Integrality Conditions for Elements in Ghost Rings of Generalized Burnside Rings\(^*\)

Robert Böttje\(^1\)
Department of Mathematics
University of California
Santa Cruz, CA 95064
U.S.A.
bottje@math.ucsc.edu

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Abstract

The Burnside ring of a finite group \(G\) is a subring of finite index in its ‘ghost ring’ of integer valued functions on the conjugacy classes of subgroups of \(G\). Dress gave a congruence criterion for the values of such a function that ensures that the function lies in the Burnside ring. We prove a different type of congruence criterion that generalizes to other rings, in particular to the ring of monomial representations of \(G\). As an application we obtain much shorter and more conceptual proofs for the integrality of canonical induction formulas for linear and trivial source modules.

Introduction

Counting the number of \(H\)-fixed points on a finite \(G\)-set for all subgroups \(H\) of a finite group \(G\) induces an injective ring homomorphism \(\rho_G: B(G) \to \hat{B}(G)\) with finite cokernel from the Burnside ring of \(G\) to the ring of integer valued functions on the set of conjugacy classes of subgroups of \(G\). Dress (see for example [LD87, Ch. 4, Thm. 5.7]) gave an equivalent condition for a function \(f \in \hat{B}(G)\) to be in the image of \(\rho_G\) by requiring that the values of \(f\) satisfy certain congruences.

The ring homomorphism \(\rho_G: B(G) \to \hat{B}(G)\) is a special case of a more general construction \(\rho_G: A^+(G) \to A^+(G)\) that starts with a restriction functor \(A\) on \(G\). Such constructions have been introduced and studied by Dress in [D71] and later by Böttje in [B98a] in connection with canonical induction formulas. Again, elements of \(A^+\) can be viewed as functions. In the Main Theorem 2.2 we give a list of congruences for the values of a function \(f \in A^+(G)\) which are

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satisfied if and only if \( f \) lies in the image of \( \rho_G \). Specializing to the Burnside ring we obtain congruences that are different from Dress'. And specializing to the ring \( D(G) \) of monomial representations of \( G \) and the corresponding map \( \rho_G: D(G) \to \hat{D}(G) \) into its ‘ghost ring’ one obtains a criterion that is very handy to shorten the quite lengthy proofs of integrality of canonical induction formulas for the trivial source ring and the linear source ring (cf. [B90], [B94], [B98b]).

For any section \( H \leq Q \leq G \) of \( G \) there are well-known fixed point maps \( \text{fix}^H_{Q,G}: B(G) \to B(Q/H) \) and \( \text{fix}^H_{Q,G}: B(G) \to B(Q/H) \) commuting with \( \rho_G \) and \( \rho_{Q/H} \). We also introduce similar maps \( \text{fix}^H_{Q,G}: D(G) \to D(Q/H) \) and \( \text{fix}^H_{Q,G}: D(G) \to D(Q/H) \) again commuting with \( \rho_G \) and \( \rho_{Q/H} \). A consequence of our main theorem is that an element \( f \in B(G) \) is integral, i.e., in the image of \( \rho_G \), if and only if \( \text{fix}^H_{Q,G}(f) \) is integral for all section with \( Q/H \) a \( p \)-group. A similar statement holds for \( f \in \hat{D}(G) \) with ‘\( p \)-group’ replaced by ‘elementary group’.

The paper is arranged as follows. In Section 1 we recall the basic definitions and constructions that lead to the map \( \rho_G: A^+(G) \to A^+(G) \). In Section 2 we prove the Main Theorem 2.2 which contains the congruence conditions for an element \( f \in A^+(G) \) to belong to the image of \( \rho_G \). After that the Main Theorem is applied to the Burnside ring (resp. the ring of monomial representations) and is extended to a local integrality detection theorem in Corollary 2.7 (resp. Corollary 2.10) using only sections of \( G \) that are \( p \)-groups (resp. elementary groups).

After recalling the definition of various canonical induction formulas, Section 3 contains a reduction of the proof of the integrality of the canonical induction formula for trivial and linear source modules to the integrality of the canonical Brauer induction formula for a \( p \)-group, by making use of the congruences established in Section 2.

## 1 The Setup

Throughout this paper we fix a finite group \( G \) and a commutative ring \( k \).

We will need the notion of a \( k \)-restriction functor \( A \) on \( G \) and for the reader's convenience we will recall its definition together with the functorial constructions \( A_+ \) and \( A^+ \) from [B98a, Sections 1 and 2]. The Burnside ring and the monomial representation ring are isomorphic to \( A_+ \) for certain choices of \( A \) and their ghost rings are isomorphic to \( A^+ \).

### 1.1 A \( k \)-restriction functor \( A \) on \( G \) is a family of \( k \)-modules \( A(H) \), \( H \leq G \), together with conjugation and restriction maps

\[
c^H_{\cdot}: A(H) \to A(\hat{H}) \quad \text{and} \quad \text{res}^H_{\cdot}: A(H) \to A(I)
\]

for all \( I \leq H \leq G \) and \( g \in G \). Here we use the abbreviation \( \hat{H} := gHg^{-1} \). Very often we will also abbreviate \( c^H_{\cdot}(a) \) by \( g^a \) for \( H \leq G \), \( a \in A(H) \), and \( g \in G \).
The conjugation and restriction maps have to satisfy the axioms
\[
\begin{align*}
c_H^h &= \text{res}^H_h = \text{id}_{A(H)}, \\
\text{res}_J^I \circ \text{res}^H_J &= \text{res}^H_I, \\
c_H^I \circ c_H^J &= c_H^{IJ}, \\
c_I \circ \text{res}^H_I &= \text{res}^H_I \circ c_H^I,
\end{align*}
\]
for all \( J \leq I \leq H \leq G \), \( h \in H \), and \( g, g' \in G \).

If \( A \) is a \( k \)-restriction functor on \( G \) with the additional property that every \( A(H) \), \( H \leq G \), is a \( k \)-algebra and if the conjugation and restriction maps are \( k \)-algebra maps, then we call \( A \) a \( k \)-algebra restriction functor on \( G \).

A morphism \( f: A \to A' \) between \( k \)-restriction functors (resp. \( k \)-algebra restriction functors) \( A \) and \( A' \) is a family \( f = (f_H)_{H \in G} \) of \( k \)-module (resp. \( k \)-algebra) homomorphisms \( f_H: A(H) \to A'(H) \) which commute with the respective conjugation and restriction maps.

1.2 For every \( k \)-restriction functor \( A \) on \( G \) one can define an associated \( k \)-restriction functor \( A_+ \) by
\[
A_+(H) := \left( \bigoplus_{K \leq H} A(K) \right)_H,
\]
for \( H \leq G \). Here we view \( \bigoplus_{K \leq H} A(K) \) as a \( kH \)-module via the conjugation maps \( c_K^h, K \leq H, h \in H \), and, for any \( kH \)-module \( M \), we use the notation
\[
M_H := M/\langle m - hm \mid m \in M, h \in H \rangle_h
\]
for the \( k \)-module of \( H \)-cofixed points of \( M \). The class of an element \( a \in A(K) \) in \( A_+(H) \) will be denoted by \([K, a]_H\). The conjugation and restriction maps are defined by
\[
c_H^a: A_+(H) \to A_+(\,^aH), \quad [K, a]_H \mapsto \left[ \,^aK, \,^a[a]_H \right],
\]
for \( H \leq G \) and \( g, g' \in G \), and by
\[
\text{res}_I^H: A_+(H) \to A_+(I), \quad [K, a]_H \mapsto \sum_{I \subset K \leq H \cap K} \left[ I \cap \,^hK, \text{res}_{I \cap \,^hK}^h([h]_I) \right],
\]
for \( I \leq H \leq G \).

