A general theory of canonical induction formulae*

Robert Boltje
Institut für Mathematik
Universität Augsburg
86135 Augsburg
Germany
email: boltje@math.uni-augsburg.de

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Abstract

So far there exist three independent constructions of two different canonical versions of Brauer's induction theorem for complex characters due to V. Snaith, P. Symonds, and the author (see [23],[27],[2]). 'Canonical' in this context means functorial with respect to restrictions along group homomorphisms. In this article we axiomatize the situation in which the above canonical induction formulae are constructed. Mackey functors and related structures arise in this way naturally as a convenient language. This approach allows to construct canonical induction formulae for arbitrary Mackey functor. In particular we obtain canonical induction formulae for the Brauer character ring, the group of projective characters, the ring of trivial source modules, and the ring of linear source modules. In most cases, it is not difficult to construct such formulae over the rational numbers. A much more subtle question is whether the constructed formula comes from a canonical induction formula defined over the integers. We give a sufficient condition in the general framework of Mackey functors for a canonical induction formula to be integral. As an application we show how canonical induction formulae allow the construction of functorial maps on representation rings in terms of functorial maps on subrings, as for example the span of linear characters in the case of the canonical Brauer induction formula. This will be used in a subsequent article in the case of Adams operations and Chern classes.

Introduction

It is always very pleasant if a question about some higher dimensional object can be reduced to one-dimensional objects, as it is the case with the splitting

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principle for vector bundles (cf. [18, 17.5]) or with Brauer's induction theorem for character rings of a finite group (cf. [8]). In both cases one knows that for each higher dimensional object there exists a family of one-dimensional objects by which the object itself or at least its interesting invariants are determined. These existence theorems have already fundamental consequences. In the case of Brauer's theorem (and this was Brauer's incentive) one deduces easily from the classically known facts for one-dimensional representations that the Artin L-function of an arbitrary complex Galois representation of a number field is meromorphically extendible to the whole complex plane and satisfies a functional equation. For this kind of qualitative questions the existence statement in Brauer's theorem is sufficient. If one wants to study more detailed properties, as for example Artin's conjecture, stating that the Artin L-function of a non-trivial irreducible representation is holomorphic, a more explicit version of this theorem would be desirable.

Another important area in number theory is the conjectured Langlands correspondence between Galois representations of a local field K and representations of $\mathrm{GL}_n(K)$ for varying $n \in \mathbb{N}$. In this case again, one has a correspondence with all the required properties for the subsets of one-dimensional Galois representations and representations of $\mathrm{GL}_1(K)$ via local class field theory. By Brauer's theorem one can associate to each Galois representation a family of one-dimensional Galois representations on finite extension fields; this determines via class field theory a family of representations of the unit groups of these extension fields, and one would like to have an induction process which associates to this family a (virtual) representation of $\mathrm{GL}_n(K)$ for some $n \in \mathbb{N}$. Let us assume for the moment that there is such a notion of induction. Then still one has the problem that Brauer's theorem is only an existence theorem, and one has to show that a construction as indicated above would not depend on the special choice of a family of one-dimensional representations.

Surprisingly there are canonical choices for Brauer's induction theorem, as Snaith proved in [23] by topological methods. Independently, using an algebraic approach the author introduced another canonical choice (cf. [2]). Both choices share the property of functoriality with respect to group homomorphisms. However, while the topological formula is not additive but gives a topological interpretation to the multiplicities of the occurring one-dimensional representations in terms of Euler characteristics (showing a priori that they are integers), the algebraic formula is additive but produces a priori only multiplicities in $\mathbb{Q}$. In [2] we gave a proof for the integrality of these coefficients which made heavy use of the fact that we are dealing with complex characters, in contrast to the construction of the coefficients which was quite formal and invited to generalization. There is also a geometric interpretation of the algebraic formula due to Symonds, cf. [27], but like the topological approach this method seems not to be apt to generalize to repesentations over other rings than $\mathbb{C}$ or $\mathbb{R}$.

There exists already a variety of applications of these canonical Brauer induction formulae, cf. [5], [24], [25], [26], and [27], and one would certainly like to have similar constructions for various other representation rings besides the character ring. The aim of this paper is to introduce the basic constructions, notions and results about a very general approach, producing canonical choices of induction formulae for many sorts of representation rings. We consider this article as a reference for future examples and applications. Therefore some parts are kept in a more general framework than necessary for the five examples of canonical induction formulae we already introduce here (see Examples 1.8 and 6.13), namely for the classical case of the character ring, the ring of Brauer characters, the Grothendieck group of projective modules for a finite group over a complete discrete valuation ring $\mathcal{O}$, and the representation rings of trivial source modules and linear source modules over $\mathcal{O}$.

