Linear Source Modules and Trivial Source Modules*

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Abstract
We consider the Grothendieck ring $L_0(G)$ of the category of linear source $OG$-modules with respect to direct sums, where $O$ is a complete discrete valuation ring of characteristic zero containing enough roots of unity and having a residue field of prime characteristic. We show that $L_0(G)$ is semisimple after tensoring with the field of fractions of $O$, and we determine all its species. Moreover, we show that there is a canonical integral induction formula for $L_0(G)$ inducing only lattices of rank 1. As a consequence we obtain similar statements for the ring of trivial source $OG$-modules.

Introduction

The representation theory of trivial source modules for a finite group $G$ over a complete discrete valuation ring $O$ of characteristic zero which is large enough for $G$ and has a residue field $F$ of characteristic $l > 0$ is well understood, see for instance [3]. The representation ring $T_0(G)$ of trivial source $OG$-modules is a subring of the Green ring $A_0(G)$ and is finitely generated as an abelian group. The study of $T_0(G)$ has had an enormous impact on the understanding of the ring $A_0(G)$. If $K$ denotes the field of fractions of $O$, then it is well-known that $K \otimes T_0(G)$ is a semisimple $K$-algebra, and its species, i.e., the $K$-algebra maps $K \otimes T_0(G) \to K$, are completely determined. But if one wishes to interpret the representation theory of $OG$ as a link between those of $FG$ and $KG$, there is a slight drawback to the ring $T_0(G)$. While the reduction modulo the radical of $O$ induces a surjective ring homomorphism $T_0(G) \to R_F(G)$ onto the ring of Brauer characters of $G$, the canonical map $T_0(G) \to R_K(G)$ to the character ring of $G$ is not surjective.

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In this paper we introduce the notion of a linear source $O\Gamma$-module as a slight generalization of a trivial source $O\Gamma$-module. The representation ring $L_\Gamma(G)$ of these modules is still finitely generated as an abelian group, it covers the character ring by tensoring such modules with $K$, and the $K$-algebra $K \otimes L_\Gamma(G)$ is semisimple. Moreover, using canonical induction formulae for $R_K(G)$ and $R_F(G)$ one can give canonical sections for the surjective maps $L_\Gamma(G) \to R_K(G)$ and $L_\Gamma(G) \to R_F(G)$.

In Section 1 we introduce the category of linear source $O\Gamma$-modules and prove some elementary facts about them. Section 2 is devoted to the semisimplicity of $K \otimes L_\Gamma(G)$. We also give a complete description of its species. Moreover, it is shown that two linear source $O\Gamma$-modules are isomorphic, if and only if their restrictions to $l$-hypo-elementary subgroups contain the same direct summands of rank 1 with the same multiplicities. In Section 3 we recall the notion of a canonical induction formula and show in Section 4 that there are canonical induction formulae for $T_\Gamma(G)$ and $L_\Gamma(G)$ using trivial source modules and linear source modules of rank 1 respectively, as already announced in [2].

I would like to thank B. Külshammer for his suggestion to study the trivial source ring and the trivial source modules of rank 1 as a potential example for a canonical induction formula.

**Notation** For a finite group $G$ we denote by $G'$ its derived subgroup, by $G_\pi$ the set of elements of $\pi$-order in $G$, and by $O_\pi(G)$ the biggest normal $\pi$-subgroup of $G$, where $\pi$ can be any set of primes. For a field $K$, the Grothendieck ring of $KG$-modules is denoted by $R_K(G)$. The notation $H < G$ indicates that $H$ is a proper subgroup of $G$. For $H \leq G$ and $g \in G$ we set $gH := gHg^{-1}$. If $k$ is a commutative ring and $V$ a $kG$-module, we denote by $\ker(V)$ the normal subgroup of $G$ consisting of all elements of $G$ which act as the identity on $V$. If $W$ is another $kG$-module, we write $W | V$ to indicate that $W$ is isomorphic to a direct summand of $V$.

## 1 Linear source modules

Throughout this paper $G$ denotes a finite group and $O$ denotes a complete discrete valuation ring with residue field $F$ of characteristic $l > 0$ and field of fractions $K$ of characteristic 0 which is large enough for $G$, i.e. contains a primitive $e$-th root of unity where $e$ is the exponent of $G$. Moreover, all other finite groups occurring in this paper are implicitly assumed to have exponent dividing $e$ so that $K$ is always a splitting field for them.

By $O\Gamma$-**lat** we denote the category of $O\Gamma$-lattices, i.e. $O\Gamma$-modules $M$ which are free as $O$-modules and of finite $O$-rank $\text{rk}_OM$. Let $A_\Gamma(G)$ be the free abelian group on the set of isomorphism classes $\{M\}$ of objects $M \in O\Gamma$-**lat** modulo the subgroup generated by the elements $\{M \oplus N\} - \{M\} - \{N\}$ for $M, N \in O\Gamma$-**lat**. By the Krull-Schmidt-Azumaya Theorem one can identify $A_\Gamma(G)$ with the free abelian group on the set of isomorphism classes of indecomposable $O\Gamma$-lattices. The element in $A_\Gamma(G)$ associated to $V \in O\Gamma$-**lat** will be denoted by $[V]$. The ring structure of $A_\Gamma(G)$ is induced by the tensor product $V \otimes_O W$ with the diagonal $G$-action for $V, W \in O\Gamma$-**lat**. In the literature, the ring $A_\Gamma(G)$ is also called the **Green**
ring of $OG$. The rings $A_O(H)$, $H \leq G$, form a Green functor on $G$, i.e. there are conjugation, restriction, and induction maps
\[
c_{g,H} : A_O(H) \to A_O(gH),
\]
\[
\text{res}^H_U : A_O(H) \to A_O(U),
\]
\[
\text{ind}^H_U : A_O(U) \to A_O(H),
\]
for $U \leq H \leq G$ and $g \in G$, satisfying certain axioms, cf. [2] for a definition of a Green functor in the form convenient for our purposes. For $H \leq G$, $g \in G$, and $V \in \mathcal{O}H\text{–lat}$ we define $gV \in \mathcal{O}gH\text{–lat}$ to be the $O$-module $V$ whose $H$-action $H \to \text{End}_O(V)$ is replaced by the $gH$-action $gH \to H \to \text{End}_O(V)$, where $gH \to H$ is the conjugation map $x \mapsto g^{-1}xg$ for $x \in gH$.

The category $OG\text{–lat}$ has as a full subcategory the category $OG\text{–per}$ of permutation $OG$-modules, i.e. $OG$-modules which have a $G$-stable finite $O$-basis. Note that the class of permutation modules is stable under conjugation, restriction, induction, and taking direct sums and tensor products. Thus, the span of the images in $A_O(H)$ of objects from $OH\text{–per}$, $H \leq G$, form a Green subfunctor $P_O \subseteq A_O$, in fact the smallest Green subfunctor of $A_O$.

There is a well-known full subcategory $OG\text{–triv}$ of $OG\text{–lat}$ which contains $OG\text{–per}$ and consists of those $OG$-lattices all of whose indecomposable direct summands have a trivial source. The objects of $OG\text{–triv}$ are called trivial source $OG$-modules. We summarize some of their basic properties in the following proposition, see for instance [3] for a proof of it.

1.1 Proposition

(a) For $V \in OG\text{–lat}$ and a Sylow $l$-subgroup $P$ of $G$ the following are equivalent:

(i) One has $V \in OG\text{–triv}$.

(ii) The module $\text{res}^P_G(V)$ is a permutation $OP$-module.

(iii) The module $V$ is isomorphic to a direct summand of a permutation $OG$-module.

(b) For $H \leq G$, $V, V' \in OG\text{–triv}$, and $W \in OH\text{–triv}$ one has:

(i) The $OG$-lattices $V \oplus V'$, $V \otimes_O V'$, and $V^* = \text{Hom}_O(V, O)$ are again trivial source $OG$-modules.

(ii) The $OH$-lattice $\text{res}^G_H(V)$ is a trivial source $OH$-module.

(iii) The $OG$-lattice $\text{ind}^G_H(W)$ is a trivial source $OG$-module.

(iv) Every direct summand of $V$ is a trivial source $OG$-module.

The images in $A_O(G)$ of objects from $OG\text{–triv}$ span a subring $T_O(G)$ of $A_O(G)$ which is the free abelian group on the set of isomorphism classes of indecomposable trivial source $OG$-modules. Note that in contrast to $A_O(G)$ the group $T_O(G)$ is
always finitely generated. In view of property (ii) in Proposition 1.1 (a), the objects in $\mathcal{O}G-\text{triv}$ are also called $I$-permutation $\mathcal{O}G$-modules.

We introduce the category $\mathcal{O}G-\text{lin}$ of \textbf{linear source $\mathcal{O}G$-modules} as the full subcategory of $\mathcal{O}G-\text{lat}$ consisting of those $\mathcal{O}G$-lattices all of whose indecomposable direct summands have a source of $I$-rank 1. Hence, one has inclusions

$$\mathcal{O}G-\text{per} \subseteq \mathcal{O}G-\text{triv} \subseteq \mathcal{O}G-\text{lin} \subseteq \mathcal{O}G-\text{lat}$$

of full subcategories.

An $\mathcal{O}G$-module $V$ is called \textbf{monomial}, if $V$ is a direct sum of $\mathcal{O}G$-lattices isomorphic to $\mathcal{O}G$-lattices of the form $\text{ind}_H^G(W)$ for a subgroup $H \leq G$ and an $\mathcal{O}H$-lattice $W$ of $I$-rank 1. Similar to Proposition 1.1 one has the following for linear source modules.

### 1.2 Proposition

(a) For $V \in \mathcal{O}G-\text{lat}$ and a Sylow $I$-subgroup $P$ of $G$ the following are equivalent:

(i) One has $V \in \mathcal{O}G-\text{lin}$.

(ii) The module $\text{res}_P^G(V)$ is a monomial $\mathcal{O}P$-module.

(iii) The module $V$ is isomorphic to a direct summand of a monomial $\mathcal{O}G$-module.

(b) For $H \leq G$, $V, V' \in \mathcal{O}G-\text{lin}$, and $W \in \mathcal{O}H-\text{lin}$ one has:

(i) The $\mathcal{O}G$-lattices $V \oplus V'$, $V \otimes \mathcal{O} V'$, and $V^\ast = \text{Hom}_\mathcal{O}(V, \mathcal{O})$ are again linear source $\mathcal{O}G$-modules.

(ii) The $\mathcal{O}H$-lattice $\text{res}_H^G(V)$ is a linear source $\mathcal{O}H$-module.

(iii) The $\mathcal{O}G$-lattice $\text{ind}_H^G(W)$ is a linear source $\mathcal{O}G$-module.

(iv) Every direct summand of $V$ is a linear source $\mathcal{O}G$-module.

**Proof**  (a) It suffices to show the equivalences for an indecomposable $\mathcal{O}G$-lattice $V$. Let $H \leq G$ be a vertex of $V$ and let $S \in \mathcal{O}H-\text{lat}$ be a source of $V$. Then $V | \text{ind}_H^G(S)$ and this shows that (i) implies (iii). If $V$ is a direct summand of a monomial $\mathcal{O}G$-module, then $\text{res}_P^G(V)$ is a direct summand of a monomial $\mathcal{O}P$-module by Mackey’s theorem. But by Green’s indecomposability theorem, $\text{ind}_H^G(W)$ is indecomposable for each $Q \leq P$ and each $W \in \mathcal{O}Q-\text{lat}$ with $\text{rk}_\mathcal{O}(W) = 1$. This shows that $\text{res}_H^G(V)$ is monomial, and that (iii) implies (ii). If $\text{res}_P^G(V)$ is monomial, then also $\text{ind}_P^G(\text{res}_P^G(V))$ is monomial, and contains $V$ (up to isomorphism) as a direct summand, since $V$ is relatively $P$-projective. Hence, $V | \text{ind}_H^G(W)$ for some $Q \leq P$ and some $W \in \mathcal{O}Q-\text{lat}$ with $\text{rk}_\mathcal{O}(W) = 1$. Since $S$ is a source of $V$, we obtain $S | \text{res}_H^G(V) | \text{res}_H^G(\text{ind}_Q^G(W))$. Now, Mackey’s theorem shows that $S | \text{ind}_H^G(W')$ for some $Q' \leq H$ and some $W' \in \mathcal{O}Q-\text{lat}$ with $\text{rk}_\mathcal{O}(W') = 1$. But since $S$ (as a source) has vertex $H$, this implies $Q' = H$ and $S = W'$, and we have shown that (ii) implies (i).
(b) Part (iv) holds by the very definition of $OG$–lin, and the parts (i), (ii), and (iii) follow from (a) (iii), since the class of monomial modules is stable under the constructions in (i), (ii), and (iii). 