We will also need the projection maps
\[
\pi_H: A_+(H) \to A(H), \quad [K, a]_H \mapsto \begin{cases} a, & \text{if } K = H, \\ 0, & \text{if } K < H, \end{cases}
\]
for \( H \leq G \), which commute with the respective conjugation maps. Note that \( \pi_H \) is well-defined since \( c_H^h = \text{id}_{A(H)} \) for \( h \in H \).
If $A$ is a $k$-algebra restriction functor on $G$, then also $A_+$ is a $k$-algebra restriction functor under the multiplication

$$
[K, a]_H \cdot [L, b]_H := \sum_{K \subseteq H, K \nsubseteq L} [K \cap hL, \res^K_{\cap L}(a) \cdot \res^K_{\cap L}(h b)]_H
$$
on $A_+ (H), H \subseteq G$. In this case, the identity element of $A_+ (H)$ is $[H, 1_{A(H)}]_H$ and $\pi_H: A_+ (H) \to A(H)$ is a $k$-algebra map, for $H \subseteq G$.

1.3 If $A$ is a $k$-restriction functor on $G$, one can form the $k$-restriction functor $A^+$ on $G$ in a dual way to the construction of $A_+$. One defines

$$
A^+ (H) := \left( \prod_{K \subseteq H} A(K) \right)^H,
$$

for $H \subseteq G$, where $\prod_{K \subseteq H} A(K)$ is viewed as $kH$-module via the conjugation maps of $A$, and where $M^H$ denotes the $k$-submodule of $H$-fixed points for a $kH$-module $M$. The conjugation and restriction maps are defined by

$$
c^H_1: A^+ (H) \to A^+ (s^H), \quad (a_K)_{K \subseteq H} \mapsto (\pi^H_1 a_K)_{K \subseteq H},
$$

for $H \subseteq G$ and $g \in G$, and by the projection maps

$$
\res^H_1: A^+ (H) \to A^+ (I), \quad (a_K)_{K \subseteq H} \mapsto (a_K)_{K \subseteq I}, \quad (1.3.a)
$$

for $I \subseteq H \subseteq G$. Note that the restriction maps of $A$ are not used in the construction of $A^+$.

If $A$ is a $k$-algebra restriction functor, then also $A^+$ is by viewing $A^+ (H)$ as a $k$-subalgebra of the $k$-algebra $\prod_{K \subseteq H} A(K)$, for $H \subseteq G$.

1.4 For every $k$-restriction functor (resp. $k$-algebra restriction functor) $A$ on $G$ there is a morphism $\rho: A_+ \to A^+$ of $k$-restriction functors (resp. $k$-algebra restriction functors) on $G$, called the mark morphism. It is defined as

$$
\rho_H := (\pi_K \circ \res^H_K)_{K \subseteq H}: A_+ (H) \to A^+ (H) = \left( \prod_{K \subseteq H} A(K) \right)^H,
$$

for $H \subseteq G$. We should note here that $A_+$ and $A^+$ also have induction maps so that they form $k$-Mackey functors on $G$ and that $\rho$ is a morphism of $k$-Mackey functors on $G$. But we will not use the Mackey functor structure in this paper.

Also, for every $H \subseteq G$, there is a $k$-linear map in the reverse direction, namely

$$
\sigma_H: A^+ (H) \to A_+ (H), \quad (a_K)_{K \subseteq H} \mapsto \sum_{L \subseteq K \subseteq H} [L] \mu_{L,K} [L, \res^K_L (a_K)]_H,
$$

where the sum runs over all pairs $(L, K)$ of subgroups of $H$ with $L \subseteq K$, and where $\mu_{L,K}$ denotes the Möbius function of the partially ordered set of subgroups.
of $G$ evaluated at $(L, K)$. The maps $\sigma_H$ and $\rho_H$, $H \leq G$, are almost inverse to each other. In fact, one has

$$
\rho_H \circ \sigma_H = |H| \cdot \text{id}_{A^+(H)} \quad \text{and} \quad \sigma_H \circ \rho_H = |H| \cdot \text{id}_{A_+(H)}.
$$

(1.4.a)

cf. [B98a, Prop. 2.4].

1.5 Definition Let $A$ be a $k$-restriction functor on $G$. A $G$-stable $k$-basis of $A$ is a collection $B = (B_H)_{H \leq G}$ with $B_H$ a $k$-basis of $A(H)$ satisfying $\delta_H^*(B_H) = B \cdot H$ for all $H \leq G$ and $g \in G$.

If $B$ is a $G$-stable $k$-basis of $A$, we associate with it the sets

$$
\mathcal{M}_H := \{(K, \psi) \mid K \leq H, \psi \in B_K\}.
$$

for $H \leq G$. Obviously, $\mathcal{M}_I \subseteq \mathcal{M}_H$ if $I \leq H \leq G$, and $\mathcal{M}_G$ is a $G$-set via conjugation:

$$
\delta^*(K, \psi) := (\delta K, \delta^* \psi),
$$

for $(K, \psi) \in \mathcal{M}_G$ and $g \in G$. For $K \leq H \leq G$ and $\varphi \in B_H$ we can write

$$
\text{res}_H^G(\varphi) = \sum_{\psi \in B_K} m_{(K, \psi)}^{(H, \varphi)} \cdot \psi
$$

with uniquely determined multiplicities $m_{(K, \psi)}^{(H, \varphi)} \in k$. For $(K, \psi), (H, \varphi) \in \mathcal{M}_G$ we define

$$(K, \psi) \leq (H, \varphi) : \iff K \leq H \text{ and } m_{(K, \psi)}^{(H, \varphi)} \neq 0.$$  

Note that if $(K, \psi) \leq (H, \varphi)$ and $g \in G$, then $\delta^*(K, \psi) \leq \delta^*(H, \varphi)$. Note also that in general $\mathcal{M}_G$ is not a partially ordered set. But it is, for instance, if $\text{res}_{I}^g(B_H) \subseteq B_I$ for all $I \leq H \leq G$, or if $k = \mathbb{Z}$ and $m_{(K, \psi)}^{(H, \varphi)} \in \mathbb{N}$ for all $(K, \psi) \leq (H, \varphi)$ in $\mathcal{M}_G$.

1.6 Remark Assume that $A$ is a $k$-restriction functor on $G$ with $G$-stable $k$-basis $B = (B_H)_{H \leq G}$. Let $\mathcal{M}_G, m_{(K, \psi)}^{(H, \varphi)}$ and $\leq$ be derived from $B$ as in Definition 1.5.

In this situation, the disjoint union $\bigcup_{K \leq H} B_K$ is a $k$-basis of $\bigoplus_{K \leq H} A(K)$, for $H \leq G$, and if $(K, \psi)$ runs through a set of representatives of the $H$-orbits of $\mathcal{M}_H$, then the elements $[K, \psi]_H$ form a $k$-basis of $A^+_H$.

Moreover, every element $f \in A^+(H)$, $H \leq G$, is of the form

$$
f = \left( \sum_{\psi \in B_K} f_{(K, \psi)} \cdot \psi \right)_{K \leq H},
$$

with unique coefficients $f_{(K, \psi)} \in k$ satisfying $f_{(K, \psi)} = f_{\delta(K, \psi)}$ for all $(K, \psi) \in \mathcal{M}_H$ and $h \in H$. Thus, we may also view $A^+(H)$ as the set of all functions $f : \mathcal{M}_H \to k$ that are constant on $H$-orbits. In particular, $A^+(H)$ is a free $k$-module.
1.7 Example Assume that $k = \mathbb{Z}$.