The language of Mackey functors on a finite group G over a commutative base ring k is perfectly suited to formulate our results and constructions, and also motivates the idea of considering *functorial* choices of induction formulae. The notion of a Mackey functor axiomatizes structures allowing induction, restriction, and conjugation maps, as for example character rings or cohomology groups of G-modules.

We introduce Mackey functors and related structures (restriction functors and conjugation functors) in Section 1, where we also provide the necessary facts about them. Restriction functors and conjugation functors arise from Mackey functors by forgetting the induction maps and then also the restriction maps. In Section 2 we define basic functors between these three categories, namely the consturctions $-_+$ and $-^+$ which arise from adjoints of the forgetful functors. The functor $-_+$ generalizes the construction of the Burnside ring from the constant restriction functor $\mathbb{Z}$, and there is also a generalization of the mark homomorphism on the Burnside ring. In Section 3 we motivate and define the notion of a canonical induction formula as a morphism of restriction functors $a\colon M \to A_+$, which splits the induction morphism $b\colon A_+ \to M$, for a Mackey functor M and a restriction subfunctor A of M, where $A_+(G)$ is a quotient of $\bigoplus_{H\leq G} A(H)$ and $b\colon A_+(G) \to M(G)$ is induced by the induction maps. If M is the character ring Mackey functor and $A \subseteq M$ is spanned by the linear characters, we obtain as an example the canonical induction formula in [2].

Section 4 gives a glimpse of future applications by extending morphisms on A to morphisms on M using a canonical induction formula. We will apply these results in subsequent papers to constructions of Adams operations and Chern classes on various representation rings different from the character ring.

In Section 5 we obtain for suitable base rings a parametrization of all morphisms of restriction functors $a\colon M \to A_+$ by the more convenient set of morphisms $p\colon M \to A$ of conjugation functors. Section 6 is devoted to the situation where the order of G is invertible in the base ring. Under this hypothesis we obtain complete answers in terms of p to the questions, if a is a splitting for b, a morphism of Mackey functors, or a ring homomorphism. In Section 7 we study the effect of a change of base rings with respect to the functors $-_+$ and $-^+$, in order to be able to pass from results over $\mathbb{Q}$, which we proved in Section 7, to result over $\mathbb{Z}$, the case we are most interested in.

We place ourselves in a more specific situation in Section 8, where we work with the base ring $\mathbb{Z}$ and impose in Hypothesis 8.1 conditions on M and A which are satisfied in all the examples we are interested in. Crucial among them for what follows is the notion of a stable basis $\mathcal{B}$ of A, which, as for example the set of one-dimensional characters, is a $\mathbb{Z}$-basis which is stable under conjugation maps and whose positive span is stable under restriction maps. For M, A, $p\colon M \to A$, and $\mathcal{B}$ as in Hypothesis 8.1 we may tensor all objects with $\mathbb{Q}$ over $\mathbb{Z}$ in order to obtain an associated map $a\colon \mathbb{Q}\otimes M \to \mathbb{Q}\otimes A_+$ by our previous results. The most interesting question then is, whether a is integral, i.e. a maps M to A_+. In Section 9 we transform an explicit alternating sum formula for a in terms of p, in which the denominator $|H|$ occurs for each subgroup $H \leq G$, by refining the index set from the simplicial complex of chains of subgroups of G to the simplicial chain complex of a poset associated to the stable basis $\mathcal{B}$. A further subtle refinement and rearrangement in this alternating sum makes apparent that under some condition $(*_\pi)$ on M, A, p, and $\mathcal{B}$ (π being a set of primes) we obtain an alternating sum formula whose denominators divide $|G|_{\pi'}$. This is the statement of Theorem 9.3 which provides a general tool for integrality proofs. We show that $(*_\pi)$ is satisfied for the set π of all primes in the cases of the character ring and the Brauer character ring, and for the set π of primes distinct from the prime characteristic l of the residue field of $\mathcal{O}$ in the case of the Grothendieck group of projective $\mathcal{O}G$-modules. In the first two cases this completes the integrality proof, and in the third case we give an argument which shows that the remaining l-power denominators also vanish. An integrality proof using Theorem 9.3 for the introduced induction formulae in the case of trivial source modules and linear source can be found in [4].