The span $L_0(G)$ of the images in $A_0(G)$ of objects from $OG$–lin is the free abelian group on the set of isomorphism classes of indecomposable linear source $OG$-modules, and one obtains subring inclusions

$$P_0(G) \subseteq T_0(G) \subseteq L_0(G) \subseteq A_0(G)$$

which induce inclusions $P_0 \subseteq T_0 \subseteq L_0 \subseteq A_0$ of Green subfunctors on $G$. Note that, since for a given subgroup $H$ of $G$ there are only finitely many $OH$-lattices of $O$-rank 1 (up to isomorphism), the group $L_0(G)$ is still finitely generated. Note also that each $OG$-lattice $W$ of $O$-rank 1 is isomorphic to $O_\varphi$ for some $\varphi \in \Hom(G, O^\times)$, where $O_\varphi = O$ as $O$-module and $g \cdot \alpha = \varphi(g)\alpha$ defines the $G$-action for $\alpha \in O$ and $g \in G$. If $\varphi = 1$ we write just $O$ instead of $O_1$. Obviously this induces a bijection between the multiplicative group $\tilde{G} := \Hom(G, O^\times)$ and the set of isomorphism classes of $OG$-lattices of rank 1 with $O_\varphi \cong O_\psi \otimes_O O_\varphi$ for $\varphi, \psi \in \tilde{G}$. It is easy to see that, for $\varphi \in \tilde{G}$, each Sylow $l$-subgroup $P$ is a vertex of $O_\varphi$ and $O_{\varphi|_P}$ is a source of $O_\varphi$. This implies that $O_\varphi$ is always a linear source module, and that $O_\varphi$ is a trivial source module, if and only if $\varphi|_P = 1$ which is equivalent to $\varphi$ lying in $\tilde{G}_l$, the $l'$-part of the abelian group $\tilde{G}$.

A finite group $H$ is called $l$-hypo-elementary, if $H/O_l(H)$ is cyclic. The $l$-hypo-elementary groups play an important role for the Green functors $P_0$, $T_0$, $L_0$, and $A_0$ as will be explained in the next section.

We will have to consider the set $\mathcal{M}^l(G)$ of pairs $(H, \varphi)$ with $H \leq G$ and $\varphi \in \tilde{H}$. Note that $\mathcal{M}^l(G)$ is a partially ordered set (poset) by defining $(U, \psi) \leq (H, \varphi)$, if and only if $U \leq H$ and $\varphi|_U = \psi$ for $(U, \psi), (H, \varphi) \in \mathcal{M}^l(G)$, and that $G$ acts on $\mathcal{M}^l(G)$ by conjugation via poset automorphisms. For $(H, \varphi) \in \mathcal{M}^l(G)$ we denote by $N_G(H, \varphi)$ the stabilizer of $(H, \varphi)$ in $G$.

1.3 Proposition Let $H$ be a finite $l$-hypo-elementary group with Sylow $l$-subgroup $P$.

(a) An indecomposable trivial source $OH$-module has vertex $P$, if and only if it is isomorphic to $O_\varphi$ for some $\varphi \in \tilde{H}$ with $\varphi|_P = 1$.

(b) If $V \in OH$–lin is indecomposable with vertex $P$ and source $O_\psi, \psi \in \tilde{P}$, then $V \cong \text{ind}_U^H(O_\psi)$, where $U = N_H(P, \psi)$ and $\varphi \in \tilde{U}$ is an extension of $\psi$. Moreover, if $V \cong \text{ind}_U^H(O_\psi')$ for some $(U, \varphi') \in \mathcal{M}^l(H)$, then $(U, \varphi)$ and $(U', \varphi')$ are $H$-conjugate. Conversely, if $\psi \in \tilde{P}$ and $U := N_H(P, \psi)$, then $\psi$ has $n = [U : P]$ extensions $\varphi, \ldots, \varphi_n \in \tilde{U}$ and $V_i := \text{ind}_U^H(O_{\varphi_i})$ is indecomposable with vertex $P$ and source $O_\psi$ for each $i = 1, \ldots, n$. Furthermore, $V_1, \ldots, V_n$ are pairwise non-isomorphic.
**Proof** (a) This follows from the decomposition

\[ \text{ind}^H_P(O) \cong \bigoplus_{\varphi \in H} O_{\varphi}. \]

(b) Let \( \psi \in \hat{P} \) and \( U := N_G(P, \psi) \). Then \( \psi \) can be extended to \( U \), since \( U/P \) centralizes \( P/\ker(\psi) \) so that \( U/\ker(\psi) \) is abelian. Moreover, there are exactly \( n = [U : P] \) such extensions \( \varphi_1, \ldots, \varphi_n \in U \) and

\[ \text{ind}^H_P(O_{\varphi}) \cong \bigoplus_{i=1}^n O_{\varphi_i}. \tag{1.3.a} \]

The module \( V_i = \text{ind}^H_P(O_{\varphi}), i = 1, \ldots, n \), is indecomposable, since \( \text{ind}^H_P(K_{\varphi_i}) \) is an irreducible \( KH \)-module by Clifford theory. By Mackey's theorem one has

\[ \text{res}^H_P(V_i) \cong \bigoplus_{h \in H/U} hO_{\varphi}. \tag{1.3.b} \]

If \( Q \leq P \) is a vertex and \( S \in \mathcal{O}Q-\text{lat} \) is a source of \( V_i \), then \( V_i \mid \text{ind}^H_Q(S) \) and therefore \( \text{res}^H_P(\text{ind}^H_Q(S)) \). Now Mackey's theorem and Green's indecomposability theorem imply \( O_{\varphi} = \text{ind}^H_P\mathcal{O}(S') \) for some \( h \in H \) and some \( \mathcal{O}(P \cap hQ) \)-module \( S' \), hence \( P = Q \) and \( O_{\varphi} \) is a source of \( V_i \) by (1.3.b). Moreover, \( V_1, \ldots, V_n \) are pairwise non-isomorphic, since \( K \otimes O V_i, i = 1, \ldots, n \), are pairwise non-isomorphic by classical Clifford theory.

Now if \( V \in \mathcal{O}H-\text{lin} \) is indecomposable with vertex \( P \) and source \( O_{\varphi} \), then

\[ V \mid \text{ind}^H_P(O_{\varphi}) \cong \bigoplus_{i=1}^n \text{ind}(O_{\varphi_i}) = \bigoplus_{i=1}^n V_i \]

by (1.3.a), so that \( V \cong \text{ind}^H_P(O_{\varphi_i}) \) for some \( i = 1, \ldots, n \). If additionally \( V \cong \text{ind}^H_P(O_{\varphi'}) \) for any pair \( (U', \varphi') \in \mathcal{M}'(H) \), then by considering \( O \)-ranks we get \( U = U' \), and by restricting \( V \) to \( U \) we obtain \( \varphi' = h\varphi_i \) for some \( h \in H \).

By \( A_{O}(G) \) (resp. \( A_{O}^a(G) \)) we define the span of the isomorphism classes \([V]\) of indecomposable \( OG \)-lattices \( V \) whose vertices are the Sylow \( l \)-subgroups (resp. are not the Sylow \( l \)-subgroups) of \( G \) and by \( A_{O}^b(G) \) we denote the span of the elements \([O_{\varphi}], \varphi \in \hat{G} \). Then,

\[ A_O(G) = A'^O_{O}(G) \oplus A'^b_{O}(G) \]

and \( A_O(G) \) splits further into \( A^a_{O}(G) \) and the span of the isomorphism classes of indecomposable \( OG \)-lattices whose vertices are the Sylow \( l \)-subgroups of \( G \), but which are not of \( O \)-rank 1. According to those decompositions we obtain projections

\[ p_{O}^{a}: A_O(G) \xrightarrow{\text{q}_{O}^{a}} A'^O_{O}(G) \xrightarrow{\text{t}_{O}^{a}} A'^a_{O}(G). \]
Similarly we define decompositions

\[ L_0(G) = L'_0(G) \oplus L^0_0(G) \quad \text{and} \quad T_0(G) = T'_0(G) \oplus T^0_0(G) \]

by intersecting \( A_0(G) \) and \( A^0_0(G) \) with \( L_0(G) \) and \( T_0(G) \). Moreover, we define

\[ L^{ab}_0 := A_0^{ab} \subseteq L_0(G) \quad \text{and} \quad T^{ab}_0 := A_0^{ab} \cap T_0(G) \subseteq T_0(G). \]

Then \( L^{ab}_0(G) \) is generated by \( [O_\varphi], \varphi \in \hat{G}, \) and \( T^{ab}_0(G) \) by \( [O_\varphi], \varphi \in \hat{G} \). By restriction we obtain projection maps

\[
\begin{align*}
p^G_0 : & \quad A_0(G) \xrightarrow{q^G_0} A'_0(G) \xrightarrow{t^G_0} A_0^{ab}(G) \\
p^G_0' & \quad L_0(G) \xrightarrow{q^G_0'} L'_0(G) \xrightarrow{t^G_0'} L_0^{ab}(G) \\
p^G_0'' & \quad T_0(G) \xrightarrow{q^G_0''} T'_0(G) \xrightarrow{t^G_0''} T_0^{ab}(G). 
\end{align*}
\]

Note that in general \( A'_0(G), L'_0(G), \) and \( T'_0(G) \) are ideals in \( A_0(G), \) \( L_0(G), \) and \( T_0(G) \) respectively, and that \( A_0^{ab}, L_0^{ab}, \) and \( T_0^{ab} \) are subrings of \( A_0(G) \).

### 1.4 Corollary

Let \( H \) be \( l \)-hypo-elementary.

(a) One has \( T'_0(H) = T^{ab}_0(H) \), the subgroups \( L'_0(H) \) and \( T'_0(H) \) are subrings of \( A_0(H) \), and the projection maps \( q^H_0 : L_0(H) \to L'_0(H) \) and \( q^H_0'' : T_0(H) \to T'_0(H) \) are ring homomorphisms.

(b) One has isomorphisms \( L'_0(H) \cong L_0'(H/O_1(H)') \cong R_K(H/O_1(H)') \) induced by inflation and by tensoring with \( K \) over \( O \).

(c) For each \( U \leq H \) with \( O_1(H) \leq U \) one has \( q^H_0 \circ \text{res}_U^H = \text{res}_U^H \circ q^H_0 \).

**Proof**

(a) The equation \( T'_0(H) = T^{ab}_0(H) \) follows from Proposition 1.3 (a). Then with \( T^{ab}_0(H) \), also \( T'_0(H) \) is a ring. From the explicit description of indecomposable linear source \( O \)-modules with vertex \( O_1(H) \) in Proposition 1.3 (b) one can easily see that the tensor product of two such modules is isomorphic to a direct sum of modules of the form \( \text{ind}_0^H(O_\varphi) \) with \( O_1(H) \leq U \leq H \) and \( \varphi \in \hat{U} \). Moreover, each indecomposable direct summand \( V \) of \( \text{ind}_0^H(O_\varphi) \) has vertex \( O_1(H) \). In fact, if \( Q \) is a vertex and \( S \in OQ-\text{lat} \) is a source of \( V \), then \( V / \text{ind}_0^H(S) \) and \( V / \text{ind}_0^H(O_\varphi) \), and restriction to \( P = O_1(H) \) yields \( ^hO_\varphi|_P \cong \text{ind}_P^H,^q_\varphi(S') \) for some \( h, k \in H \) and \( S' \in O(P \cap kQ)-\text{lat} \), which implies \( P = Q \).

Since \( L'_0(H) \) and \( T'_0(H) \) are subrings, the projection onto them is a ring homomorphism.