(a) Consider the constant restriction functor $A$ on $G$ with $A(H) = \mathbb{Z}$ for all $H \leq G$ and with all conjugation and restriction maps being the identity map. Then $B_H := \{1\}$, $H \leq G$, defines a $G$-stable $\mathbb{Z}$-basis of $A$ and the ring $A_+(H)$ is isomorphic to the Burnside ring $B(H)$ by identifying the basis element $[K, 1]_H$ of $A_+(H)$ with the basis element $[H/K]$ of $B(H)$. Under this identification, the conjugation and restriction maps on both rings coincide. Moreover, the $K$-component of the mark homomorphism $\rho_H : A_+(H) \to A^+(H) = (\prod_{K \leq H} \mathbb{Z})^H$ is induced by taking the number of $K$-fixed points of an $H$-set, so that $\rho_H$ corresponds to the usual imbedding $B(H) \to B(H) := (\prod_{K \leq H} \mathbb{Z})^H$. One usually calls an element of $f \in B(G)$ integral if it is in the image of $\rho_G$.

(b) Let $C$ be an abelian group and, for $H \leq G$, set $\bar{H} := \text{Hom}(H, C)$. Then $\bar{H}$ is a group under multiplication and $A(\bar{H}) := \mathbb{Z}_H$, the group ring of $\bar{H}$, gives rise to a restriction functor $A$ with $G$-stable $\mathbb{Z}$-basis $B_{\bar{H}} := \bar{H}$, $H \leq G$. The conjugation maps are defined by $(c^H_{\bar{H}}(\varphi))(x) := \varphi(g^{-1}xg)$ and the restriction maps by $\text{res}^H_{\bar{H}}(\varphi) := \varphi|_I$ for $g \in G$, $I \leq H \leq G$, $x \in g\bar{H}$, and $\varphi \in B_{\bar{H}}$. Note that $(K, \psi) \leq (H, \varphi)$ for $(K, \psi), (H, \varphi) \in \mathcal{M}_G$ if and only if $K \leq H$ and $\psi = \varphi|_K$, and that the multiplicity $m^{(H, \psi)}_{K, \varphi}$ is equal to 1 in this case.

If $C$ is the multiplicative group $\mathbb{C}^\times$ of the field $\mathbb{C}$ of complex numbers, the algebra restriction functor $A_+$ was used in [B90] and [B98a] in the construction of the canonical Brauer induction formula on the character ring of $G$. If $C = \mathbb{C}^\times$, the group of units of a complete discrete valuation ring $\mathcal{O}$ that is large enough for $G$, the algebra restriction functor $A_+$ was used in [B98a] for the construction of a canonical induction formula for the linear source ring and the trivial source ring of $G$, and for the Grothendieck group of projective $\mathcal{O}G$-module.

These induction formulas where defined via natural maps into $A^+$ and the most crucial part of the proof was to show that the resulting elements in $A^+$ were integral, i.e., contained in the image of $\rho : A_+ \to A^+$. In the next section we will give a criterion for an element of $A^+(G)$ to be in the image of $\rho_G$ and use this criterion in Section 3 to give a shortened and more conceptual integrality proof of all these induction formulas. Note that in all the above examples the map $\rho_G$ is injective with finite cokernel (cf. (1.4.a)), since $A_+(G)$ and $A^+(G)$ are free $\mathbb{Z}$-modules, cf. 1.6.

2 The Integrality Conditions

Throughout this section we assume that $A$ is a $k$-restriction functor on $G$ with stable $k$-basis $B$, that $\mathcal{M}_G$ is the associated $G$-set, and that $m^{(H, \psi)}_{K, \varphi} \in k$ are the multiplicities as defined in 1.5.
A. The Main Theorem

2.1 Proposition Let

\[ f = \left( \sum_{\psi \in B_k} f(K, \psi) \cdot \psi \right)_{K \in G} \in A^+(G) \]

and assume that \( f \in \text{im}(\rho_Q) \). Then, for every \((H, \varphi) \in \mathcal{M}_G\) and every intermediate subgroup \( H \leq Q \leq G \) of \( G \) one has the congruence

\[
\sum_{(H, \varphi) \leq (I, \psi) \in \mathcal{M}_Q} \mu_{H, I} \cdot m^{(I, \psi)}_{(H, \varphi)} \cdot f(I, \psi) \equiv 0 \pmod{[N_Q(H, \varphi) : H]} \quad (2.1.\text{a})
\]

in \( k \), where the sum runs over all pairs \((I, \psi) \in \mathcal{M}_Q \) with \((H, \varphi) \leq (I, \psi) \).

Proof Let \( x \in A^+(G) \) be such that \( f = \rho_Q(x) \). Then, since \( \rho_Q \) commutes with restrictions and since \( \sigma_Q \circ \rho_Q = |Q| \cdot \text{id}_{A^+(Q)}, \) cf. (1.4.a), we have

\[
|Q| \cdot \text{res}_Q^G(x) = \sigma_Q(\rho_Q(\text{res}_Q^G(x))) = \sigma_Q(\text{res}_Q^G(\rho_Q(x))) \\
= \sigma_Q(\text{res}_Q^G(f)) = \sigma_Q(\left( \sum_{\psi \in B_k} f(K, \psi) \cdot \psi \right)_{K \leq Q}) \\
= \sum_{L \leq I \leq Q} |L| \cdot \mu_{L, I} \cdot \left[ L, \text{res}_L^I(\sum_{\psi \in B_k} f(K, \psi) \cdot \psi) \right]_Q \\
= \sum_{L \leq I \leq Q} |L| \cdot \mu_{L, I} \cdot \sum_{\psi \in B_I} f(I, \psi) \cdot \left[ L, \text{res}_L^I(\psi) \right]_Q \\
= \sum_{L \leq I \leq Q} |L| \cdot \mu_{L, I} \cdot \sum_{\psi \in B_I} f(I, \psi) \cdot \sum_{\lambda \in B_K} m^{(I, \psi)}_{(L, \lambda)} \cdot [L, \lambda]_Q \\
= \sum_{(L, \lambda) \leq (I, \psi) \in \mathcal{M}_Q} |L| \cdot \mu_{L, I} \cdot m^{(I, \psi)}_{(L, \lambda)} \cdot f(I, \psi) \cdot [L, \lambda]_Q.
\]

We want to determine the coefficient of the basis element \([H, \varphi]_Q\) in the last expression. We have \([L, \lambda]_Q = [H, \varphi]_Q\) if and only if \((L, \lambda)\) and \((H, \varphi)\) are conjugate in \( Q \). If they are, the expression for \((L, \lambda)\) and \((H, \varphi)\) in the last sum are the same and we obtain that the coefficient is given by

\[
\frac{|Q|}{[N_Q(H, \varphi)]} \sum_{(H, \varphi) \leq (I, \psi) \in \mathcal{M}_Q} |H| \cdot \mu_{H, I} \cdot m^{(I, \psi)}_{(H, \varphi)} \cdot f(I, \psi).
\]

Comparing this with the beginning of the above sequence of equations we obtain that the last expression is an integer which is divisible by \( |Q| \). Now, the result follows.

The following theorem, our Main Theorem, includes a converse of Proposition 2.1. In fact, not all the congruences of the type (2.1.a) are needed to
conclude that an element of $A^+(G)$ is contained in the image of $\rho_G$. Before we state the theorem we define

$$N_G := \{(H, \varphi, Q) \mid (H, \varphi) \in \mathcal{M}_G, H \leq Q \leq N_G(H, \varphi)$$

such that $Q/H \in \text{Syl}(N_G(H, \varphi)/H)$,

where $\text{Syl}(X)$ denotes the set of Sylow subgroups of a finite group $X$ (for all primes). Note that $N_G$ is a $G$-set under conjugation.

2.2 Theorem Let

$$f = \left( \sum_{\psi \in B_k} f_{(K, \psi)} \cdot \psi \right)_{K \in G} \in A^+(G).$$

Then the following are equivalent:

(i) The element $f$ is in the image of $\rho_G$.