In Section 10 we adopt a global point of view by considering M and A to be defined on all finite groups instead of only the subgroups of a given finite group, and we show how our previous results can be extended to this point of view. Moreover, we assume that there are restriction maps for all group homomorphisms, not only the subgroup inclusions, and show that the canonical induction formulae also commute with these restriction maps.

Finally, in Section 11 we give a method of computing a canonical induction formula $a\colon M \to A_+$ in the standard situation of Hypothesis 8.1 by giving a matrix equation $\Gamma \cdot \alpha(m) = \beta(m)$ for the coefficients $\alpha(m)$ of $a_G(m) \in A_+(G)$, $m \in M(G)$, with respect to a special basis in $A_+(G)$, where Γ is an upper triangular quadratic matrix with coefficients in $\mathbb{Z}$ which is independent of m.

Notation

Let G be a finite group, $g \in G$, and $K, H \leq G$ be subgroups of G. We set ${}^gH = gHg^{-1}$ and $H^g = g^{-1}Hg$, write $H < G$ if H is a proper subgroup of G, and $K =_G H$ if K and H are conjugate under G. For a set π of primes we denote by g_π the element of $< g >$ whose order is precisely the π-part of the order of g. The exponent of G is denoted by $\exp(G)$ and the set of complex irreducible characters of G by $\mathrm{Irr}(G)$. For a G-set S we denote by S^G the set of G-fixed points. If k is a commutative ring, kG denotes the group ring and $I(kG)$ the augmentation ideal of kG, i.e. the k-submodule of kG generated by the elements $g-1$, $g \in G$. All rings and algebras are unitary, and homomorphisms of rings and algebras always preserve unities. For a ring R its group of units is denoted by $R^\times$, the category of R-modules by $R-\mathbf{Mod}$, and the full subcategory of finitely generated R-modules by $R-\mathbf{mod}$. For the rank of a free R-module $M \in R-\mathbf{mod}$ we write $\mathrm{rk}_R M$. Unadorned tensor products are taken over $\mathbb{Z}$.

1 Mackey functors and related structures

Throughout this section G denotes a finite group and k a commutative ring.

1.1 Definition (a) A ***k-conjugation-functor*** (resp. ***k-algebra conjugation functor***) on G is a pair (X, c) consisting of a family of k-modules (resp. k-algebras) $X(H)$, $H \leq G$, and a family of k-module (resp. k-algebra) homomorphisms $c_{g,H}\colon X(H) \to X({}^gH)$, the ***conjugation*** maps, for $H \leq G$ and $g \in G$, satisfying the axioms

(C1) $c_{h,H} = \mathrm{id}_{X(H)}$ (Triviality),

(C2) $c_{g'g,H} = c_{g',{}^gH} \circ c_{g,H}$ (Transitivity),

for all $h \in H \leq G$ and $g, g' \in G$.

(b) A ***k-restriction functor*** (resp. ***k-algebra restriction functor***) on G is a triple (A, c, res) consisting of a k-conjugation (resp. k-algebra conjugation) functor (A, c) on G together with a family of k-module (resp. k-algebra) homomorphisms $\mathrm{res}^H_K\colon A(H) \to A(K)$, the ***restriction*** maps, for $K \leq H \leq G$, satisfying the axioms

(R1) $\mathrm{res}^H_H = \mathrm{id}_{A(H)}$ (Triviality),

(R2) $\mathrm{res}^K_L \circ \mathrm{res}^H_K = \mathrm{res}^H_L$ (Transitivity),

(R3) $c_{g,K} \circ \mathrm{res}^H_K = \mathrm{res}^{{}^gH}_{{}^gK} \circ c_{g,H}$ (G-equivariance),

for all $L \leq K \leq H \leq G$ and $g \in G$.

(c) A ***k-Mackey functor*** on G is a quadruple $(M, c, \mathrm{res}, \mathrm{ind})$ consisting of a k-restriction functor (M, c, res) and a family of k-module homomorphisms $\mathrm{ind}^H_K\colon M(K) \to M(H)$, the ***induction*** maps, for $K \leq H \leq G$, satisfying the axioms

(M1) $\mathrm{ind}^H_H = \mathrm{id}_{M(H)}$ (Triviality),

(M2) $\mathrm{ind}^H_K \circ \mathrm{ind}^K_L = \mathrm{ind}^H_L$ (Transitivity),

(M3) $c_{g,H} \circ \mathrm{ind}^H_K = \mathrm{ind}^{{}^gH}_{{}^gK} \circ c_{g,K}$ (G-equivariance),

(M4) $\mathrm{res}^H_U \circ \mathrm{ind}^H_K = \sum_{h \in U\backslash H/K} \mathrm{ind}^U_{U\cap {}^hK} \circ \mathrm{res}^{{}^hK}_{U\cap {}^hK} \circ c_{h,K}$ (Mackey-formula),

for all $L \leq K \leq H \leq G$, $U \leq H$, and $g \in G$, where in (M4), h runs through a set of representatives in H for the double cosets $U\backslash H/K$.