(b) Obviously \( O_1(H)' \) acts trivially on every indecomposable \( V \in O-\text{lin} \) with vertex \( O_1(H) \). Therefore, the inflation map \( L'_0(H/O_1(H)') \to L'_0(H) \) is an isomorphism. Moreover, by the explicit description of the indecomposable linear source \( H/O_1(H)' \)-modules with vertex \( O_1(H)/O_1(H)' \) one can easily check that tensoring with \( K \) over \( O \) sets them into bijective correspondence to the irreducible characters of \( H/O_1(H)' \).
(c) Let $O_l(H) \leq U \leq H$ and let $V \in \mathcal{O}_l^\text{lin}$ be indecomposable. If $V$ has vertex $O_l(H)$, so have all indecomposable direct summands of $\text{res}^H_{O_l(H)}(V)$ by the explicit description of $V$. If $V$ has vertex strictly smaller than $O_l(H)$, so have all indecomposable direct summands by general considerations.

If $G$ is an $l$-elementary group, i.e. if $G \cong P \times C$ for an $l$-group $P$ and a cyclic group $C$, then the following proposition extends Proposition 1.3 to a complete overview of all indecomposable linear source $\mathcal{O}G$-modules.

1.5 Proposition Let $G = P \times C$ be $l$-elementary with Sylow $l$-subgroup $P$. Then for each indecomposable module $V \in \mathcal{O}G^\text{lin}$ there is a pair $(H, \varphi) \in \mathcal{M}^l(G)$, unique up to $G$-conjugacy, such that $V = \text{ind}^H_P(\mathcal{O}_\varphi)$. Moreover, $H$ contains $C$ and $\text{rk}_G(V)$ is an $l$-power. Conversely, if $(H, \varphi) \in \mathcal{M}^l(G)$ with $C \leq H$, then $V := \text{ind}^H_P(\mathcal{O}_{\varphi})$ is an indecomposable linear source $\mathcal{O}G$-module with vertex $Q := H \cap P$ and source $\mathcal{O}_{\varphi Q}$.

Proof First let $V \in \mathcal{O}G^\text{lin}$ be indecomposable with vertex $Q \leq P$ and source $\mathcal{O}_\psi, \psi \in Q$. Then, with $H := Q C$, $\text{ind}_Q^H(\mathcal{O}_\psi)$ is the direct sum of all $\mathcal{O}_\psi$ with $\psi \in H$ extending $\psi$, and we have

$$V | \text{ind}_Q^H(\mathcal{O}_\psi) \cong \bigoplus_{\varphi \in H, \psi | Q = \psi} \text{ind}_H^Q(\mathcal{O}_\varphi).$$

Since $\text{ind}_H^Q(\mathcal{O}_\varphi)$ is indecomposable for all $\varphi \in H$ by Green's indecomposability theorem, this implies $V = \text{ind}_H^Q(\mathcal{O}_\varphi)$ for some $\varphi \in H$ with $\varphi | Q = \psi$. If, at the same time, $V = \text{ind}_H^Q(\mathcal{O}_{\varphi'})$ for some $(H', \varphi') \in \mathcal{M}^l(G)$, then $[G : H'] = \text{rk}_G(V) = [G : H]$ is an $l$-power, so $H'$ contains $C$. We have

$$\mathcal{O}_\psi | \text{res}_H^G(\text{ind}_H^Q(\mathcal{O}_\varphi)) \cong \text{res}_H^G(\text{ind}_{H'}^Q(\mathcal{O}_{\varphi'})) \cong \bigoplus_{g \in H \cap G / H'} \text{ind}_{H'|H \cap G/H'}^H(\mathcal{O}_{(g \cdot \varphi) |_{H \cap G / H'}}),$$

and since $C \leq H \cap g H'$ for all $g \in G$, Green's indecomposability theorem yields that $\mathcal{O}_{\varphi'}$ is isomorphic to one of the last summands above. This shows that $H \cap g H' = H$ and $g \varphi \in H \cap g H' = \varphi$ for some $g \in G$, which implies $(H, \varphi) = (g H', \varphi')$.

Finally, if $(H, \varphi) \in \mathcal{M}^l(G)$ with $C \leq H$, then $V := \text{ind}_H^Q(\mathcal{O}_\varphi)$ is indecomposable by Green's indecomposability theorem, and the statement about the vertex and source of $V$ follows from the discussion in the first part of the proof and the unicity of $(H, \varphi)$.

2 Detection theorems and species

2.1 For any Green functor $A$ on $G$, a subgroup $H$ of $G$ is called coprimordial for $A$, if

$$\bigcap_{U \leq H} \ker(\text{res}_U^H : A(H) \to A(U)) \neq \{0\}.$$
We denote the set of coprimordial subgroups for $A$ by $C(A)$. If $A(H)$ is a torsion free abelian group for all $H \leq G$, then obviously $C(A) = C(K' \otimes A)$ for any field $K'$ of characteristic zero. Moreover, if $A \subseteq A'$ is an inclusion of Green functors on $G$ and $A'(H)$ is also torsion free for all $H \leq G$, then $C(A) = C(K' \otimes A) = C(K' \otimes A') = C(A')$ for all fields $K'$ of characteristic zero (see for instance [2, Proposition 6.2]).

It is well-known that the coprimorial subgroups for $\mathbb{Q} \otimes T_O$ are just the $l$-hypoelementary subgroups of $G$. In fact, if $H \leq G$ is coprimorial for $\mathbb{Q} \otimes T_O$, then $H$ is $l$-hypoelementary by Conlon's induction theorem, cf. [4, Cor. 80.61]. Conversely, if $H \leq G$ is $l$-hypoelementary, then one can produce an element $x \in \mathbb{Q} \otimes T_O(H)$ with $\text{res}_{H}^G(x) = 0$ for all $U < H$, by taking the image $x$ of the primitive idempotent $\varepsilon_H \in \mathbb{Q} \otimes \Omega(H)$, whose marks are zero for proper subgroups of $H$ and $1$ for $H$, under the canonical map $\mathbb{Q} \otimes \Omega(H) \to \mathbb{Q} \otimes T_O(H)$ (where $\Omega(H)$ denotes the Burnside ring of $H$), and by applying [4, Lemma 81.28].

If $\mathcal{H} = \mathcal{H}_G$ denotes the set of $l$-hypoelementary subgroups of $G$, then the above discussion shows that

$$C(A) = C(\mathbb{Q} \otimes A) = C(K \otimes A) = \mathcal{H}$$


2.2 Let $A$ be a Green subfunctor of $A_O$. A $K$-species of $A(G)$ is a ring homomorphism $s: A(G) \to K$. Note that if $t: A(H) \to K$ is a $K$-species of $A(H)$ for some $H \leq G$, then $t \circ \text{res}_{H}^G$ is a $K$-species of $A(G)$. For any element $g \in G$ we have a $K$-species $s_g: A(O) \to K$ by defining $s_g([V])$ for $V \in \mathcal{O}_G-$lat as the value of the character of the $KG$-module $K \otimes K V$ at $g$.

For each $H \in \mathcal{H}$ and $h \in H$ we may define a species $s_{(H,h)}$ on $L_O(G)$ by

$$s_{(H,h)} := s_h \circ q_H^T \circ \text{res}_H^G: L_O(G) \to L_O(H) \to L_O(H) \to K,$$

since $q_H^T$ is a ring homomorphism by Corollary 1.4 (a). Restricting this species to $T_O(G)$ we obtain a $K$-species

$$s_{(H,h)} = s_h \circ q_H^T \circ \text{res}_H^G: T_O(G) \to T_O(H) \to T_O(H) \to K$$

on $T_O(G)$. By Corollary 1.4 (c), we have $s_{(H,h)} = s_{(U,h)}$ for $U = \langle O_l(H), h \rangle$. Therefore, we can restrict our attention to pairs $(H,h)$ with $H \in \mathcal{H}$ and $h \in H$ such that $\langle O_l(H), h \rangle = H$. In this case, one can see from Proposition 1.3 that

$$s_{(H,h)} = s_h \circ p_H^T \circ \text{res}_H^G,$$

since the character value of $\text{ind}_U^H(\varphi)$ at $h$ is zero for all $O_l(H) \leq U < H$ and all $\varphi \in \hat{U}$.

Part (b) of the following proposition is well-known (see for instance [1, Section 2.13]), but it can be proved along with part (a) without additional effort.

2.3 Proposition Let $H, \bar{H} \in \mathcal{H}$, $h \in H$, and $\bar{h} \in \bar{H}$ with $\langle O_l(H), h \rangle = H$ and $\langle O_l(\bar{H}), \bar{h} \rangle = \bar{H}$.
(a) One has \( s_{(H,K)} = s_{(\tilde{H}, \tilde{K})} : \text{LO}(G) \to K \), if and only if there is some \( g \in G \) such that \( g\tilde{H} = H \) and \( g\tilde{O}_1(H)' = hO_1(H)' \).

(b) One has \( s_{(H,K)} = s_{(\tilde{H}, \tilde{K})} : \text{TLO}(G) \to K \), if and only if there is some \( g \in G \) such that \( g\tilde{H} = H \) and \( g\tilde{O}_1(H) = hO_1(H) \).

Proof. It is obvious that the conditions in (a) and (b) are sufficient for \( s_{(H,K)} = s_{(\tilde{H}, \tilde{K})} \), since conjugation doesn’t change the species, and since \( O_1(H)' \) (resp. \( O_1(H) \)) acts trivially on any indecomposable linear source (resp. trivial source) \( OH \)-module \( H \) with vertex \( O_1(H) \).

First we calculate \( s_{(U,u)} ([\text{ind}_I^G(O_\varphi)]) \) for \( U \in \mathcal{H}, \ u \in U \) with \( \langle O_1(U), u \rangle = U, \ I \leq G, \) and \( \varphi \in \hat{I} \):

\[
s_{(U,u)} ([\text{ind}_I^G(O_\varphi)]) = (s_u \circ q_U^0) \circ (\text{res}_U^G([\text{ind}_I^G(O_\varphi)]))
= \sum_{g \in U \setminus I} (s_u \circ q_U^0) ([\text{ind}_{U \cap I}^I(O_{\varphi})_{U \cap I}])
= \sum_{g \in U \setminus I, O_1(U) \leq U \cap I} \sum_{g \in U \setminus I, U \cap I = \langle u \rangle} (s_u)((\text{ind}_{U \cap I}^I(O_{\varphi}))_{U \cap I})
= \sum_{g \in U \setminus I} (g \varphi)(u).
\]

Let us assume that \( s_{(H,K)} = s_{(\tilde{H}, \tilde{K})} \) holds on \( \text{TLO}(G) \). Then evaluation at \([\text{ind}_I^G(O)]\) yields \([N_G(H) : H] = \{g \in \tilde{H} \setminus H \mid \tilde{H} \leq gH\}\). Hence, there exists \( g' \in G \) with \( \tilde{H} \leq g'H \). Similarly, evaluation at \([\text{ind}_I^G(O)]\) yields the existence of some \( g \in G \) with \( H \leq g\tilde{H} \). This implies \( H = g\tilde{H} \), and we may assume (after conjugation with \( g \)) that \( H = \tilde{H} \). Then we have to show that some conjugate of \( \tilde{h} \) under \( N_G(H) \) and \( h \) lie in the same coset of \( H/O_1(H)' \) for part (a), and in the same coset of \( H/O_1(H) \) for part (b).

To prove part (a) we apply \( s_{(H,K)} = s_{(\tilde{H}, \tilde{K})} \) to \([\text{ind}_I^G(O_\varphi)]\) for all \( O_1(H) \leq I \leq H \) and all \( \varphi \in \hat{I} \). Then we obtain by the above calculation

\[
s_{(H,K)} ([\text{ind}_I^G(O_\varphi)]) = \sum_{g \in H \setminus I, H \leq gI} (g \varphi)(h)
= \begin{cases} 
0, & \text{if } I < H, \\
\sum_{g \in N_G(H)/H} (g \varphi)(h), & \text{if } I = H, \\
\sum_{g \in N_G(H)/H} g([\text{ind}_I^H(\varphi)])(h), & \text{if } I > H.
\end{cases}
\]

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and similarly for $s_{(H, h)}$. This implies that
\[
\sum_{g \in N_G(H)/H} (\overline{g}_{\chi})(h) = \sum_{g \in N_G(H)/H} (\overline{g}_{\chi})(\overline{h})
\]
for all irreducible characters $\chi$ of $H$ with $\ker(\chi) \supseteq O_t(H)'$, since each such $\chi$ is of the form $\text{ind}_H^G(\varphi)$ with $I$ and $\varphi$ as above, cf. Proposition 1.3 (b) and Corollary 1.4 (b). This implies that (after dividing out the trivial $O_t(H)'$-action), for all $N_G(H)/H$-stable characters $\chi$ of $H/O_t(H)'$, we have $\chi(hO_t(H)) = \chi(hO_t(H))$. Now the result follows from the next lemma.