(ii) The congruence (2.1.a) holds for all $(H, \varphi) \in \mathcal{M}_G$ and for $Q := N_G(H, \varphi)$ (in which case one has $N_Q(H, \varphi) = N_G(H, \varphi)$).

(iii) The congruence (2.1.a) holds for all $(H, \varphi, Q) \in N_G$ (in which case one has $N_Q(H, \varphi) = Q$).

(iv) The congruence (2.1.a) holds for all $(H, \varphi) \in \mathcal{M}_G$ and all $H \leq Q \leq G$.

Proof Obviously, by Proposition 2.1, the statement in (i) implies (ii), (iii), and (iv). Also, (iv) implies (ii) and (iii) for trivial reasons.

(ii)$\Rightarrow$ (i): Let $S_f := \{(K, \psi) \in \mathcal{M}_G \mid f_{(K, \psi)} \neq 0\}$ be the support of $f$. If $S_f$ is not empty we set

$$m(f) := \max\{|K| \mid (K, \psi) \in S_f\}.$$

If $S_f$ is empty we set $m(f) := 0$. We prove that $f \in \text{im}(\rho_G)$ by induction on $m(f)$. If $m(f) = 0$, then $f = 0$ and $f \in \text{im}(\rho_G)$. So we assume that $m(f) > 0$ and we choose representatives $(H_1, \varphi_1), \ldots, (H_n, \varphi_n)$ of the $G$-orbits of elements $(H, \varphi) \in S_f$ with $|H| = m(f)$. The congruence (2.1.a) for $(H_i, \varphi_i)$ and $Q = N_G(H_i, \varphi_i)$ yields $f_{(H_i, \varphi_i)} = \alpha_i [N_G(H_i, \varphi_i) : H_i]$ for some $\alpha_i \in k$ and $i = 1, \ldots, n$. We consider the element

$$f' := f - \sum_{i=1}^n \rho_G(\alpha_i \cdot [H_i, \varphi_i]_G).$$

By the definition of $\rho_G$, the value at $(K, \psi)$ of the function $\rho_G(\alpha_i \cdot [H_i, \varphi_i]|_G)$ is zero for all $(K, \psi)$ with $|K| \geq m(f)$, except for the value at $(H_i, \varphi_i)$ and its $G$-conjugates where it is equal to $\alpha_i \cdot [N_G(H_i, \varphi_i) : H_i]$. This implies that $m(f') < m(f)$. Since $f$ and $\rho_G(\alpha_i \cdot [H_i, \varphi_i]|_G)$ satisfy the congruences of part (ii) (the latter by Proposition 2.1), also $f'$ satisfies the congruences of part (ii).

By induction, we have $f' \in \text{im}(\rho_G)$, and then also $f \in \text{im}(\rho_G)$.

(iii)$\Rightarrow$(i): We proceed again by induction on $m(f)$ to show that $f \in \text{im}(\rho_G)$ and we may again assume that $m(f) > 0$. Let $(H_i, \varphi_i), i = 1, \ldots, n$, be chosen as
above and fix \( i \in \{1, \ldots, n\} \). If we can show that \( f_{(H_i, \varphi_i)} \in k \cdot [N_G(H_i, \varphi_i) : H_i] \), we can proceed as above and the proof is complete. But, if \( H_i = N_G(H_i, \varphi_i) \), there is nothing to show, and if \( H_i < N_G(H_i, \varphi_i) \), then the congruence in part (iii) for \((H_i, \varphi_i)\) and \(Q/H_i\) a non-trivial Sylow \( p\)-subgroup of \( N_G(H_i, \varphi_i)/H_i\) implies that \( f_{(H_i, \varphi_i)} \in k \cdot [Q : H_i] \). Since this holds for all primes \( p \) dividing \([N_G(H_i, \varphi_i) : H_i]\), the proof is complete. \( \square \)

2.3 Remark (a) The condition in Theorem 2.2(ii) can be simplified by observing that if \( I/H \) is a \( p\)-group, then \( \mu_{H,I} = 0 \) unless \( I/H \) is elementary abelian. And if \( I/H \) is elementary abelian of rank \( r \), then \( \mu_{H,I} = (-1)^r p^r \), cf. [R64, 5, Example 2].

(b) Set

\[
\mathcal{M}_G^* := \{(H, \varphi) \in \mathcal{M}_G \mid [N_G(H, \varphi) : H] \not\in k^\times\}
\]

and

\[
\mathcal{N}_G^* := \{(H, \varphi, Q) \in \mathcal{N}_G \mid [Q : H] \not\in k^\times\}.
\]

Note that both sets are stable under \( G\)-conjugation and that they depend on \( k \). It is easy to check that the congruences in Theorem 2.2(ii) (resp. Theorem 2.2(ii)) for two \( G\)-conjugate pairs in \( \mathcal{M}_G \) (resp. triples in \( \mathcal{N}_G \)) are equivalent. Moreover, it is obvious that the congruences are satisfied for pairs in \( \mathcal{M}_G \setminus \mathcal{M}_G^* \) (resp. triples in \( \mathcal{N}_G \setminus \mathcal{N}_G^* \)). Therefore, it suffices to check the conditions in (ii) (resp. (iii)) for a set \( \mathcal{R}_G \) (resp. \( \mathcal{S}_G \)) of representatives of the \( G\)-orbits of \( \mathcal{M}_G^* \) (resp. \( \mathcal{N}_G^* \)).

The next proposition shows that the congruences in Theorem 2.2(ii) (resp. Theorem 2.2(ii)) for \( \mathcal{R}_G \) (resp. \( \mathcal{S}_G \)) form a minimal set of congruences that ensures that the element \( f \) lies in the image of \( \rho_G \).

Before we state the proposition we need to introduce the maps

\[
\pi(H, \varphi) : A^+(G) \to k/[N_G(H, \varphi) : H]k,
\]

\[
(\sum_{\psi \in K} f_{(K, \psi)} \cdot \psi)_{K \subseteq G} \mapsto \sum_{(H, \psi) \leq (I, \psi) \in \mathcal{M}_G \cap \mathcal{N}_G} \mu_{H,I} \cdot m_{(H, \psi)}^{(I, \psi)} : f_{(I, \psi)} + [N_G(H, \varphi) : H]k,
\]

for each \((H, \varphi) \in \mathcal{M}_G\), and

\[
\pi(H, \varphi, Q) : A^+(G) \to k/[Q : H]k,
\]

\[
(\sum_{\psi \in K} f_{(K, \psi)} \cdot \psi)_{K \subseteq G} \mapsto \sum_{(H, \psi) \leq (I, \psi) \in \mathcal{M}_Q \cap \mathcal{N}_G} \mu_{H,I} \cdot m_{(H, \psi)}^{(I, \psi)} : f_{(I, \psi)} + [Q : H]k,
\]

for each \((H, \varphi, Q) \in \mathcal{N}_G\). Note that \( \pi_{(H, \varphi)} = \pi(H, \varphi) \) and \( \pi'_{(H, \varphi, Q)} = \pi'(H, \varphi, Q) \) for all \((H, \varphi) \in \mathcal{M}_G\), \((H, \varphi, Q) \in \mathcal{N}_G\), and \( g \in G \). We denote by \( \pi := (\pi_{(H, \varphi)})_{(H, \varphi) \in \mathcal{R}_G} \) and \( \pi' := (\pi'_{(H, \varphi, Q)})_{(H, \varphi, Q) \in \mathcal{S}_G} \) the maps from \( A^+(G) \) into the respective direct products.
2.4 Proposition With the above notation one has short exact sequences

\[ 0 \xrightarrow{} \text{im}(\rho_G^* \otimes \pi) \xrightarrow{\iota} A^+(G) \xrightarrow{\pi} \bigoplus_{(H, \varphi) \in \mathcal{R}_G} k/[N_G(H, \varphi) : H]k \xrightarrow{} 0 \]

and

\[ 0 \xrightarrow{} \text{im}(\rho_G^* \otimes \pi') \xrightarrow{\iota'} A^+(G) \xrightarrow{\pi'} \bigoplus_{(H, \varphi, Q) \in \mathcal{S}_G} k/[Q : H]k \xrightarrow{} 0. \]

of \( k \)-modules, where \( \iota \) denotes the inclusion map.