A ***k-Green functor*** on G is a k-Mackey functor $(M, c, \mathrm{res}, \mathrm{ind})$ on G such that each $M(H)$, $H \leq G$, is a k-algebra, the conjugation and restriction maps are k-algebra homomorphisms and the axioms

(M5) $x \cdot \mathrm{ind}^H_K(y) = \mathrm{ind}^H_K(\mathrm{res}^H_K(x) \cdot y)$,

$$\mathrm{ind}_K^H(y) \cdot x = \mathrm{ind}_K^H(y \cdot \mathrm{res}_K^H(x)), \text{ (Frobenius axioms)}$$

are satisfied for all $K \le H \le G$, $x \in M(H)$, and $y \in M(K)$.

(d) A ***morphism*** $f\colon X \to Y$ of k-conjugation (resp. k-restriction, resp. k-Mackey functors) X and Y on G (by abuse of notation we often write X instead of (X, c), etc.) is a family of k-module homomorphisms $f_H\colon X(H) \to Y(H)$, $H \le G$, commuting with conjugation maps (resp. conjugation and restriction maps, resp. conjugation, restriction, and induction maps). For a morphism of k-algebra conjugation functors (resp. k-algebra restriction functors, resp. k-Green functors) on G we require additionally that f_H is a k-algebra homomorphism for all $H \le G$.

1.2 Remark (a) From Definition 1.1 we obtain categories $k\mathbf{-Con}(G)$, $k\mathbf{-Con}_{\mathrm{alg}}(G)$, $k\mathbf{-Res}(G)$, $k\mathbf{-Res}_{\mathrm{alg}}(G)$, $k\mathbf{-Mack}(G)$, and $k\mathbf{-Mack}_{\mathrm{alg}}(G)$. In each of these six categories we have an obvious notion of ***subfunctors***. ***Injectivity*** (resp. ***surjectivity***) of a morphism $f\colon X \to Y$ means injectivity (resp. surjectivity) of all $f_H\colon X(H) \to Y(H)$, $H \le G$. The categories $k\mathbf{-Con}(G)$, $k\mathbf{-Res}(G)$, and $k\mathbf{-Mack}(G)$ are abelian. For any map of commutative rings $k \to k'$ there are scalar extension functors

$$\begin{aligned} k' \otimes_k -\colon k\mathbf{-Con}(G) \to k'\mathbf{-Con}(G), \quad \ldots \quad , \\ k\mathbf{-Mack}_{\mathrm{alg}}(G) \to k'\mathbf{-Mack}_{\mathrm{alg}}(G). \end{aligned}$$

(b) Let X be an object in one of the six categories in part (a). Then the axioms (C1) and (C2) imply that the conjugation maps are isomorphisms and that they provide each $X(H)$, $H \le G$, with a module structure over the group ring $kN_G(H)/H$. In the sequel we will often write ${}^g x$ instead of $c_{g,H}(x)$ for $g \in G$, $H \le G$, and $x \in X(H)$. If $\mathcal{R}_G$ denotes a set of representatives for the conjugacy classes of subgroups of G, then clearly $k\mathbf{-Con}(G) \cong \prod_{H \in \mathcal{R}_G} kN_G(H)/H - \mathbf{Mod}$.

1.3 Example (a) The Burnside rings $\Omega(H)$ and the cohomology groups $\mathrm{H}^n(H, V)$, $H \le G$, for fixed $n \in \mathbb{N}_0$ and $V \in \mathbb{Z}G\mathbf{-Mod}$, are examples of $\mathbb{Z}$-Mackey functors on G (even of a $\mathbb{Z}$-Green functor in the Burnside ring case). We refer to [12, §80] for the notation and basic results concerning the Burnside ring.

(b) There is a constant k-algebra restriction functor $\underline{k}$ with $\underline{k}(H) = k$ for all $H \le G$, and with all conjugation and restriction maps being the identity. Obviously $\underline{k}$ is an initial object in the category $k\mathbf{-Res}_{\mathrm{alg}}(G)$. In Section 2 we will see that the Burnside ring Green functor $k \otimes \Omega$ arises from $\underline{k}$ by a functor $-_+$.