To prove part (b) we apply $s_{(H, h)} = s_{(H, h)}$ to all $[\text{ind}_H^G(\varphi)]$ with $\varphi \in \tilde{H}_r$. Then we obtain
\[
\sum_{g \in N_G(H)/H} (\overline{g}_{\varphi})(h) = \sum_{g \in N_G(H)/H} (\overline{g}_{\varphi})(\overline{h})
\]
for all $\varphi \in \tilde{H}_r$. This implies that (after dividing out the trivial $O_t(H)$-action), for all $N_G(H)/H$-stable characters $\chi$ of $H/O_t(H)$, we have $\chi(hO_t(H)) = \chi(hO_t(H))$. And again the result follows from the next lemma.

2.4 Lemma Let $G$ and $S$ be finite groups and let $S$ act on $G$ by group automorphisms. If $g$ and $\tilde{g}$ are elements of $G$ such that $\chi(g) = \chi(\tilde{g})$ for all $S$-stable characters of $G$, then there is some $s \in S$ such that $g$ and $\tilde{g}$ are conjugate in $G$.

Proof We consider the character table of $G$ as an $m \times m$-matrix $M$, the rows indexed by the irreducible characters of $G$, the columns by the conjugacy classes of elements of $G$. The subgroup $R_K(G)^S$ of $S$-fixed points of $R_K(G)$ has as a basis the sums of $S$-conjugate irreducible characters of $G$. Let $\chi_1, \ldots, \chi_n$ denote this basis. If we consider the $m \times n$-matrix $M'$ which arises from $M$ by adding up the rows indexed by $S$-conjugate irreducible characters, then the rows are still linearly independent, and $M'$ has rank $n$. Obviously $\chi_i(g) = \chi_i(s^g)$ for all $i = 1, \ldots, n, s \in S$, and $g \in G$. Hence, the columns of $M'$ belonging to $S$-conjugate conjugacy classes of $G$ coincide, and we may omit these repetitions in $M'$ to obtain a matrix $M''$ which still has rank $n$. By [5, Satz V 13.5 (b)], there are as many $S$-orbits of irreducible characters of $G$ as there are $S$-orbits of conjugacy classes of $G$. Therefore $M''$ is an $n \times n$-matrix with rank $n$, and the conjugacy classes of $g$ and $\tilde{g}$ have to lie in the same $S$-orbit.

2.5 Let $\mathcal{H}_L = \mathcal{H}_L^H$ be the set of all pairs $(H, c)$ with $H \in \mathcal{H}$ and $c \in H/O_t(H)'$ such that $(c) = H/O_t(H)$. Each element $(H, c)$ of $\mathcal{H}_L$ then determines a $K$-species $s_{(H, c)}^L \colon L(G) \to K$ by setting $s_{(H, c)}^L(h) = s_{(H, h)}$ for any $h \in H$ with $hH = c$. By Proposition 2.3 this species is well-defined. Similarly, let $\mathcal{H}_T = \mathcal{H}_T^H$ be the set of all pairs $(H, c)$ with $H \in \mathcal{H}$ and $c \in H/O_t(H)$ such that $(c) = H/O_t(H)$. Again each $(H, c) \in \mathcal{H}_T$ determines a well-defined $K$-species $s_{(H, c)}^T \colon T(G) \to K$ by setting $s_{(H, c)}^T(h) = s_{(H, h)}$ for any $h \in H$ with $hO_t(H) = c$. The sets $\mathcal{H}_L$ and $\mathcal{H}_T$ are $G$-sets via the obvious conjugation action and, by Proposition 2.3, we have for $(H, c), (\tilde{H}, \tilde{c}) \in \mathcal{H}_L$ (resp. $(H, c), (\tilde{H}, \tilde{c}) \in \mathcal{H}_T$) the equation $s_{(H, c)}^T = s_{(\tilde{H}, \tilde{c})}^T$. 

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\( (\text{resp. } s^T_{(H,c)} = s^T_{(H,c')}), \) if and only if \((H,c)\) and \((H,c')\) are \(G\)-conjugate. Note that mapping \((H, hO(H))\) to \((H, hO(H))\) defines a surjective morphism \(\mathcal{H}^L \rightarrow \mathcal{H}^T\) of \(G\)-sets and that \(s(H, hO(H)) \mid_{T_O(G)} = \mathcal{s}(H, hO(H))\). Therefore, we obtain a commutative diagram of \(K\)-algebra homomorphisms.

\[
\begin{align*}
K \otimes T_O(G) & \quad \subset \quad K \otimes L_O(G) \\
\downarrow s^T = (s_{(H,c)}) & \quad \downarrow s^L = (s_{(H,c)}) \\
(\prod_{(H,c) \in \mathcal{H}^T} K)^G & \quad \rightarrow \quad (\prod_{(H,c) \in \mathcal{H}^L} K)^G,
\end{align*}
\]

where we consider the products of copies of \(K\) as \(G\)-sets via the \(G\)-action on the index sets \(\mathcal{H}^T\) and \(\mathcal{H}^L\), and where the lower embedding is induced by the canonical map \(\mathcal{H}^L \rightarrow \mathcal{H}^T\). Of course, the \(G\)-fixed points of the products in Diagram (2.5.a) are again isomorphic to the products of copies of \(K\), indexed by representatives for the \(G\)-orbits of \(\mathcal{H}^T\) and \(\mathcal{H}^L\) respectively. Since the species attached to such representatives are pairwise distinct by Proposition 2.3, they are \(K\)-linearly independent by the lemma of Dedekind, and therefore, their collections form surjective maps into the products of copies of \(K\). This implies that the maps \(s^T\) and \(s^L\) in Diagram (2.5.a) are surjective.

The part concerning \(s^T\) in the following proposition is well-known.

2.6 Proposition The maps \(s^T\) and \(s^L\) in Diagram (2.5.a) are isomorphisms. In particular, the \(K\)-algebras \(K \otimes T_O(G)\) and \(K \otimes L_O(G)\) are semisimple and their species are given by \(s^T_{(H,c)}(H,c) \in \mathcal{H}^T / G\) (resp. \(s^L_{(H,c)}(H,c) \in \mathcal{H}^L / G\)).

Proof We only have to show the injectivity of \(s^L\). The injectivity of \(s^T\) then follows from the commutativity of Diagram (2.5.a).

It suffices to show that for all \(H \in \mathcal{H}\) and all \(x \in K \otimes L_O(H)\) with

\[
(s_u \circ \text{res}^H_U)(x) = 0
\]

for all \(O_l(H) \leq U \leq H\) and all \(u \in U\) with \(\langle O_l(U), u \rangle = U\),

one has \(x = 0\). In fact, if \(y \in K \otimes L_O(G)\) is in the kernel of \(s^L\), we have to show that \(y = 0\), which is equivalent to \(\text{res}^H_{O_l}(y) = 0\) for all \(H \in \mathcal{H}\). But, since \(y \in \text{ker}(s^L)\), the element \(x := q^L_H(\text{res}^H_{O_l}(y))\), for \(H \in \mathcal{H}\), satisfies the condition in (2.6.a) by Corollary 1.4 (c):

\[
0 = (s_u \circ q^L_U \circ \text{res}^H_U)(y) = (s_u \circ \text{res}^H_U \circ q^L_H \circ \text{res}^H_{O_l})(y) = (s_u \circ \text{res}^H_U)(x).
\]

So let \(H \in \mathcal{H}\) and \(x = \sum \alpha_i [V_i] \in K \otimes L_O'(G)\), where \(V_1, \ldots, V_n \in \mathcal{O}H-\text{lin}\) are representatives of the isomorphism classes of indecomposable linear source \(O_H\)-modules with vertex \(O_l(H)\), and \(\alpha_1, \ldots, \alpha_n \in K\), such that (2.6.a) holds. Let \(\chi_1, \ldots, \chi_n\) be the characters of \(V_1, \ldots, V_n\). Then, by Corollary 1.4 (b), the characters \(\chi_1, \ldots, \chi_n\) are precisely the irreducible characters of \(H/O_l(H)\), and (2.6.a) just states that the function \(\alpha_1 \chi_1 + \ldots + \alpha_n \chi_n: H/O_l(H) \rightarrow K\) vanishes on each \(uO_l(H)'\), \(u \in H\), since
for each $u \in H$ we may choose $U$ as $(O(H), u)$. This implies $\alpha_1 = \ldots = \alpha_n = 0$, and the proof is complete. 

2.7 Corollary The collections of maps

$$r_G^L = (p_H^T \circ \text{res}_H^G)_{H \in \mathcal{H}} : L_O(G) \rightarrow \prod_{H \in \mathcal{H}} L^{ab}_O(H)$$

and

$$r_G^T = (p_H^T \circ \text{res}_H^G)_{H \in \mathcal{H}} : T_O(G) \rightarrow \prod_{H \in \mathcal{H}} T^{ab}_O(H)$$

are injective.

Proof It suffices to prove the injectivity of $r_G^L$, since $r_G^T$ is the restriction of $r_G^L$. By (2.2.a) we have a commutative diagram

$$L_O(G) \xrightarrow{s_L^*} \prod_{(H, o) \in \mathcal{H}^L} K$$

$$r_G^L \Bigg\downarrow \quad \Bigg\downarrow$$

$$\prod_{H \in \mathcal{H}} L^{ab}_O(H)$$

in which the right diagonal map is defined by mapping an element of $L^{ab}_O(H)$, $H \leq G$, into the components indexed by $(H, hO(H)^r)$ with the different allowed choices for $hO(H)^r$ via the maps $s_h : L^{ab}_O(H) \rightarrow K$. Now the injectivity of $r_G^L$ follows from the injectivity of $s_L^*$. 

2.8 Remark It is easy to see that the species of the different Green functors are related by the following commutative diagram

$$K \otimes \Omega(G) \rightarrow K \otimes P_O(G) \leftrightarrow K \otimes T_O(G) \leftrightarrow K \otimes L_O(G) \rightarrow K \otimes R_K(G)$$

$$\downarrow (s_H) \quad \downarrow (s^P_H) \quad \downarrow (s^T_{(H, o)}) \quad \downarrow (s^*_H) \quad \downarrow (s_g)$$

$$(\prod_{H \in \mathcal{H}} K)^G \rightarrow (\prod_{H \in \mathcal{H}} K)^G \leftrightarrow (\prod_{(H, o) \in \mathcal{H}^T} K)^G \leftrightarrow (\prod_{(H, o) \in \mathcal{H}^T} K)^G \rightarrow (\prod_{g \in G} K)^G$$

whose ingredients we have to explain. By $S = S_G$ we denote the $G$-set of subgroups of $G$. For $H \in S$, the map $s_H : \Omega(G) \rightarrow \mathbb{Z}$ denotes the mark homomorphism with respect to $H$ which maps the class of a finite $G$-set $X$ to the number $|X^H|$ of $H$-fixed points. The kernel of the surjective map $K \otimes \Omega(G) \rightarrow K \otimes P_O(G)$ is precisely the $K$-span of the primitive idempotents $\varepsilon_U$, $U \in \mathcal{S} \cap \mathcal{H}$, of $K \otimes \Omega(G)$ which are given by $s_H(\varepsilon_U) = 1$ if $H$ and $U$ are conjugate, and $s_H(\varepsilon_U) = 0$ otherwise, cf. [4, Lemma 81.28]. Therefore the mark homomorphisms $s_H$ with $H \in \mathcal{H}$ induce morphisms $s_H^* : K \otimes P_O(G) \rightarrow K$ whose collection forms an isomorphism onto the $G$ fixed points of $\prod_{H \in \mathcal{H}} K$. For $g \in G$ the map $s_g : R_K(G) \rightarrow K$ is defined by
evaluating a character of $G$ at $g$. We still have to explain the lower horizontal maps. Each of the $K$-spaces is of the form $([\prod_{g \in R} K]_G)$ for a $G$-set $R$. We can obviously identify this $K$-space with the $K$-space $\text{Hom}_G(R, K)$ of $G$-equivariant maps to $K$, where we consider $K$ as a trivial $G$-set. Then the lower maps are induced by the morphisms of $G$-sets $G \to \mathcal{H}$, $g \mapsto ((g), g)$, $\mathcal{H} \to \mathcal{H}'$, $(H, hO_1(H)) \mapsto (H, hO_1(H))$, $\mathcal{H} \to \mathcal{H}$, $(H, hO_1(H)) \mapsto H$, and the inclusion $\mathcal{H} \to \mathcal{S}$. It is not difficult to see that the squares are commutative.