Proof The exactness at the middle terms follows immediately form the equivalences (i) \( \iff \) (ii) and (i) \( \iff \) (iii) in Theorem 2.2 together with Remark 2.3(b).

We show that the image of the maps \( \pi \) and \( \pi' \) is contained in the direct sum (rather than the direct product). In fact, \( A^+(G) \) is the free \( k \)-module on the characteristic functions \( \varepsilon_{(L, \lambda)} = (\sum_{\psi \in K} f_{(K, \psi)} \cdot \psi)_{K \in G} \) with \( f_{(K, \psi)} = 1 \) if \((K, \psi) \) and \((L, \lambda) \) are \( G \)-conjugate and \( f_{(K, \psi)} = 0 \) otherwise. Since there are only finitely many pairs \((H, \varphi) \in \mathcal{M}_G \) with \((H, \varphi) \leq \lambda(L, \lambda) \) for any \( g \in G \), our claim follows.

Next we show that the map \( \pi \) is surjective. We will prove that for every \((H, \varphi) \in \mathcal{R}_G \), the element \( e_{(H, \varphi)} \in \bigoplus_{(H, \varphi) \in \mathcal{R}_G} k/[N_G(H, \varphi) : H]k \), which has entry \( 1 + [N_G(H, \varphi)]k \) in the \((H, \varphi)\)-component and 0 everywhere else, lies in the image of \( \pi \). We proceed by induction on \(|H| \). If \( H = 1 \), we have \( \pi(e_{(H, \varphi)}) = e_{(H, \varphi)} \). Now let \((H, \varphi) \in \mathcal{R}_G \) with \(|H| > 1 \). We consider the element \( \pi(e_{(H, \varphi)}) \).

By the definition of the multiplicities \( m_{(K, \psi)}^{(H, \varphi)} \), we have \( \pi_{(K, \psi)}(e_{(H, \varphi)}) = 0 \) for all \((K, \psi) \in \mathcal{M}_G \) unless \((K, \psi) \leq \lambda(H, \varphi) \) for some \( g \in G \). Moreover, we have \( \pi_{(H, \varphi)}(e_{(H, \varphi)}) = 1 + [N_G(H, \varphi) : H]k \). Therefore, by induction, \( e_{(H, \varphi)} - \pi(e_{(H, \varphi)}) \) lies in the image of \( \pi \) and so does \( e_{(H, \varphi)} \).

Similarly one shows that \( \pi' \) is surjective by replacing the elements \( e_{(H, \varphi)} \) and \( e_{(H, \varphi), Q} \) by elements \( e_{(H, \varphi), Q} \) and \( e_{(H, \varphi), Q} \) for \((H, \varphi, Q) \in \mathcal{S}_G \), where \( e_{(H, \varphi), Q} \) has \((H, \varphi, Q)\)-component \( 1 + [Q : H]k \) and vanishes everywhere else, and where \( e_{(H, \varphi), Q} \) considered as \( G \)-invariant function on \( \mathcal{M}_G \) vanishes outside the \( G \)-orbit of \((H, \varphi) \) and is constant on the \( G \)-orbit of \((H, \varphi) \) with value \( n \cdot 1_k \), where \( n \in \mathbb{Z} \) is chosen with the property that \( n \cdot 1_k + [Q : H]k = 1_k + [Q : H]k \) and \( n \cdot 1_k + [Q' : H]k = 0 + [Q' : H]k \) for all \((H, \varphi, Q') \in \mathcal{S}_G \) with \( Q' \neq Q \). (Note that \( Q'/H \) and \( Q/H \) are non-trivial Sylow subgroups of \( N_G(H, \varphi)/H \) for different primes.) The surjectivity of \( \pi' \) now follows the same argument as for \( \pi \).

B. Application to the Burnside Ring

The following corollary is an immediate consequence of Theorem 2.2 for the Burnside ring \( B(G) \) and the mark homomorphism \( \rho_G : B(G) \to \hat{B}(G) \), cf. Example 1.7(a).
2.5 Corollary Let \( f = (f_K)_{K \leq G} \in \hat{B}(G) \). Then the following are equivalent:

(i) The element \( f \) is in the image of \( \rho_G \).

(ii) The congruence

\[
\sum_{H \leq l \leq N_G(H)} \mu_{H,l} \cdot f_l \equiv 0 \mod [N_G(H) : H]
\]

holds for all \( H \leq G \).

(iii) The congruence

\[
\sum_{H \leq l \leq Q} \mu_{H,l} \cdot f_l \equiv 0 \mod [Q : H]
\]

holds for all \( H \leq G \) and all Sylow subgroups \( Q/H \) of \( N_G(H)/H \).

(iv) The congruence

\[
\sum_{H \leq l \leq Q} \mu_{H,l} \cdot f_l \equiv 0 \mod [N_Q(H) : H]
\]

holds for all \( H \leq Q \leq G \).

2.6 As a consequence of Corollary 2.5 we will obtain an integrality detection result for the Burnside ring of a different flavor: Integrality is detected via taking fixed points for sections of \( G \) which are \( p \)-groups.

Recall that a pair \((H, Q)\) of subgroups of \( G \) is called a section of \( G \) if \( H \leq Q \leq G \). Let \((H, Q)\) be a section of \( G \). Taking \( H \)-fixed points of a \( Q \)-set induces a ring homomorphism

\[
\text{def}^H_Q : B(Q) \to B(H/Q), \quad [K, 1]_H \mapsto \begin{cases} [K/H, 1]_{Q/H}, & \text{if } H \leq K, \\ 0, & \text{if } H \nleq K, \end{cases}
\]

which we call deflation. Similarly, we have a deflation map

\[
\text{def}^H_Q : \hat{B}(Q) \to \hat{B}(Q/H), \quad (a_K)_{K \leq Q} \mapsto (a_K)_{K/H \leq Q/H}.
\]

It is straightforward to check that \( \rho_{Q/H} \circ \text{def}^H_Q = \text{def}^H_Q \circ \rho_Q \). Thus, we obtain ring homomorphisms

\[
\text{fix}^H_{Q,G} := \text{def}^H_Q \circ \text{res}_Q^G : B(G) \to B(H/Q),
\]

and

\[
\begin{align*}
\text{fix}^H_{Q,G} := \text{def}^H_Q \circ \text{res}_Q^G : \hat{B}(G) & \to \hat{B}(Q/H), \\
(a_K)_{K \leq G} & \mapsto (a_K)_{K/H \leq Q/H}.
\end{align*}
\]

(cf. (1.3.a)) such that

\[
\text{fix}^H_{Q,G} \circ \rho_G = \rho_{Q/H} \circ \text{fix}^H_{Q,G}.
\]
We call a section \((H, Q)\) of \(G\) a \textit{p-section}, if \(Q/H\) is a \(p\)-group, and we call it a \textit{maximal \(p\)-section} if for every \(p\)-section \((H', Q')\) of \(G\) with \(H' \leq H \leq Q \leq Q'\) one has \(H' = H\) and \(Q' = Q\).

### 2.7 Corollary
For \(f = (f_K)_{K \leq G} \in \hat{B}(G)\) the following are equivalent:

(i) The element \(f\) is in the image of \(\rho_G\).

