formulae for the other representation rings we cannot derive a formula in terms of the canonical basis given by the classes of indecomposable modules.

3.7 Finally, we will derive a second formula for the idempotents of \( K \otimes R(G) \) and \( K \otimes R^p(G) \) from the idempotents of \( K \otimes L^p(G) \) and \( K \otimes T^p(G) \).
Applying Equation (3.2.b) to \( \gamma: \mathbb{L}^p \to \mathbb{R} \) we obtain for \( g \in \mathbb{G} \) the formula

\[
e_{\mathbb{G}}(G)^{R(G)} = \gamma^*_G(e_{\langle (g), g \rangle}^{\mathbb{L}^p(G)}) = \frac{1}{|C_G(g)|} \sum_{\varphi \in \hat{H}} \varphi(g^{-1}) \text{ind}_H^G(\varphi),
\]

where \( H := \langle g \rangle \), since the condition \( H_0 \cap \{ g \} \neq \emptyset \) implies that \( H_0 = H_n = H \). If \( \varphi \) denotes a generator of \( \hat{H} \) and \( \varphi(g^{-1}) = \zeta \), then we obtain

\[
e_{\mathbb{G}}(G)^{R(G)} = \frac{1}{|C_G(g)|} \sum_{i=0}^{[H]-1} \zeta^i \cdot \text{ind}_H^G(\varphi^i).
\]

Similarly we obtain for \( g \in \mathbb{G}' \) the formula

\[
e_{\mathbb{G}}(G)^{R(G)} = \gamma^*_G(e_{\langle (g), g \rangle}^{\mathbb{T}^p(G)}) = \frac{1}{|C_G(g)|} \sum_{\varphi \in \hat{H}} \varphi(g^{-1}) \text{ind}_H^G(\varphi).
\]

where again \( H := \langle g \rangle \), \( \varphi \) is a generator of \( \hat{H} \), and \( \zeta = \varphi(g^{-1}) \).

References


24
<table>
<thead>
<tr>
<th>Ref</th>
<th>Author</th>
<th>Title</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fe82</td>
<td>W. Feit</td>
<td>The representation theory of finite groups</td>
<td>North Holland, Amsterdam, 1982.</td>
</tr>
</tbody>
</table>