3 Canonical induction formulae

For the reader’s convenience we will recall the basic facts about canonical induction formulae from [2].

A conjugation functor $X$ on $G$ over a commutative ring $k$ consists of a family of $k$-modules $X(H)$, $H \leq G$, together with $k$-linear maps $c_{g,H} : X(H) \to X(gH)$, the conjugation maps, for $g \in G$ and $H \leq G$. These maps are required to be transitive with respect to composition and the multiplication in $G$. Moreover, it is required that $c_{g,H} = \text{id}_{X(H)}$ if $g \in H$. Then $c_{g,H}$ is an isomorphism with inverse $c_{g^{-1},H}$. We frequently write $kx$ instead of $c_{g,H}(x)$ for $g \in G$, $H \leq G$, and $x \in X(H)$. A morphism $f : X \to Y$ between conjugation functors is a family of $k$-linear maps $f_H : X(H) \to Y(H)$, $H \leq G$, which commute with the conjugation maps.

A $k$-conjugation functor $A$ on $G$ together with $k$-linear maps $\text{res}_H^U : A(H) \to A(U)$, the restriction maps, for $U \leq H \leq G$, is called a $k$-restriction functor on $G$, if the restriction maps are transitive with respect to subgroup towers, commute with conjugation maps, and are the identity for $U = H$. A morphism $f : A \to B$ of $k$-restriction functors is a morphism of the underlying $k$-conjugation functors which additionally commutes with the restriction maps.

If $M$ is a $k$-restriction functor with additional $k$-linear maps $\text{ind}_H^U : M(U) \to M(H)$, the induction maps, for $U \leq H \leq G$, which are transitive with respect to subgroup towers, commute with conjugations, are the identity for $U = H$, and which satisfy the Mackey double coset formula for induction followed by restriction, then $M$ is called a $k$-Mackey functor on $G$. A morphism between $k$-Mackey functors is a morphism between the underlying $k$-restriction functors, whose maps commute also with the induction maps.

3.1 The examples we consider in this paper all satisfy Hypothesis 8.1 of [2], namely that we have

- a Mackey functor $M$ on $G$ such that $M(H)$ is a free abelian group for all $H \leq G$,
- a restriction subfunctor $A \subseteq M$, i.e. a family of subgroups $A(H) \subseteq M(H)$, stable under conjugation and restriction,
- and a family $B = \{ B(H) \}_{H \leq G}$ of $\mathbb{Z}$-bases $B(H)$ of $A(H)$ for all $H \leq G$ which is stable under conjugation maps, and whose positive span (i.e. the linear combinations with non-negative coefficients) is stable under restriction maps.
3.2 Under the assumptions in 3.1, one can form new Mackey functors $A^+$ and $A_+$ from $A$ by setting
\[
A^+(H) := \left( \prod_{U \leq H} A(U) \right)^H \quad \text{and} \quad A_+(H) := \left( \bigoplus_{U \leq H} A(U) \right)^H ,
\]
for $H \leq G$, where we consider $\prod_{U \leq H} A(U)$ and $\bigoplus_{U \leq H} A(U)$ as being endowed with the $H$-action coming from the conjugation maps $c_{h,U} : A(U) \to A(hU)$ for each $h \in H$ and $U \leq H$, and where the exponent $H$ denotes taking invariants and the index $H$ denotes taking coinvariants under the $H$-action. If we write $[U, a]_H \in A_+(H)$ for the class of the element $a \in A(U)$ in $A_+(H)$, we can define the structure maps on $A_+(H)$, $H \leq G$, by
\[
c_{+g,H} : A_+(H) \to A_+(gH), \quad [U, a]_H \mapsto [gU, g^*a]_H ,
\]
\[
\text{res}_{+I}^H : A_+(H) \to A_+(I), \quad [U, a]_H \mapsto \sum_{h \in I \cap [H/U]} [I \cap hU, \text{res}_{hU}^{h_0U}(h^*a)]_I ,
\]
\[
\text{ind}_{+I}^H : A_+(I) \to A_+(H), \quad [J, b]_I \mapsto [J, b]_H ,
\]
for $I \leq H \leq G, U \leq H, J \leq I, a \in A(U), b \in A(J)$. Since we don’t need the structure maps of the Mackey functor $A^+$, we omit their definition.

If, for $H \leq G$, we define $\mathcal{M}(H)$ as the set of monomial pairs $(U, a)$ with $U \leq H$ and $a \in B(U)$, then $\mathcal{M}(H)$ is an $H$-set via the conjugation action, and also a poset by defining $(J, b) \preceq (U, a)$, if and only if $J \leq U$ and $b$ occurs in $\text{res}_{U}^{J}(a)$ with respect to the basis $B(J)$. The $H$-action on $\mathcal{M}(H)$ respects this partial order, and the set $\Delta(\mathcal{M}(H))$ of strictly increasing chains in $\mathcal{M}$ is again an $H$-set. If $(U, a) \in \mathcal{M}(H)/H$ runs through a set of representatives for the $H$-orbits of $\mathcal{M}(H)$, then the elements $[U, a]_H$ form a $\mathbb{Z}$-basis of $A_+(H)$.

In order to avoid confusion we should mention that, as free $\mathbb{Z}$-modules, the groups $A_+(H)$ and $A^+(H)$ are isomorphic with obvious bases indexed by $\mathcal{M}(H)/H$. Only the structure maps as Mackey functors are different, if we want to identify these two groups by identifying these bases. Nevertheless, their Mackey functor structures are related by the more complicated map $\rho^A$ defined below. Although we could have replaced the direct product in the definition of $A^+(H)$ with a direct sum (as for $A_+(H)$), we prefer the product, because it underlines the dual nature of the two constructions, one being a limit and the other a colimit in the categorical sense.

If for each $H \leq G$ the group $A(H)$ is a ring, and if conjugation maps and restriction maps for $A$ are ring homomorphisms, then $A_+(H)$ is again a ring and also an $A(H)$-module by the definition $b \cdot [U, a]_H := [U, \text{res}_{U}^{H}(b) \cdot a]_H$ for $U \leq H$, $a \in A(U)$, and $b \in A(H)$, cf. [2, 2.2].

There is a morphism $\rho^A : A_+ \to A^+$ of Mackey functors given by
\[
\rho^A_H := (\pi^A_0 \circ \text{res}_{+U}^H)_{U \leq H} : A_+(H) \to \left( \prod_{U \leq H} A(U) \right)^H ,
\]
where the projection $\pi^A_0 : A_+(H) \to A(H)$, for $H \leq G$, is defined on a basis element
$[U,a]_H \in A_+(H)$, $(U,a) \in \mathcal{M}(H)$, by

$$\pi_A^H([U,a]_H) := \begin{cases} a, & \text{if } U = H, \\ 0, & \text{if } U < H. \end{cases}$$

Then $\rho^A$ is injective and induces an isomorphism $\rho^A : \mathbb{Q} \otimes A_+ \to \mathbb{Q} \otimes A^+$ of $\mathbb{Q}$-Mackey functors on $G$, see [2, Prop. 2.4].

3.3 Let $M$, $A$, and $B$, be given as in 3.1. There is a morphism of Mackey functors on $G$,

$$b^{M,A} : A_+ \to M,$$

defined by $b^{M,A}_U([U,a]_H) = \text{ind}^H_U(a)$ for all $U \leq H \leq G$ and $a \in A(U)$. A canonical induction formula for $M$ from $A$ is a section of $b^{M,A}$ in the category of restriction functors, i.e., a family of maps $a_H : M(H) \to A_+(H)$, $H \leq G$, which commutes with conjugation and restriction maps and which satisfies $b_H \circ a_H = \text{id}_{M(H)}$ for all $H \leq G$.

There is a bijection between the set of morphisms $a : \mathbb{Q} \otimes M \to \mathbb{Q} \otimes A_+$ of $\mathbb{Q}$-restriction functors and the set of morphisms

$$p : \mathbb{Q} \otimes M \to \mathbb{Q} \otimes A_+, \quad (3.3.a)$$

of $\mathbb{Q}$-conjugation functors. If $a$ and $p$ are in correspondence, then they determine each other by the commutative diagrams

$$\begin{array}{ccc} \mathbb{Q} \otimes M(H) & \xrightarrow{\alpha_H} & \mathbb{Q} \otimes A_+(H) \\ \downarrow{p_H} & & \downarrow{\pi_H} \\ \mathbb{Q} \otimes A(H) & & \end{array} \quad (3.3.b)$$

and

$$\begin{array}{ccc} \mathbb{Q} \otimes M(H) & \xrightarrow{\alpha_H} & \mathbb{Q} \otimes A_+(H) \\ \downarrow{r_H = (p_U \circ \text{res}^H_U)_{U \leq H}} & & \downarrow{\rho_H} \\ \mathbb{Q} \otimes A^+(H) & & \end{array} \quad (3.3.c)$$

for $H \leq G$, see [2, Cor. 5.4 (iii)]. Actually, the commutativity of each one of the two previous diagrams for all $H \leq G$ is equivalent to the commutativity of the other one for all $H \leq G$. Using an explicit inverse of $\rho^*_H$ one obtains from Diagram (3.3.c) the explicit formula

$$a_H(m) = \frac{1}{|H|} \sum_{H_0 < \cdots < H_n} (-1)^n[H_0][H_0, (\text{res}^H_{H_0} \circ p_{H_0} \circ \text{res}^H_{H_0})_H(m)]_H \quad (3.3.d)$$
for all $H \leq G$ and all $m \in M(H)$, where the sum runs over all strictly increasing chains of subgroups of $H$, cf. Equation (6.1a) in [2].

If $p$ is chosen as in (3.3.a), there arise two interesting questions, namely:

(i) Is $a: \mathbb{Q} \otimes M \to \mathbb{Q} \otimes A_+$ a section of $b^{M,A}: \mathbb{Q} \otimes A_+ \to \mathbb{Q} \otimes M$, i.e. is $a$ an induction formula for $\mathbb{Q} \otimes M$ from $\mathbb{Q} \otimes A$?

(ii) Is $a$ integral, i.e. is $a_H(M(H)) \subseteq A_+(H)$ for all $H \leq G$?

For a positive answer to Question (i) we have necessary and sufficient conditions, and in case of Question (ii) we at least can give conditions which imply a positive answer, see [2, Prop. 6.4 and Thm. 9.3]. We will need the following particular result on the integrality of $a$ later.

3.4 Theorem ([2, Thm. 9.3 and Cor. 9.4]) Let $M$, $A$, and $B$ be given as in 3.1, let $p: \mathbb{Q} \otimes M \to \mathbb{Q} \otimes A$ be given as in (3.3.a) with $p_H(M(H)) \subseteq A(H)$ for all $H \leq G$, and let $a: \mathbb{Q} \otimes M \to \mathbb{Q} \otimes A_+$ be the morphism of restriction functors associated to $p$ via Diagram (3.3.c). Assume further that for a set of primes $\pi$ the following condition holds:

\[ (*_\pi) \quad \text{For all } U \subseteq H \leq G \text{ such that } H/U \text{ is a cyclic } \pi\text{-group and all } \psi \in B(U) \text{ which are fixed under } H, \text{ the coefficients of } \psi \text{ with respect to the basis } B(U) \text{ in } \text{res}_U^H(p_H(\vartheta)) \text{ and } p_U(\text{res}_U^H(\vartheta)) \text{ coincide for all } \vartheta \in M(H). \]

Then

\[ |H|_{\pi^1} \cdot a_H(M(H)) \subseteq A_+(H) \]

for all $H \leq G$. If furthermore, $\text{res}_U^H(B(H)) \subseteq B(U)$ for all $U \leq H \leq G$, then one has for all $H \leq G$ and $m \in M(H)$ the explicit formula

\[ a_H(m) = \frac{1}{|H|} \sum_{\sigma = ((H_0, \varphi) < \cdots < (H_s, \varphi_+)) \in \Delta(M(H))/H} (-1)^n \frac{|N_H(\sigma)/H_0|}{|N_H(\sigma)/H_0|} \times \]

\[ \times m_{\varphi_+}(\text{res}_{H_s}^H(m))[H_0, \varphi_+]_H , \]

where $M(H)$ is the $H$-poset of monomial pairs as defined in 3.2, the sum runs over a set of representatives $\sigma$ for the $H$-orbits of strictly increasing chains in $\Delta(M(H))$, $N_H(\sigma)$ denotes the stabilizer of $\sigma$ in $H$, and where for $U \leq H$, $\varphi \in B(U)$, and $m' \in M(U)$ the term $m_{\varphi}(m')$ denotes the coefficient of $\varphi$ in the element $p_U(m') \in A(U)$ with respect to the basis $B(U)$.