(ii) The element \(\text{fix}^H_{Q/G}(f) \in \hat{B}(Q/H)\) is in the image of \(\rho_{Q/H}\) for all maximal \(p\)-sections \(H \leq Q \leq G\) of \(G\), where \(p\) is a prime divisor of \(G\).

**Proof** (i)\(\Rightarrow\)(ii): This follows immediately from (2.6.b).

(ii)\(\Rightarrow\)(i): We show that \(f \in \text{im}(\rho_G)\) using the criterion in Corollary 2.5(iii).

So let \((H, Q)\) be a \(p\)-section of \(G\) and let \((H', Q')\) be a maximal \(p\)-section of \(G\) with \(H' \leq H \leq Q \leq Q'\). By (ii) we know that \(f' := \text{fix}^H_{Q/G}(f) \in \text{im}(\rho_{Q/H})\) and (2.6.b) implies that

\[
f'' := \text{fix}^H_{Q/H.H/G}(f') \in \text{im}(\rho_{Q/H}(H/H)).\]

But \(f''\) can be viewed as an integer valued \(Q\)-invariant function on the set of subgroups \(K\) with \(H \leq K \leq Q\), and as such it is in the image of \(\rho_{Q/H}\). Moreover, by (2.6.a), the function \(f''\) is just the restriction of the function \(f\). Now, using Corollary 2.5(i)\(\Rightarrow\)(iv) applied to \(f''\), \(H/H\), and \(Q/H\), we obtain the required congruence. \(\square\)

### C. Application to Monomial Representation Rings

For the remainder of this section we assume that \(O\) is an integral domain and we define the algebra restriction functor \(A\) as in Example 1.7(b) with \(C := O^\times\), i.e., \(A(H) := \mathbb{Z}H\) with \(H := \text{Hom}(H, O^\times)\), for \(H \leq G\). We set \(D_O := A_0\) and \(D_O := A_+\). The ring \(D_O(G)\) is called the \textit{monomial representation ring} of \(G\) over \(O\), and \(\hat{D}_O(G) = (\prod_{K \leq G} \mathbb{Z}K)^G\) is called its ghost ring. Recall that we have the mark homomorphism \(\rho_H: D_O(H) \to D_O(H)\) for every \(H \leq G\). We will use the notations introduced in Definition 1.5 with respect to the stable basis \(\mathcal{B}_H := \hat{H}, H \leq G\).

The following corollary is again immediate from Theorem 2.2.

### 2.8 Corollary
For \(f = \left(\sum_{\psi \in K} f_{(K, \psi)} \cdot \psi\right)_{K \leq G} \in \hat{D}_O(G)\) the following are equivalent:

(i) The element \(f\) is in the image of \(\rho_G\).

(ii) The congruence

\[
\sum_{(H, \varphi) \leq (I, \psi) \in \mathcal{M}_G(H, \varphi)} \mu_{H, I} \cdot f_{(I, \psi)} \equiv 0 \mod [N_G(H, \varphi) : H]
\]

holds for all \((H, \varphi) \in \mathcal{M}_G\).
(iii) The congruence

$$ \sum_{(H, \varphi) \leq (I, \psi) \in \mathcal{M}_Q} \mu_{H, I} \cdot f_{(I, \psi)} \equiv 0 \mod [Q : H] $$

holds for all \((H, \varphi) \in \mathcal{M}_G\) and all Sylow subgroups \(Q/H\) of \(N_G(H, \varphi)/H\).

(iv) The congruence

$$ \sum_{(H, \varphi) \leq (I, \psi) \in \mathcal{M}_Q} \mu_{H, I} \cdot f_{(I, \psi)} \equiv 0 \mod [N_Q(H, \varphi) : H] $$

holds for all \((H, \varphi) \in \mathcal{M}_G\) and all \(H \leq Q \leq G\).

2.9 Our next goal is a detection result, analogous to Corollary 2.7, for the image of the ring homomorphism \(\rho_G : D_G(G) \to \hat{D}_G(G)\). Let \((H, \varphi) \in \mathcal{M}_G\). Since \(G\) is an integral domain, \(H/\ker(\varphi)\) is a cyclic group, and, for all \(H \leq Q \leq N_G(H, \varphi)\) such that \(Q/H\) is a \(p\)-group, the group \(Q/\ker(\varphi)\) is \(p\)-elementary, i.e., a direct product of a \(p\)-group and a cyclic \(p'\)-group. Recall that a group is called elementary if it is \(p\)-elementary for some prime \(p\).

For \((H, \varphi) \in \mathcal{M}_G\) and any \(H \leq Q \leq N_G(H, \varphi)\) we define the deflation maps

$$ \text{def}_Q^{(H, \varphi)} : D_G(Q) \to D_G(Q/\ker(\varphi)), $$

$$ [K, \psi]_Q \mapsto \begin{cases} [\hat{K}, \hat{\psi}]_Q & \text{if } (K, \psi) \geq (H, \varphi), \\ 0 & \text{if } (K, \psi) \not\geq (H, \varphi), \end{cases} $$

and

$$ \text{def}_Q^{(H, \varphi)} : D_G(Q) \to D_G(Q/\ker(\varphi)), $$

$$ \left( \sum_{\psi \in \hat{K}} f_{(K, \psi)} \cdot \psi \right)_{K \leq Q} \mapsto \left( \sum_{\psi \in \hat{K}} \tilde{f}_{(K, \psi)} \cdot \tilde{\psi} \right)_{K \leq Q}. $$

Here, \(\hat{K}\) denotes the image of an intermediate group \(\ker(\varphi) \leq K \leq N_G(H, \varphi)\) modulo \(\ker(\varphi)\) and \(\tilde{\psi} \in \hat{K}\) denotes the induced homomorphism of an element \(\psi \in \hat{K}\) with \(\psi|_{\ker(\varphi)} = 1\). The elements \(\tilde{f}_{(K, \psi)} \in \mathbb{Z}\) are defined by

$$ \tilde{f}_{(K, \psi)} := \begin{cases} f_{(KH, \psi^* \varphi)} & \text{if } \psi|_{K \cap H} = \varphi|_{K \cap H}, \\ 0 & \text{if } \psi|_{K \cap H} \neq \varphi|_{K \cap H}, \end{cases} $$

where, in the first case, \(\psi^* \varphi\) is defined as the unique homomorphism \(\lambda \in \text{Hom}(KH, G)\) with \(\lambda|_K = \psi\) and \(\lambda|_H = \varphi\). Note that

$$ \tilde{f}_{(K, \psi)} = f_{(K, \psi)} \quad (2.9, a) $$

for all \((K, \psi) \in \mathcal{M}_{N_G(H, \varphi)}\) with \((K, \psi) \geq (H, \varphi)\). It is straightforward to show that

$$ \rho_{Q/\ker(\varphi)} \circ \text{def}_Q^{(H, \varphi)} = \text{def}_Q^{(H, \varphi)} \circ \rho_Q. $$
Therefore, we obtain maps
\[ \text{fix}_{Q,G}^{(H,\varphi)} := \text{def}_{Q}^{(H,\varphi)} \circ \text{res}_{Q}^{G} : D_{G}(G) \to D_{G}(Q/\ker(\varphi)) \]
and
\[ \text{fix}_{Q,G}^{(H,\varphi)} := \text{def}_{Q}^{(H,\varphi)} \circ \text{res}_{Q}^{G} : \hat{D}_{G}(G) \to \hat{D}_{G}(Q/\ker(\varphi)) \]
satisfying
\[ \text{fix}_{Q,G}^{(H,\varphi)} \circ \rho_{G} = \rho_{Q/\ker(\varphi)} \circ \text{fix}_{Q,G}^{(H,\varphi)} . \tag{2.9.b} \]
Note that if \( f \in \hat{D}_{G}(G) \) is considered as \( G \)-invariant function \( f : M_{G} \to \mathbb{Z} \), and if \( f' := \text{fix}_{Q,G}^{(H,\varphi)}(f) \in \hat{D}(Q/\ker(\varphi)) \) is considered as \( Q \)-invariant function on \( \{(K,\psi) \in M_{Q} \mid (\ker(\varphi),1) \leq (K,\psi)\} \), then
\[ f|_{\tilde{M}} = f'|_{\tilde{M}}, \quad \text{with } \tilde{M} := \{(K,\psi) \in M_{Q} \mid (K,\psi) \geq (H,\varphi)\} , \tag{2.9.c} \]
by (1.3a) and (2.9a).