1.4 Definition (cf. [15], [31], [28])

(a) Let $X, Y, Z \in k\mathbf{-Mack}(G)$. A ***pairing*** $X \otimes_k Y \to Z$ is a family of k-module homomorphisms

$$X(H) \otimes_k Y(H) \to Z(H), \quad x \otimes_k y \mapsto x \cdot y,$$

satisfying the axioms

(P1) ${}^g(x \cdot y) = {}^gx \cdot {}^gy$ (G-equivariance),

(P2) $\mathrm{res}^H_K(x \cdot y) = \mathrm{res}^H_K(x) \cdot \mathrm{res}^H_K(y)$ (compatibility with restrictions),

(P3) $\mathrm{ind}^H_K(a) \cdot b = \mathrm{ind}^H_K(a \cdot \mathrm{res}^H_K(b))$,
$a' \cdot \mathrm{ind}^H_K(b') = \mathrm{ind}^H_K(\mathrm{res}^H_K(a') \cdot b')$, (Frobenius axioms),

for $K \leq H \leq G$, $g \in G$, $a', x \in X(H)$, $b, y \in Y(H)$, $a \in X(K)$, $b' \in Y(K)$.

(b) Let A be a k-Green functor on G. A k-Mackey functor M on G together with a pairing $A \otimes_k M \to M$ is called an A-***module***, if $A(H) \otimes_k M(H) \to M(H)$ provides each $M(H)$, $H \leq G$, with an $A(H)$-module structure.

If $A(H)$ is commutative for all $H \leq G$, then an A-***algebra*** is a k-Green functor B on G together with a morphism $i\colon A \to B$ of k-Green functors such that each $B(H)$, $H \leq G$, is an $A(H)$-algebra via i_H.

For a finite left G-set S we denote the class of S in the Burnside ring by $[S] \in \Omega(G)$. The elements $[G/H] \in \Omega(G)$, where H runs through a set of representatives for the conjugacy classes of subgroups of G, form a $\mathbb{Z}$-basis of $\Omega(G)$ and also a k-basis of $k \otimes \Omega(G)$. The following proposition (whose proof is left to the reader) points out the distinguished role the Burnside ring functor $k \otimes \Omega \in k\mathrm{-}\mathbf{Mack}_{\mathrm{alg}}(G)$ plays for the categories $k\mathrm{-}\mathbf{Mack}(G)$ and $k\mathrm{-}\mathbf{Mack}_{\mathrm{alg}}(G)$, namely the same role that $\mathbb{Z}$ plays for the categories of abelian groups (i.e. $\mathbb{Z}$-modules) and the categories of rings (i.e. $\mathbb{Z}$-algebras).

1.5 Proposition ([15, Prop. 4.2], [31, Ex. 2.11], [28, Prop. 6.1])

(i) *Every k-Mackey functor M on G has a unique structure of a $k \otimes \Omega$-module, namely*

$$(k \otimes \Omega(H)) \otimes_k M(H) \to M(H), \quad [H/K] \otimes_k m \mapsto \mathrm{ind}^H_K(\mathrm{res}^H_K(m)),$$

for $K \leq H \leq G$ and $m \in M(H)$.

(ii) *For every k-Green functor A on G there is a unique morphism $i\colon k \otimes \Omega \to A$ of k-Green functors on G, namely*

$$i_H\colon k \otimes \Omega(H) \to A(H), \quad [H/K] \mapsto \mathrm{ind}^H_K(1_{A(K)}),$$

for $K \leq H \leq G$, i.e. $k \otimes \Omega$ is an initial object in $k\mathrm{-}\mathbf{Mack}_{\mathrm{alg}}(G)$. In particular, A has a unique $k \otimes \Omega$-algebra structure. Moreover, i_H induces the unique $k \otimes \Omega$-module structure of A. □

1.6 For a k-Mackey functor M on G and $H \leq G$ we define the k-submodule

$$\mathcal{I}(M)(H) := \sum_{K<H} \mathrm{ind}^H_K(M(K)) = \sum_{K<H} \mathrm{im}(\mathrm{ind}^H_K\colon M(K) \to M(H))$$

of $M(H)$. Axiom (M3) implies that $\mathcal{I}(M)$ is a k-conjugation subfunctor of M on G. Since morphisms of k-Mackey functors commute with induction maps, these submodules are preserved under such morphisms, and we obtain a functor

$$\mathcal{I}\colon k\mathrm{-}\mathbf{Mack}(G) \to k\mathrm{-}\mathbf{Con}(G).$$

Note that if $M \in k-\mathbf{Mack}_{\mathrm{alg}}(G)$, then $\mathcal{I}(M)(H)$ is an ideal of $M(H)$ for $H \leq G$ by the Frobenius axiom (M5).