4 Canonical induction formulae for linear source modules

4.1 Example We will give some examples for a Mackey functor $M$, a restriction subfunctor $A \subseteq M$, a $\mathbb{Z}$-basis $B(H)$ of $A(H)$ for all $H \leq G$, and a morphism $p: M \to A$ of $\mathbb{Q}$-conjugation functors such that $M, A$, and $B$ satisfy the hypotheses
in 3.1. Of course, $p$ then induces a morphism of $\mathbb{Q}$-conjugation functors as in (3.3 a), such that $p(M(H)) \subseteq A(H)$ for all $H \leq G$, and we obtain an associated morphism $\alpha : \mathbb{Q} \otimes M \to \mathbb{Q} \otimes A_+$ of $\mathbb{Q}$-restriction functors. As in 3.2 we also have a resulting set of monomial pairs $M(H)$ for each $H$, which is an $H$-poset.

(a) Let $M := L_{\mathcal{O}}$, $A := L_{\mathcal{O}}^{ab}$, and $B(H) := \{[\mathcal{O}_\varphi] \mid \varphi \in \hat{H}\}$ for $H \leq G$. We identify $B(H)$ with $\hat{H}$ (i.e., if we often write $\varphi$ instead of $[\mathcal{O}_\varphi]$) and obtain as set of monomial pairs $M(H) = M^r(H)$ as defined in Section 1. The maps $p_H^r : L_{\mathcal{O}}(H) \to L_{\mathcal{O}}^a(H)$, $H \leq G$, form a morphism of conjugation functors and we will denote the associated morphism of $\mathbb{Q}$-restriction functors by $\alpha^r : \mathbb{Q} \otimes L_{\mathcal{O}} \to \mathbb{Q} \otimes L_{\mathcal{O}}^{ab}$. The aim of this section is to show that $\alpha^r$ is an integral induction formula for $L_{\mathcal{O}}$ from $L_{\mathcal{O}}^{ab}$.

(b) Let $M := T_{\mathcal{O}}$, $A := T_{\mathcal{O}}^{ab}$, and $B(H) := \{[\mathcal{O}_\varphi] \mid \varphi \in \hat{H}\}$ for $H \leq G$. Then the set of monomial pairs $M(H)$, $H \leq G$ for this choice of $B$ is given as the $H$-subposet $M^T(H)$ of $M^L(H)$ consisting of all those pairs $(U, \varphi)$ with $U/\ker(\varphi)$ being an $\ell'$-group. In section 1 we defined a morphism $p^T : T_{\mathcal{O}} \to T_{\mathcal{O}}^{ab}$ of conjugation functors on $G$. We denote the associated morphism of restriction functors by $\alpha^T : \mathbb{Q} \otimes T_{\mathcal{O}} \to \mathbb{Q} \otimes T_{\mathcal{O}}^{ab}$. Note that $T_{\mathcal{O}}^{ab}$ can be considered as a Mackey subfunctor of $L_{\mathcal{O}}^{ab}$ by identifying the $\mathbb{Z}$-basis $[U, \varphi]_H$, $(U, \varphi) \in M^T(H)/H$, in the obvious way with a subset of the basis $[U, \varphi]_H$, $(U, \varphi) \in M^L(H)/H$, of $L_{\mathcal{O}}^{ab}(H)$ for $H \leq G$. Since $p^T$ is the restriction of $p^r$, we obtain the commutative diagram

$$
\begin{array}{ccc}
\mathbb{Q} \otimes L_{\mathcal{O}} & \xrightarrow{\alpha^T} & \mathbb{Q} \otimes T_{\mathcal{O}} \\
\cup & & \cup \\
\mathbb{Q} \otimes T_{\mathcal{O}} & \xrightarrow{\alpha^T} & \mathbb{Q} \otimes T_{\mathcal{O}}^{ab} \\
\end{array}
$$

see [2, Prop. 6.11]. Hence, if $\alpha^r$ is an integral canonical induction formula for $L_{\mathcal{O}}$ from $L_{\mathcal{O}}^{ab}$, also $\alpha^T$ is an integral canonical induction formula for $T_{\mathcal{O}}$ from $T_{\mathcal{O}}^{ab}$.

(c) In order to show that $\alpha^r$ is integral, we will need another set of data. Let $M = L_{\mathcal{O}}$, and for $H \leq G$ let $B^a(H)$ be the set of isomorphism classes of indecomposable linear source $\mathcal{O}H$-modules $V$ with $H/\ker(V)$ being $\ell$-elementary. We define $L_{\mathcal{O}}^a(H) \subseteq L_{\mathcal{O}}(H)$ as the span of $B^a(H)$. Then $L_{\mathcal{O}}^a$ is a restriction subfunctor of $L_{\mathcal{O}}$ and the projection map $p^a : L_{\mathcal{O}} \to L_{\mathcal{O}}^a$ which maps the class of an indecomposable module to itself, if it is in $B^a$, and to zero otherwise, is a morphism of conjugation functors. We denote the set of monomial pairs of this example by $M^a(H)$, $H \leq G$. The associated morphism of $\mathbb{Q}$-restriction functors will be denoted by $\alpha^a : \mathbb{Q} \otimes L_{\mathcal{O}} \to \mathbb{Q} \otimes L_{\mathcal{O}}^a$.

4.2 Before we can state the next theorem we need to introduce the following notions and recall some facts from [2]. For $H \leq G$, $V \in \mathcal{O}H\text{-lin}$, and $\varphi \in \hat{H}$ we define

$$m_{\varphi}(V) \in \mathbb{N}_0$$

as the multiplicity of $\mathcal{O}_\varphi$ as a direct summand in $V$. Recall from 3.2 that $L_{\mathcal{O}}(H)$ and $L_{\mathcal{O}}^{ab}(H)$ are both $L_{\mathcal{O}}^{ab}(H)$-modules for all $H \leq G$. Note also that for any
ring automorphism $\sigma : \mathcal{O} \to \mathcal{O}$ we have a Galois action on $L_0^b(H)$ for all $H \leq G$, induced by composing a representation $H \to \text{GL}_n(\mathcal{O})$ with the induced group homomorphism $\sigma : \text{GL}_n(\mathcal{O}) \to \text{GL}_n(\mathcal{O})$. The subrings $L_0^b(H)$, $H \leq G$, are stable under this Galois action, and one obtains an induced Galois action on the groups $L_0^b(H)$, $H \leq G$. For $H \leq G$, a module $V \in \mathcal{O}H-\text{lin}$, and $(U, \phi) \in \mathcal{M}^b(H)$ we define $a_H^b(V) \in \mathbb{Q}$ as the coefficient of the basis element $[U, \phi]_H$ in $a_H^b(V) \in L_0^b(H)$, i.e.

$$a_H^b(V) = \sum_{(U, \phi) \in \mathcal{M}^b(H)/H} \alpha_{U}^b([V])[U, \phi]_H.$$ 

For any homomorphism of finite groups $f : I \to G$ one has a ring homomorphism $\text{res}_f : L_0^b(G) \to L_0^b(I)$ induced by composing a representation $G \to \text{GL}_n(\mathcal{O})$ with the homomorphism $f$, since a monomial $\mathcal{O}G$-module is still monomial as $\mathcal{O}I$-module. There is also a ring homomorphism

$$\text{res}_+ f : L_0^b(G)_{\mathcal{O}G} \to L_0^b(I)_{\mathcal{O}I},$$

given on a basis element $[H, \phi]_{\mathcal{O}G}$ of $L_0^b(G)_{\mathcal{O}G}$ by

$$\text{res}_+ f([H, \phi]_{\mathcal{O}G}) = \sum_{\sigma \in f(I) \setminus G/H} [f^{-1}(\sigma H), (\sigma \phi) \circ f]_I,$$

cf. [2, 10.2]. Finally we abbreviate $b_{\mathcal{O}G, L_0^b}$ by $b^L$, $b^L_{\mathcal{O}G, L_0^b}$ by $b^T$, and $b_{\mathcal{O}G, L_0^b}$ by $b^T$.  

4.3 Theorem  Asume the notation of 4.1 and 4.2.  
(i) For each $H \leq G$ one has $b^T_H \circ a^L_H = \text{id}_{L_0^b(\mathcal{O})}$.  
(ii) The morphism $a^L$ is integral, i.e. for each $H \leq G$ one has $a^L_H(L_0^b(\mathcal{O})) \subseteq L_0^b(H)$.  
(iii) For each $H \leq G$ the map $a^L_H$ is $L_0^b(H)$-linear.  
(iv) The morphism $a^L$ commutes with the Galois action.  
(v) For each $H \leq G$ and $\phi \in \hat{H}$ one has $a^L_H([\mathcal{O}, \phi]) = [H, \phi]_H$.  
(vi) The morphism $a^L$ is given explicitly by

$$a^L_H([V]) = \sum_{\sigma \in \{(H_{\sigma}, \phi_{\sigma}) < \cdots < (H_0, \phi_0)\} \in \Delta(M^b(H)/H)} (-1)^n \frac{|(N_H(\sigma)/H_0)\nu|}{|N_H(\sigma)/H_0|} \times \times m_\phi\sigma(\text{res}_{H_\sigma}^H(V))[H_0, \phi_0]_H,$$  

for $H \leq G$ and $V \in \mathcal{O}H-\text{lin}$, where the sum runs over a set of representatives $\sigma$ for the $H$-conjugacy classes of chains in the poset $\mathcal{M}^b(H)$ and $N_H(\sigma)$ denotes the stabilizer of $\sigma$.  
(vii) For all $H \leq G$, $V \in \mathcal{O}H-\text{lin}$, and $(U, \phi) \in \mathcal{M}^b(H)$ one has

$$m_\phi(\text{res}_{H}^H(V)) = 0 \implies a_{U, \phi}^H(V) = 0.$$
(viii) For all $H \leq G$, $(U, \varphi) \in \mathcal{M}^l(H)$, and all indecomposable modules $V \in \mathcal{O}H - \text{lin}$ one has $a^H_{U(\varphi)}([V]) = 0$, unless a Sylow $l$-subgroup of $U$ is contained in some vertex of $V$.

(ix) Let $f : I \to G$ be a homomorphism of finite groups. Then the diagram

$$
\begin{array}{ccc}
L_O(G) & \xrightarrow{a^f} & L_{\overline{O}_+}(G) \\
\text{res}_{f} & \downarrow & \downarrow \text{res}_{+f} \\
L_O(I) & \xrightarrow{a^f} & L_{\overline{O}_+}(I)
\end{array}
$$

is commutative.

**Proof** The most elaborate part is to prove (ii). We postpone the proof of part (ii) to the end and first establish all the other results.

(iii) This follows immediately from [2, Prop. 6.10], since $p_H : L_O(H) \to L_{\overline{O^0}}(H)$ is $L_{\overline{O^0}}(H)$-linear for all $H \leq G$.

(iv) This follows from the commutativity of Diagram (3.3,c) since $p_U$, $\text{res}_H^f$, $\text{res}_H^U$, and $\text{res}_U^H$ commute with the Galois action.

(v) This follows immediately from [2, Prop 6.12], since

$$
\text{res}_H^f(p_H([O_\varphi])) = [O_\varphi_{|U}] = p_U(\text{res}_U^H([O_\varphi]))
$$

for all $\varphi \in \hat{H}$.