If \( \varphi = 1 \) we just write \( \text{def}_{Q}^{H} \) and \( \text{fix}_{Q,G}^{H} \) instead of \( \text{def}_{Q}^{(H,1)} \) and \( \text{fix}_{Q,G}^{(H,1)} \).

In order to state the following corollary we call a section \((H,Q) \) of \( G \) \( p \)-elementary if \( Q/H \) is \( p \)-elementary, and we call it maximally \( p \)-elementary if it is \( p \)-elementary and for every \( p \)-elementary section \((H',Q') \) of \( G \) with \( H' \leq H \leq Q \leq Q' \) on has \( H' = H \) and \( Q' = Q \). Similarly we define (maximal) elementary sections of \( G \).

**2.10 Corollary** For \( f = (\sum_{\psi \in K} f(K,\psi) \cdot \psi)_{K \in G} \in \hat{D}_{G}(G) \) the following are equivalent:

(i) The element \( f \) is in the image of \( \rho_{G} \).

(ii) The element \( \text{fix}_{Q,G}^{(H,\varphi)}(f) \) is in the image of \( \rho_{Q/\ker(\varphi)} \) for every \((H,\varphi) \in M_{G} \) and every Sylow \( p \)-subgroup \( Q/H \) of \( N_{G}(H,\varphi)/H \).

(iii) The element \( \text{fix}_{Q,G}^{H}(f) \) is in the image of \( \rho_{Q/H} \) for every maximal elementary section \((H,Q) \) of \( G \).  

**Proof** (i)\(\Rightarrow\)(ii): This follows immediately from (2.9.b) with \( \varphi = 1 \).

(ii)\(\Rightarrow\)(iii): Let \((H,\varphi) \in M_{G} \) and let \( Q/H \) be a Sylow \( p \)-subgroup of \( N_{G}(H,\varphi)/H \). Then \( Q/\ker(\varphi) \) is \( p \)-elementary and there exists a maximal elementary section \((H',Q') \) of \( G \) with \( H' \leq \ker(\varphi) \leq Q \leq Q' \). From (iii) we know that \( f' := \text{fix}_{Q,G}^{H}(f) \) lies in the image of \( \rho_{Q/H} \), and therefore, by (2.9.b), also
\[ f'' := \text{fix}_{Q/H,Q/H}^{(H'/H,\varphi)} (f') \in \text{im}(\rho_{Q/H}/(\ker(\varphi)/H)) , \]
where \( \varphi \in \text{Hom}(H/H',O^{\circ}) \) denotes the homomorphism induced by \( \varphi \). The element \( f'' \) can be viewed as an integer valued \( Q \)-invariant function on the set \( \{(K,\psi) \in M_{Q} \mid (K,\psi) \geq (\ker(\varphi),1)\} \) and as such it is in the image of \( \rho_{Q/\ker(\varphi)} \). Moreover, by (2.9.c), the functions \( f'' \) and \( f \) coincide on the set \( \{(K,\psi) \in M_{Q} \mid (K,\psi) \geq (H,\varphi)\} \). Now, we apply Corollary 2.8(i)\(\Rightarrow\)(iv) to obtain the required congruence in (ii).
(ii)→(i): We use Corollary 2.8(iii)→(i) in order to show that \( f \in \text{im}(\rho_Q) \). So let \((H, \varphi) \in \cM_G\) and let \(Q/H\) be a Sylow \(p\)-subgroup of \(N_G(H, \varphi)/H\). We know by (ii) that \( f' := \text{fix}_{Q, G}^H(f) \in \text{im}(\rho_Q/\ker(\varphi)) \), and by (2.9.c) that \( f' \) considered as \(Q\)-invariant function on \( \{(K, \psi) \in \cM_Q \mid (K, \psi) \geq (\ker(\varphi), 1)\} \) coincides with \( f \) on \( \cM \). Applying Corollary 2.8(i)→(iv) to \( f' \), \((H/\ker(\varphi), \tilde{\varphi})\) (with \( \tilde{\varphi} \) the homomorphism on \( H/\ker(\varphi) \) induced by \( \varphi \)), and \( Q/\ker(\varphi) \), we then obtain the required congruence.

\[ \square \]

3 Application to the Integrality of Canonical Induction Formulas

In this section we give a shortened and more conceptual integrality proof for the canonical induction formulas for the ring of linear source modules and the ring of trivial source modules of \( G \).

3.1 We first recall the definition of the canonical induction formulas mentioned above. For that purpose let \( \cO \) denote a complete discrete valuation ring with residue characteristic \( p > 0 \) and field of fractions \( \cK \) of characteristic 0. We assume that \( \cO \) contains a primitive root of unity whose order is the exponent of \( G \).

A **linear** (resp. **trivial** source \( \cO G \)-module \( V \) is an \( \cO G \)-module which is isomorphic to a direct summand of a monomial (resp. permutation) \( \cO G \)-module. Here, a monomial (resp. permutation) \( \cO G \)-module is an \( \cO G \)-module which is isomorphic to a finite direct sum of modules of the form \( \text{Ind}^G_H(\cO, \varphi) \) (resp. \( \text{Ind}^G_H(\cO) \)) with \( H \leq G \) and \( \varphi \in \hat{H} := \text{Hom}(H, \cO^*) = \text{Hom}(H, \cK^*) \). Here, \( \cO, \varphi \) denotes the \( \cO H \)-module with underlying \( \cO \)-module \( \cO \) and with \( H \)-action given by \( h \cdot x := \varphi(h)x \) for \( h \in \hat{H} \) and \( x \in \cO \). The Grothendieck ring \( L(G) \) of the semiring of isomorphism classes of linear source \( \cO G \)-modules with respect to direct sums and tensor products, is called the **linear source ring**. For a linear source module \( V \) we denote the corresponding element in \( L(G) \) by \([V]\). If \( V \) runs through a set of representatives of the isomorphism classes of indecomposable linear source \( \cO G \)-modules, then the elements \([V]\) form a finite \( \mathbb{Z} \)-basis of \( L(G) \).

The subgroups generated by the elements \([V]\), where \( V \) is an (indecomposable) trivial source (resp. projective) \( \cO G \)-module form the trivial source ring \( T(G) \) (resp. the ideal \( P(G) \)). We have \( P(G) \subseteq T(G) \subseteq L(G) \).