Following Thévenaz (cf. [28]), we call a subgroup H of G ***primordial*** for M, if $\mathcal{I}(M)(H) \neq M(H)$, i.e. if there is an element in $M(H)$ which can not be obtained as a sum of properly induced elements. We denote the set of primordial subgroup for M by $\mathcal{P}(M)$. Note that for $H \leq G$ one has $M(H) = \sum_{K \leq H, K \in \mathcal{P}(M)} \mathrm{ind}_K^H(M(K))$.

1.7 Dually to 1.6 we define for $A \in k-\mathbf{Res}(G)$ and $H \leq G$ the k-submodule

$$\mathcal{K}(A)(H) := \bigcap_{K<H} \ker(\mathrm{res}_K^H \colon A(H) \to A(K))$$

of $A(H)$. These submodules form a k-conjugation subfunctor of A and they are preserved under morphisms of restriction functors on G. Hence, we obtain a functor

$$\mathcal{K} \colon k-\mathbf{Res}(G) \to k-\mathbf{Con}(G).$$

Note that if $A \in k-\mathbf{Res}_{\mathrm{alg}}(G)$, then $\mathcal{K}(A)(H)$ is an ideal in $A(H)$, since the restriction maps are k-algebra homomorphisms.

A subgroup $H \leq G$ is called ***coprimordial*** for A, if $\mathcal{K}(A)(H) \neq 0$, i.e. if the elements in $A(H)$ are not uniquely determined by proper restriction maps. We denote the set of coprimordial subgroups for A by $\mathcal{C}(A)$. Note that for $H \leq G$ two elements $x, y \in A(H)$ are equal if and only if $\mathrm{res}_K^H(x) = \mathrm{res}_K^H(y)$ for all $K \leq H$ with $K \in \mathcal{C}(A)$. More about coprimordial subgroups and the connection to primordial subgroups can be found in [1, III.1.11–1.18].

1.8 Example For a ring A and a group H we set $\widehat{H}(A) := \mathrm{Hom}(H, A^\times)$, and regard it as a multiplicative abelian group in the case that A is commutative. In this paper we will mainly be interested in the following examples.

(a) The character rings $R(H)$, $H \leq G$, i.e. the free abelian groups on the sets $\mathrm{Irr}(H)$ of complex irreducible characters of H, form a $\mathbb{Z}$-Green functor on G with the usual conjugation, restriction and induction maps. For $H \leq G$ let $R^{\mathrm{ab}}(H) \subseteq R(H)$ denote the $\mathbb{Z}$-span of the subset $\widehat{H} := \widehat{H}(\mathbb{C}) \subset \mathrm{Irr}(H)$ of linear characters. Then R^{ab} with the inherited conjugation and restriction maps is a $\mathbb{Z}$-algebra restriction functor on G. Note that, since an induced linear character may have non-linear constituents, $R^{\mathrm{ab}} \subseteq R$ is not a Mackey subfunctor. The set $\mathcal{C}(R)$ of coprimordial subgroups for R consists of the cyclic subgroups and the set $\mathcal{P}(R)$ of primordial subgroups consists of the elementary subgroups of G.

(b) Let F be an algebraically closed field of prime characteristic $l > 0$. The Grothendieck rings $R_F(H)$ of $FH-\mathbf{mod}$, $H \leq G$, with respect to short exact sequences, which we identify with the free abelian group on the set of isomorphism classes $[V]$ of irreducible FH-modules V, form a $\mathbb{Z}$-Green functor on G, and the subrings $R_F^{\mathrm{ab}}(H)$, $H \leq G$, generated by the isomorphism classes $[V]$ of FH-modules V with $\dim_F V = 1$ form a $\mathbb{Z}$-algebra restriction subfunctor of R_F. For a homomorphism $\varphi\colon H \to F^\times$ we denote by F_φ the F-vector space F endowed with the H-action $h \cdot \alpha = \varphi(h)\alpha$ for $h \in H$, $\alpha \in F$. For $\varphi = 1$ we write F instead of F_1. Then the map $\varphi \mapsto F_\varphi$ induces a bijection between $\widehat{H}(F)$ and the set of isomorphism classes of one-dimensional FH-modules, and we will often identify φ with $[F_\varphi]$, thus considering $\widehat{H}(F)$ as $\mathbb{Z}$-basis of $R_F^{\mathrm{ab}}(H)$. From the theory of Brauer characters it follows that the set $\mathcal{C}(R_F)$ of coprimordial subgroups is the set of cyclic l'-subgroups.