(vi) The formula for $a^H_U([V])$ follows from Theorem 3.4, since condition $(*)_0$ is satisfied for the set $\pi$ of primes distinct from $l$. In fact, let $U \leq H \leq G$ be such that $H/U$ is a cyclic $l$-group, and let $\psi \in \hat{U}$ be stable under $H$. Let furthermore $V \in \mathcal{O}H - \text{lin}$ be indecomposable. If $r_{K}(V) = 1$, then $p_U(\text{res}_H^U([V])) = \text{res}_U^H([V]) = \text{res}_U^H(p_H([V]));$ in particular, $(*)_1$ holds for $V$ in this case. If $r_{K}(V) > 1$, then $p_U([V]) = 0$, and we have to show that $O_\varphi | \text{res}_U^H(V)$ leads to a contradiction. Since $[H : U]$ is an $l$-number, $V$ is relatively $U$-projective, and there is some indecomposable module $W \in \mathcal{O}U - \text{lin}$ with $V \mid \text{ind}_I^H(W)$. So $O_\varphi | \text{res}_U^H(V)$ implies

$$
O_\varphi | \text{res}_U^H(V) \mid \text{res}_U^H(\text{ind}_I^H(W))
$$

$$
\cong \bigoplus_{h \in H/U} \text{ind}_U^U h_U(\text{res}_U^H h_U h^h(W)) \cong \bigoplus_{h \in H/U} h^h W.
$$

Therefore, the module $O_\varphi$ is isomorphic to $h^h W$ for some $h \in H$, and, since $\psi$ is $H$-stable, this implies $O_\varphi \cong W$. Hence,

$$
V \mid \text{ind}_I^H(O_\varphi) \cong \bigoplus_{\varphi \in H_1, \varphi_{|U} = \psi} O_\varphi,
$$

which leads to the desired contradiction, since $r_{K}(V) > 1$.

(vii) This follows immediately from the explicit formula in part (vi).

(viii) This follows from the explicit formula in part (vi). In fact, if the multiplicity $m_{\varphi_\psi}(\text{res}_H^U(V))$ is non-zero, i.e. $O_{\varphi_\psi} \mid \text{res}_H^U(V)$, then some vertex of $O_{\varphi_{U}}$ is
contained in a vertex of $V$. In particular, some Sylow $l$-subgroup of $H_0$ is contained in a vertex of $V$.

(ix) This follows immediately from [2, Prop. 10.3].

(i) We first show that

$$p_H^L \circ b_H^L \circ a_H^L = p_H^L : \mathbb{Q} \otimes L_C(H) \to \mathbb{Q} \otimes L_{ab}^H(H) \tag{4.3.a}$$

holds for all $l$-hypo-elementary subgroups $H$ of $G$. In fact, let $V \in \mathbb{O}H - \text{lin}$ be indecomposable. If $rk_\mathbb{O}(V) = 1$, then the above equation holds after evaluation at $[V]$ by part (v). From now on we assume that $rk_\mathbb{O}(V) > 1$. Let $P$ denote the Sylow $l$-subgroup of $H$. If the vertices of $V$ are strictly contained in $P$, then $\alpha_{(U, \varphi)}^H([V]) = 0$ for all $(U, \varphi) \in \mathbb{M}_l^u(H)$ with $P \leq U$ by part (viii). On the other hand, for all $(U, \varphi) \in \mathbb{M}_l^u(H)$ with $P \not\leq U$ one has $p_H^L([\text{ind}_H^U(\mathcal{O}_\varphi)]) = 0$. Hence, $(p_H^L \circ b_H^L \circ a_H^L)([V]) = 0$, and we are left with the case that $V$ has vertex $P$. Let $\mathcal{O}_\psi$ be a source of $V$ with $\psi \in \hat{P}$. Then by the same argument as above it suffices to show that

$$p_H^L \left( \varphi_{(U, \varphi)}^H([V]) \cdot \text{ind}_H^U(\text{ind}(\mathcal{O}_\varphi)) \right) = 0$$

for all $(U, \varphi) \in \mathbb{M}_l^u(H)$ with $P \leq U$. By part (vii) we may assume that $\mathcal{O}_\varphi \mid \text{res}_H^U(V)$, since otherwise $\alpha_{(U, \varphi)}^H([V]) = 0$. In this case we have

$$\text{ind}_H^U(\mathcal{O}_\varphi) \mid \text{ind}_H^U(\text{res}_H^U(V)) \mid \text{ind}_H^U(\text{res}_H^U(\text{ind}_H^U(\mathcal{O}_\varphi))))$$

$$\cong \text{ind}_H^U \left( \bigoplus_{h \in H/U} \text{ind}_H^U(\mathcal{O}_{\psi^h}) \right) \bigoplus_{h \in H/U} \text{ind}_H^U(\mathcal{O}_{\psi^h}).$$

Since $rk_\mathbb{O}(V) > 1$, the element $\psi \in \hat{P}$ is not $H$-stable (cf. Proposition 1.3 (b)). Again by Proposition 1.3 (b), we see that each module $\text{ind}_H^U(\mathcal{O}_{\psi^h})$, $h \in H$, splits into indecomposable $\mathbb{O}H$-modules of $\mathbb{O}$-rank strictly bigger than 1. Therefore, also $\text{ind}_H^U(\mathcal{O}_\varphi)$ contains no direct summand of $\mathbb{O}$-rank 1, and so is annihilated by $p_H^L$.

In order to prove the equation $b^L \circ a^L = \text{id}_{\mathbb{Q} \otimes L_C}$, it suffices to show that $b_H^L \circ a_H^L = \text{id}_{\mathbb{Q} \otimes L_C(H)}$ for all $l$-hypo-elementary subgroups $H$ of $G$, since $b^L$ and $a^L$ commute with restrictions, and since $\mathbb{C}(\mathbb{Q} \otimes L_C)$ consists precisely of these subgroups. So let $H \leq G$ be $l$-hypo-elementary. Since

$$r_H^L = (p_H^L \circ \text{res}_H^U)_{U \leq H} : L_C(H) \to \prod_{U \leq H} L_{ab}^H(U)$$

is injective by Corollary 2.7, it suffices to show for each $U \leq H$, that $p_H^L \circ \text{res}_H^U \circ b_H^L \circ a_H^L = p_H^L \circ \text{res}_H^U$. Since $a^L$ and $b^L$ commute with restrictions, this follows from Equation (4.3.a).

(ii) In the proof of part (vi) we showed that condition $(*)_l$ in Theorem 3.4 is satisfied in the present situation. Hence, we have

$$|H| \cdot a_H(L_C(H)) \subseteq L_{ab}^H(H)$$

21
for all $H \leq G$. In the sequel we will show that also $|H|_U \cdot a_H(L_O(H)) \leq L_{O+}^{ab}(H)$ for all $H \leq G$.

**4.4 Lemma** Let $V \in O\text{lin} G$ be indecomposable such that $G/\ker(V)$ is $l$-elementary. Then there exists $(H, \varphi) \in M^l(G)$, unique up to $G$-conjugacy, such that $V \cong \text{Ind}_H^G(O_\varphi)$. Moreover, $[G : H]$ is an $l$-power and one has $\ker(V) \leq H$. In particular, $\text{rk}_{O_\varphi}(V)$ is an $l$-power.

**Proof** The existence of $(H, \varphi) \in M^l(G)$ with $V \cong \text{Ind}_H^G(O_\varphi)$ follows from Proposition 1.5 and also the uniqueness follows from the uniqueness statement of Proposition 1.5 provided that we can show that, for any such $(H, \varphi)$, we have $\ker(V) \leq \ker(\varphi) (\leq H)$. Let $U := \ker(V)$, then $V \cong \text{Ind}_H^G(O_\varphi)$ implies

$$O \mid \text{res}^G_U(V) \cong \bigoplus_{g \in U \cap H} \text{Ind}_{U \cap H}^{U/H} \left( \text{res}^{U/H}_{U \cap H} \left( \text{Ind}_{U \cap H}^{H} (O_\varphi) \right) \right).$$

Hence, $O \in \text{IO} \text{lin}$ is a direct summand of one of the last summands above. Since the $l$-Sylow subgroups of $U$ are the vertices of $O$, this implies that there is some $g \in G$ such that $[U : U \cap gH]$ is an $l'$-number. On the other hand, $[U : U \cap gH] = [U : gH]$ is an $l'$-power, since $[G : gH] = \text{rk}_O(V)$ is an $l'$-power by Proposition 1.5. Therefore, $U \cap gH = U$ and $\varphi|_U = 1$. Since $U$ is normal in $G$, this implies $U \leq H$ and $\varphi|_U = 1$. □

For each $U \leq G$ we define a map

$$\eta_U : L_{O+}^e(U) \to L_{O+}^{ab}(U)$$

which maps the basis element $[I, [V]]_U$, where $I \leq U$ and $V \in \text{IO} \text{lin}$ is an indecomposable module such that $I/\ker(V)$ is $l$-elementary, to $[H, \varphi]_U \in L_{O+}^{ab}(U)$, where $(H, \varphi) \in M^l(I)$ is the (uniquely up to $I$-conjugacy determined) element with $V \cong \text{Ind}_H^I(O_\varphi)$.

**4.5 Lemma** The maps $\eta_U : L_{O+}^e(U) \to L_{O+}^{ab}(U)$, $H \leq G$, form a morphism of $\mathbb{Q}$-restriction functors on $G$, and the diagram

$$
\begin{array}{ccc}
\mathbb{Q} \otimes L_O & \xrightarrow{a^e} & \mathbb{Q} \otimes L_{O+}^e \\
\downarrow \text{id} & & \downarrow \text{id} \\
\mathbb{Q} \otimes L_{O+}^{ab} & \xrightarrow{b^e} & \mathbb{Q} \otimes L_O \\
\end{array}
$$

is commutative. In particular $b^e \circ a^e = \text{id}_{\mathbb{Q} \otimes L_O}$.

**Proof** First we show that $\eta$ is a morphism of restriction functors. Obviously $\eta$ commutes with conjugation maps. Now let $U' \leq U \leq G$, let $I \leq U$ and $V \in \text{IO} \text{lin}$ be indecomposable such that $I/\ker(V)$ is $l$-elementary, and let $(H, \varphi) \in M^l(I)$ be chosen such that $V \cong \text{Ind}_H^I(O_\varphi)$. Then we have

$$\text{res}_{U'}(\eta_U([I, [V]]_U)) = \text{res}_{U'}(\eta_U([H, \varphi]_U) = \sum_{w \in U' / H} [U' \cap wH, \text{res}_{U' \cap H}^H (\varphi)]_U$$

is commutative. In particular $b^e \circ a^e = \text{id}_{\mathbb{Q} \otimes L_O}$. □
\[ \eta_U^\circ \left( \text{res}_U^I ([I, [V]]_U) \right) = \sum_{x \in U \setminus U/I} \eta_U^\circ \left( [U' \cap [I] \text{res}_U^J ([V]) \right) \cdot \]

For \( x \in U \) we have
\[
\text{res}_{U \cap [I]}^J ([V]) \cong \text{res}_{U \cap [I] \cap [I']}^J (\text{ind}_{H}^J (\mathcal{O}_{\mathcal{F}}))
\]
\[
\cong \bigoplus_{y \in U \cap [I] \cap [I']} \text{ind}_{U \cap [I \cap [I'))^J}^J \left( \text{res}_{U \cap [I \cap [I']}^J (\mathcal{O}_{\mathcal{F}}) \right)
\]
\[
\cong \bigoplus_{y \in U \cap [I] \cap [I']} \text{ind}_{U \cap [I \cap [I')}^J (\mathcal{O}_{\mathcal{F}}) \right).
\]

Each of the summands in the last sum is indecomposable by Green's indecomposability theorem. In fact, let \( C \leq H \leq I \) be the subgroup with \( C/\ker(V) = O_V(I/\ker(V)) \). Then \( C \) is normal in \( I \) and \([I : C]\) is a power of \( I \). Hence, \( U' \cap [I] \) is normal in \( U' \cap [I \cap [I'] \cap [I') \text{ of } \) power index, and \( U' \cap [I] \leq U' \cap [I \cap [I') \leq U' \cap [I'), \) so that Green's theorem can be applied. Moreover, \( U' \cap [I \cap [I') \leq [U' \cap [I \cap [I') \cap \ker(V) \) and \( (U' \cap [I \cap [I') \cap \ker(V) \) is \( \ell \)-elementary. Therefore, by the definition of \( \eta \), we obtain for each \( x \in U \):
\[
\eta_U^\circ \left( [U' \cap [I] \text{res}_U^J ([V]) \right) \cdot \sum_{y \in U \cap [I] \cap [I')} \left[ U' \cap [I \cap [I' \cap [I')]^J \text{res}_U^J (\mathcal{O}_{\mathcal{F}}) \right]_U.
\]

Summing up over \( x \in U \setminus U/I \) we therefore obtain exactly the above expansion of \( \text{res}_{U \setminus U/I}^J (\eta_U^\circ ([I, [V]]_U)) \), since the elements \( yx \) run through a set of representatives for \( U \setminus U/H \), if \( x \) runs through a set of representatives for \( U \setminus U/I \), and, for each \( x \), \( y \) runs through a set of representatives for \( U' \cap [I \cap [I') \cap [I') \). Hence, \( \eta \) is a morphism of restriction functors.