Finally, let \( R(G) \) denote the ring of \( \cK \)-characters of \( G \). The groups \( P(H), T(H), L(H), R(H), H \leq G \), are Mackey functors with the usual conjugation, restriction, and induction maps. The canonical induction formulas that we want to recall are all defined in a similar way and have values in \( D := D_{\mathcal{O}} \), cf. 2.C. We will also abbreviate \( D_{\mathcal{O}} \) by \( D \). For \( H \leq G \), for a linear source \( \mathcal{O}H \)-module \( V \), and for a pair \((K, \psi) \in \mathcal{M}_H \), let \( m_V(K, \psi) \) denote the multiplicity
of the $\mathcal{O}K$-module $\mathcal{O}_\psi$ as a direct summand in $\text{Res}^H_K(V)$ and define the map
\[r^H_H: L(H) \to \hat{D}(H), \quad [V] \mapsto \left( \sum_{\psi \in K} m_V(K, \psi) \cdot \psi \right)_{K \leq H}.\]

Obviously, $r^L: L \to \hat{D}$ is a morphism of restriction functors on $G$. Since the maps $\rho_H: \mathbb{Q} \otimes D(H) \to \mathbb{Q} \otimes \hat{D}(H)$, $H \leq G$, form an isomorphism of restriction functors (cf. 1.4), we obtain a morphism
\[a^L := \rho^{-1} \circ r^L: L \to \mathbb{Q} \otimes D\]
of restriction functors. This is the canonical induction formula for the linear source ring as introduced in [B98a]. It is called an induction formula, since it is a section of the $\mathbb{Q}$-tensored version of the maps
\[b_H: D(H) \to L(H), \quad [K, \psi]_H \mapsto \text{Ind}^H_K(\mathcal{O}_\psi),\]
for $H \leq G$.

The canonical induction formulas for $T$ and $P$ can be defined as the restrictions $a^T$ and $a^P$ of $a^L: L \to \mathbb{Q} \otimes D$, cf. [B98a, Prop. 6.1].

Similarly, the canonical Brauer induction formula $a^K: R \to \mathbb{Q} \otimes D$ was defined using
\[r^R_H: R(H) \to \hat{D}(H), \quad \chi \mapsto \left( \sum_{\psi \in K} m_\chi(K, \psi) \cdot \psi \right)_{K \leq H},\]
for $H \leq G$, with $m_\chi(K, \psi) := (\chi|_K, \psi)$, the multiplicity of the one-dimensional character $\psi$ as a constituent in $\text{res}^H_K(\chi)$. Again, the maps $a^K_H := \rho_H^{-1} \circ r^R_H: R(H) \to \mathbb{Q} \otimes D(H)$, $H \leq G$, form a morphism of restriction functors and a section of the $\mathbb{Q}$-tensored version of the maps
\[b_H: D(H) \to R(H), \quad [K, \psi]_H \mapsto \text{Ind}^H_K(\psi),\]
for $H \leq G$.

It was proved in [B98b] that $r^L_H(L(H)) \subseteq \rho_H(D(H))$ and in [B90] that $r^R_H(R(H)) \subseteq \rho_H(D(H))$ for all $H \leq G$ so that the canonical induction formulas $a^L$ and $a^K$ are integral:
\[a^L_H(L(H)) \subseteq D(H), \quad a^K_H(R(H)) \subseteq D(H),\]
for all $H \leq G$. Consequently, also the restrictions $a^T$ and $a^P$ of $a^L$ are integral. We present a new proof of the integrality of $a^L$ in the next subsection.

3.2 The goal of this subsection is to prove the statement
\[r^L_G([V]) \subseteq \rho_G(D(G))\] (3.2.a)
for all linear source $\mathcal{O}G$-modules $V$. Of course, by replacing $G$ with any subgroup $H$, we then also proved $r^L_H(L(H)) \subseteq \rho_H(D(H))$. 

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Let \((H, Q)\) be a section of \(G\) and let \(V\) be a linear source \(OG\)-module. Then, by [BK00, Thm. 3.3], there exists a decomposition \(V = W \oplus X\) into linear source \(OG\)-modules such that \(\text{Res}_H^G(W) = O \oplus \cdots \oplus O\) and such that \(\text{Res}_H^G(X)\) has no direct summand isomorphic to \(O\). We set \(\text{def}_Q^H([V]) := [W]\) and obtain a deflation map
\[
\text{def}_Q^H: L(H) \rightarrow L(H/Q)
\]
satisfying
\[
a_{Q/H} \circ \text{def}_Q^H = \text{def}_Q^H \circ a_H,
\]
by the definition of \(r_H^G\), cf. also 2.9. Composing this map with the restriction map we obtain a map
\[
\text{fix}_Q^H := \text{def}_Q^H \circ \text{res}_H^G: L(G) \rightarrow L(Q/H)
\]
satisfying
\[
a_{Q/H} \circ \text{fix}_Q^H = \text{fix}_Q^H \circ a_G.
\]
Using Corollary 2.10, it suffices now to prove (3.2.a) in the case that \(G\) is a \(q\)-elementary group \(G = C \times Q\) with a \(q\)-group \(Q\) and a cyclic \(q'\)-group \(C\). We assume from now on that \(G = C \times Q\).

Note that every indecomposable linear source \(OG\)-module \(V\) is of the form \(O \lambda \otimes W\) with \(\lambda \in \hat{G}\) such that \(\lambda|_Q = 1\) and \(W\) an indecomposable linear source \(OG\)-module such that \(\text{Res}_H^G(W) = O \oplus \cdots \oplus O\). It is easy to see, by applying \(\rho_G\), that \(a_G([O \lambda \otimes W]) = [G, \lambda|_G \cdot a_G([W])].\) Therefore, it suffices to show that \(a_G([W]) \in D(G)\). But \(W\) is the image \(\inf_{G/C}^G(W)\) for some indecomposable linear source \(OG/C\)-module \(W\) under the inflation map
\[
\inf_{G/C}^G: L(G/C) \rightarrow L(G).
\]

There are also inflation maps
\[
\inf_{G/C}^G: D(G/C) \rightarrow D(G), \quad [H/C, \varphi]_{G/C} \mapsto [H, \varphi]_G;
\]
and
\[
\inf_{G/C}^G: \hat{D}(G/C) \rightarrow \hat{D}(G),
\]
\[
\left( \sum_{\tilde{v} \in K/C} \tilde{f}(K/C, \tilde{v}) \cdot \tilde{v} \right)_{K/C \leq G/C} \mapsto \left( \sum_{\psi \in \hat{K}} \tilde{f}(K, \psi) \cdot \psi \right)_{K \leq G},
\]
with
\[
\tilde{f}(K, \psi) := \begin{cases} 
\tilde{f}(K/C, \tilde{v}), & \text{if } \psi|_{K \cap C} = 1, \\
0, & \text{if } \psi|_{K \cap C} \neq 1.
\end{cases}
\]

It is easy to verify that the maps \(r_H^G\) and \(\rho_G\) commute with the respective inflation maps so that also
\[
a_G \circ \inf_{G/C}^G = \inf_{G/C}^G \circ a_G/C.
\]
holds. Therefore, it suffices to prove (3.2.a) in the case that $G$ is a $q$-group which we assume from now on. We also assume that $V$ is an indecomposable linear source $O_G$-module. We have to distinguish two cases.

If $q = p$, then $V \cong \text{Ind}_{H}^{G}(O_{\varphi})$ for some $(H, \varphi) \in \mathcal{M}_G$, by Green’s indecomposability theorem. Then one can verify immediately that

$$a_G([V]) = [H, \varphi]_G$$

by applying $\rho_G$ to both sides. This implies that (3.2.a) holds in this case.

If $q \neq p$, then tensoring with $\mathcal{K}$ over $\mathcal{O}$ induces an isomorphism $L(G) \rightarrow R(G)$ which commutes with the maps $r_G^L: L(G) \rightarrow D(G)$ and $r_G^R: R(G) \rightarrow D(G)$. Thus, (3.2.a) holds for all linear source $O_G$-modules $V$ if and only if $r_G^R([V]) \in \rho_G(D(G))$ hold for all $\mathcal{K}G$-modules $V$. This way the integrality of $a_G^R$ is reduced to the integrality of $a_G^L$ in the case where $G$ is a $p$-group. Now one can proceed as in [B90].

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References