(c) Let $\mathcal{O}$ be a complete discrete valuation ring of characterstic zero containing a primitive $\exp(G)$-th root of unity and assume that the residue field $\mathcal{O}/\mathrm{rad}(\mathcal{O})$ is of positive characteristic l and algebrically closed. A ***linear source*** $\mathcal{O}G$-module M is an $\mathcal{O}G$-module, free of finite rank as $\mathcal{O}$-module, such that the source of any of its indecomposable direct summands is a module of $\mathcal{O}$-rank one. Equivalently, M is isomorphic to a direct summand of a ***monomial*** $\mathcal{O}G$-module, i.e. a direct sum of modules of the form $\mathrm{ind}_H^G(\mathcal{O}_\varphi)$ for some $H \leq G$ and $\varphi \in \widehat{H}(\mathcal{O})$. Still equivalent is the condition that $\mathrm{res}_P^G(M)$ is a monomial $\mathcal{O}P$-module for all l-subgroups (or equivalently for a Sylow l-subgroup) P of G. These equivalences can be proved like the analogous statements on trivial source modules in [10]. Together with the usual conjugation, restriction, and induction maps, the Grothendieck rings $L_\mathcal{O}(H)$ with respect to direct sums of the categories $\mathcal{O}H-\mathbf{lin}$ of linear source $\mathcal{O}H$-modules, $H \leq G$, form a $\mathbb{Z}$-Green functor on G. Since direct summands of linear source modules are again linear source modules, $L_\mathcal{O}(G)$ is free on the isomorphism classes of indecomposable linear source $\mathcal{O}G$-modules.

Note that, as in (b), the map $\varphi \mapsto \mathcal{O}_\varphi$, for $\varphi \in \widehat{G}(\mathcal{O})$, induces a bijection between $\widehat{G}(\mathcal{O})$ and the set of isomorphism classes of linear source $\mathcal{O}G$-modules of $\mathcal{O}$-rank one. For $H \leq G$ we denote the span of $[\mathcal{O}_\varphi] \in L_\mathcal{O}(H)$, $\varphi \in \widehat{H}(\mathcal{O})$, by $L_\mathcal{O}^{\mathrm{ab}}(H)$, and obtain a $\mathbb{Z}$-algebra restriction subfunctor $L_\mathcal{O}^{\mathrm{ab}} \subseteq L_\mathcal{O}$. For more on linear source modules, i.e. a proof of the semisimplicity of $\mathbb{C} \otimes L_\mathcal{O}(G)$ and the determination of its species, i.e. algebra homomorphisms $\mathbb{C} \otimes L_\mathcal{O}(G) \to \mathbb{C}$, see [4].

(d) Let $\mathcal{O}$ be as in (c). There is a full subcategory $\mathcal{O}G-\mathbf{triv} \subseteq \mathcal{O}G-\mathbf{lin}$ of ***trivial source*** $\mathcal{O}G$-modules, i.e. linear source $\mathcal{O}G$-modules, all of whose indecomposable summands have trivial source. For $H \leq G$ the Grothendieck ring $T_\mathcal{O}(H)$ of $\mathcal{O}H-\mathbf{triv}$ is a subring of $L_\mathcal{O}(H)$, and they form a $\mathbb{Z}$-Green subfunctor $T_\mathcal{O}$ of $L_\mathcal{O}$. It is easy to see that an $\mathcal{O}H$-module of the form $\mathcal{O}_\varphi$, $\varphi \in \widehat{H}(\mathcal{O})$, is a trivial source module, if and only if $H/\ker(\varphi)$ is an l'-group. We define $T_\mathcal{O}^{\mathrm{ab}}(H) \subseteq L_\mathcal{O}^{\mathrm{ab}}(H)$ as the span of the elements $[\mathcal{O}_\varphi]$, $\varphi \in \widehat{H}(\mathcal{O})_{l'}$. Clearly, $T_\mathcal{O}^{\mathrm{ab}}$ is a $\mathbb{Z}$-algebra restriction subfunctor of $T_\mathcal{O}$.

Note that in the literature trivial source $\mathcal{O}G$-modules are often called l-permutation modules, since they are exactly those $\mathcal{O}G$-modules M whose restriction $\mathrm{res}_P^G(M)$ is a permutation module for all l-subgroups (or equivalently for a Sylow l-subgroup) P of G. Also equivalent is the condition that M is isomorphic to a direct summand of a permutation module. Since a direct summand of a trivial source module is again a trivial source module, $T_\mathcal{O}(G)$ is a free abelian group on the set of isomorphism classes of indecomposable trivial source $\mathcal{O}G$-modules.