Next we show that \( b_U^I \circ \eta = b^{I} : \mathbb{Q} \otimes L_{\mathcal{O}}^I \rightarrow \mathbb{Q} \otimes L_{\mathcal{O}}^I \). Let \( I \leq U \leq G \), \( V \in \mathcal{O}I-\text{lin} \), and \( (H, \varphi) \in \mathcal{M}^I(I) \) be given as in the previous paragraph. Then we have
\[
b_U^I (\eta_U^\circ ([I, [V]]_U)) = b_U^I ([H, \varphi]_U) = \text{ind}_H^I (\mathcal{O}_{\mathcal{F}}),
\]
\[
= \text{ind}_H^I (\mathcal{O}_{\mathcal{F}}) = \text{ind}_H^I ([V]).
\]

Finally, we show that \( \eta \circ a^\delta = a^\delta : \mathbb{Q} \otimes L_{\mathcal{O}} \rightarrow \mathbb{Q} \otimes L_{\mathcal{O}}^I \). We write \( \pi \) instead of \( \pi^{ab} : L_{O^+}^b \rightarrow L_{O^+}^b \) and \( \rho_U^I \) instead of \( \rho_U^b \). Since \( \rho_U^I : \mathbb{Q} \otimes L_{O^+}^b (U) \rightarrow \mathbb{Q} \otimes L_{O^+}^I (U) \) is injective, it suffices to show that \( \pi \circ \text{res}_U^I \circ \eta_U = \pi \circ \text{res}_U^I \circ a^\delta = \pi \circ \text{res}_U^I \circ a^\delta \). for all \( I \leq U \leq G \), and since \( \eta, a^\delta \), and \( a^\delta \) are morphisms of restriction functors and since \( \pi_U \circ a^\delta = p_U^I \) (cf. Diagram (3.3.b)), it suffices to show that
\[
\pi_U \circ \eta_U = p_U^I.
\]

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for all $U \leq G$. Let $V \in OU-\text{lin}$ be indecomposable. If $V \cong \mathcal{O}_G$ for some $\varphi \in \hat{U}$, then $a^U_\varphi([V]) = [U, \mathcal{O}_G]_U$ by [2, Prop. 6.12] (since $\text{res}^U_I(p^U_\varphi([\mathcal{O}_G])) = p^I_I(\text{res}^I_I([\mathcal{O}_G]))$ for all $I \leq U$), and we have

$$(\pi_U \circ \eta_U \circ a^U_\varphi)([V]) = [\mathcal{O}_G] = p^U_\varphi([V]).$$

If $\text{rk}_G(V) > 1$ but $U/\ker(V)$ is $l$-elementary, then still by the same argument, we have $a^U_\varphi([V]) = [U, [V]]_U$, and $\eta_U([U, [V]]_U) = [H, \varphi]_U$ for some $(H, \varphi) \in \mathcal{M}^l(H)$ with $H < U$. Hence, $[V]$ is mapped to zero by $\pi_U \circ \eta_U \circ a^U_\varphi$ as well as by $p^U_\varphi$. Finally, if $U/\ker(V)$ is not $l$-elementary, then, by Equation (3.3.d), $a^U_\varphi([V])$ is a linear combination of elements $[I, [W]]_U$ with $(I, [W]) \in \mathcal{M}^l(U)$ such that $I < U$. Hence, $\eta_U \circ a^U_\varphi$ maps $[V]$ to a linear combination of elements $[H, \varphi]_U$ with $(H, \varphi) \in \mathcal{M}^l(U)$ such that $H < U$, and therefore, $[V]$ lies in the kernel of $\pi_U \circ \eta_U \circ a^U_\varphi$ as well as in the kernel of $p^U_\varphi$. This completes the proof of the Lemma.

The proof of part (ii) of Theorem 4.3 will be completed by showing that $L_C$, $L^a_C$, $B^a$, and $p^a$ satisfy condition ($\ast$) in Theorem 3.4. In fact, then Theorem 3.4 implies that $[H]_l \cdot a^H_\varphi(L_C(H)) \subseteq L^a_C(H)$ for all $H \leq G$, and we obtain with Lemma 4.5 that

$$[H]_l \cdot a^H_\varphi(L_C(H)) = \eta_U([H]_l \cdot (\eta_U \circ a^H_\varphi)(L_C(H)) \subseteq \eta_U([H]_l \cdot a^H_\varphi(L_C(H)) \subseteq \eta_U(L^a_C(H)) \subseteq L^a_C(H).$$

On the other hand, we already know that $[H]_l \cdot a^H_\varphi(L_C(H)) \subseteq L^a_C(H)$ so that we have $a^H_\varphi(L_C(H)) \subseteq L^a_C(H)$ for all $H \leq G$.

So let $U \leq H \leq G$ be given such that $H/U$ is a cyclic $l$-group, let $V \in \mathcal{O}_H-\text{lin}$ be indecomposable, and let $W \in OU-\text{lin}$ be indecomposable and $H$-stable such that $U/\ker(W)$ is $l$-elementary. If $H/\ker(V)$ is $l$-elementary, then

$$p^H_\varphi(\text{res}^H_I([V])) = \text{res}^H_I([V]) = \text{res}^H_I(p^I_I([V]))$$

and condition ($\ast$) is satisfied in this case. If $H/\ker(V)$ is not $l$-elementary, then $p^H_\varphi([V]) = 0$, and we have to show that $W$ does not occur as a direct summand in $\text{res}^H_I(V)$. This follows from the next Lemma.

**4.6 Lemma** Let $U \leq H \leq G$ be such that $H/U$ is an $l$-group, and let $W \in OU-\text{lin}$ be indecomposable such that $W$ is $H$-stable and $U/\ker(W)$ is $l$-elementary. Then, for each indecomposable module $V \in \mathcal{O}_H-\text{lin}$ with $W \mid \text{res}^H_I(V)$, also $H/\ker(V)$ is $l$-elementary.

**Proof** The subgroup $I := \ker(W) \leq U$ is normal in $H$, since $W$ is $H$-stable. First we show that $H/I$ is again $l$-elementary. In fact, let $I \leq P, C \leq U$ be intermediate groups with $C/I = O_i(U/I)$ and $P/I = O_i(U/I)$, hence, $U/I = P/I \times C/I$. Then $W \cong \text{ind}_I^U(O_C)$ for some $C \leq J \leq H$ and some $\lambda \in J$ by Lemma 4.4. Since $I = \ker(\text{ind}_I^U(O_C))$, and $C/I$ is cyclic, one also has $I = \ker(\lambda|_C)$. By the uniqueness part of Lemma 4.4, for each element $h \in H$, there is some element $u \in U$ with $h(J, \lambda) = u(J, \lambda)$. Therefore, we have $h(\lambda|_C) = u(\lambda|_C) = \lambda|_C$, since $U/I$ centralizes...
$C/I$. This shows that $H/I$ centralizes $C/I$ and that $H/I = C/I \times P^I/I$ for a Sylow $l$-subgroup $P^I/I$ of $H/I$. Hence, $H/I$ is $l$-elementary.

Now let $V \in \mathcal{O}H-\mathsf{lin}$ be indecomposable with $W \mid \text{res}^H_0(V)$. It suffices to show that $I \leq \ker(V)$.

Let $Q$ be a vertex of $V$ and let $O_\psi \in \mathcal{O}Q-\mathsf{lin}$ be a source of $V$ with $\psi \in \hat{Q}$. Then, there exists an indecomposable module $V' \in \mathcal{O}(IQ)-\mathsf{lin}$ with $V' \mid \text{ind}^{IQ}_Q(O_\psi)$ and $V \mid \text{ind}^{IQ}_Q(V')$. We will show that $V' \cong O_{\psi'}$ for an extension $\psi' \in \hat{I}Q$ of $\psi$ with $\psi'|_I = 1$. Then $\ker(V) \geq I$. We have $O \mid \text{res}_I^{IQ}(V')$, since

\[
\begin{align*}
O \mid \text{res}_I^{IQ}(W) \mid \text{res}_I^{IQ}(\text{res}_0^{H}(V)) & \cong \text{res}_I^{H}(V) \mid \text{res}_I^{H}(\text{ind}_Q^{IQ}(V')) \\
& \cong \bigoplus_{h \in I \cap H/IQ} \text{ind}_I^{I \cap H/IQ}(\text{res}_{I \cap H/IQ}^{IQ}(hV')) \cong \bigoplus_{h \in I \cap H/IQ} h(\text{res}_I^{IQ}(V')).
\end{align*}
\]

From

\[
\begin{align*}
O \mid \text{res}_I^{IQ}(V') \mid \text{res}_I^{IQ}(\text{ind}_Q^{IQ}(O_\psi)) & \cong \bigoplus_{x \in I \cap IQ} \text{ind}_I^{I \cap IQ}(\text{res}_{I \cap IQ}^{IQ}(O_{\langle x \rangle})) \cong \text{ind}_I^{I \cap IQ}(O_{\langle x \rangle})
\end{align*}
\]

we see that $I \cap Q$ contains a Sylow $l$-subgroup of $I$. Therefore, $Q$ is a Sylow $l$-subgroup of $IQ$. Moreover, we obtain from $O \mid \text{ind}_I^{I \cap IQ}(O_{\langle x \rangle})$ that $\psi|_{I \cap Q} = 1$. Therefore, we can extend $\psi \in \hat{Q}$ to $\psi' \in \hat{I}Q$ with $\psi'|_I = 1$. Since the multiplicity of $O$ in

\[
\text{res}_I^{IQ}(\text{ind}_Q^{IQ}(O_\psi)) = \text{ind}_I^{I \cap IQ}(O)
\]

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equals 1, there is exactly one indecomposable summand $X$ of $\text{ind}^{\mathcal{O}}_{\mathcal{O}'}(\mathcal{O}'_\psi)$ with $\mathcal{O} | \text{res}_{\mathcal{O}}^{\mathcal{O}'}(X)$. But both $V'$ and $\mathcal{O}'_\psi$ have this property. Hence, $V' \cong \mathcal{O}'_\psi$, and the proof is complete.

This completes the proof of Theorem 4.3.

4.7 Remark (i) For the canonical induction formula $a^T$ for $T_{\mathcal{O}}$ from $T_{\mathcal{O}}^{ab}$ arising from Example 4.1 (b) a theorem analogous to Theorem 4.3 holds. This follows from the commutative diagram in 4.1 (b). Since there is an isomorphism of Mackey functors $T_{\mathcal{O}} \cong T_F$ for the residue field $F$ of $\mathcal{O}$, we also obtain a canonical induction formula for $T_F$ from $T_{\mathcal{O}}^{ab}$ by transporting everything via this isomorphism.

(ii) By Lemma 4.5 we know that also $a^{a^T}$ is a canonical induction formula. We haven't examined whether $a^{a^T}$ is integral. We used $a^{a^T}$ only to show that $a^{a^T}$ is integral.

(iii) In a subsequent paper we will construct Adams operations on $L_{\mathcal{O}}(G)$ using the canonical induction formula $a^{a^T}$ and the obvious Adams operations on $L_{\mathcal{O}}^{ab}(G)$. This construction makes use of the general results in [2, §4].

References