We remark that for a field F of characteristic l as in (b), the similarly defined Green functor T_F is isomorphic to $T_{\mathcal{O}}$, and T_F^{ab} is isomorphic to $T_{\mathcal{O}}^{\mathrm{ab}}$, cf. [10]. In fact, reduction modulo rad($\mathcal{O}$) induces an isomorphism $T_{\mathcal{O}} \cong T_{\mathcal{O}/\mathrm{rad}(\mathcal{O})}$.

Since $T_{\mathcal{O}}(H)$ and $L_{\mathcal{O}}(H)$ are $\mathbb{Z}$-free for all $H \leq G$, we have equalities $\mathcal{C}(T_{\mathcal{O}}) = \mathcal{C}(\mathbb{Q} \otimes T_{\mathcal{O}})$ and $\mathcal{C}(L_{\mathcal{O}}) = \mathcal{C}(\mathbb{Q} \otimes L_{\mathcal{O}})$. Moreover, in Proposition 6.2 we will show that $\mathcal{C}(\mathbb{Q} \otimes T_{\mathcal{O}}) = \mathcal{C}(\mathbb{Q} \otimes L_{\mathcal{O}})$, since $T_{\mathcal{O}} \subseteq L_{\mathcal{O}}$ is an inclusion of Green functors. From Conlon's induction theorem (cf. [12, Cor. 81.32]) it follows that $\mathcal{C}(\mathbb{Q} \otimes T_{\mathcal{O}})$ is the set of l-hypo-elementary subgroups, i.e. subgroups H, whose biggest normal l-subgroup has a cyclic l'-group as factor group, i.e. $H/O_l(H)$ is cyclic.

(e) Let $\mathcal{O}$ be as in (c) and let $\mathcal{O}G-\mathbf{proj}$ be the category of finitely generated projective $\mathcal{O}G$-modules. Since each projective $\mathcal{O}G$-module is a direct summand of a free module, $\mathcal{O}G-\mathbf{proj}$ is a full subcategory of $\mathcal{O}G-\mathbf{triv}$. The Grothendieck groups $P_{\mathcal{O}}(H)$, of $\mathcal{O}H-\mathbf{proj}$, $H \leq G$, with respect to direct sums, form a Mackey subfunctor $P_{\mathcal{O}} \subseteq T_{\mathcal{O}} \subseteq L_{\mathcal{O}}$ and are free abelian groups on the isomorphism classes of indecomposable projective $\mathcal{O}H$-modules. A rank-one module $\mathcal{O}_\varphi$, $\varphi \in \widehat{H}(\mathcal{O})$, $H \leq G$, is projective, if and only if H is an l'-group. Hence, we set $P_{\mathcal{O}}^{\mathrm{ab}}(H) = T_{\mathcal{O}}^{\mathrm{ab}}(H) = L_{\mathcal{O}}^{\mathrm{ab}}(H)$ for l'-subgroups $H \leq G$, and $P_{\mathcal{O}}^{\mathrm{ab}}(H) = 0$ otherwise, and obtain a $\mathbb{Z}$-restriction subfunctor $P_{\mathcal{O}}^{\mathrm{ab}} \subseteq P_{\mathcal{O}}$.

Note again that if F is a field of characteristic $l > 0$ as in (b), then for the similarly defined Mackey functor P_F and restriction functor P_F^{ab} we have $P_F \cong P_{\mathcal{O}}$ and $P_F^{\mathrm{ab}} \cong P_{\mathcal{O}}^{\mathrm{ab}}$.

Moreover, by [22, Théorèmes 34, 36], the Mackey functor P_F is isomorphic to the subfunctor of the character ring functor R consisting of those virtual characters vanishing on l-singular elements, i.e. elements of order divisible by l. This shows that $\mathcal{C}(P_F) = \mathcal{C}(P_{\mathcal{O}})$ is the set of cyclic l'-subgroups of G.

2 The two plus-constructions and the mark morphism

Let G and k be given as in Section 1. We are going to define two functors

$$-^+ \colon k-\mathbf{Con}(G) \to k-\mathbf{Mack}(G)$$

and

$$-_+ \colon k-\mathbf{Res}(G) \to k-\mathbf{Mack}(G)$$

together with a natural transformation

$$\rho^A \colon A_+ \to A^+$$

for every $A \in k-\mathbf{Res}(G)$. These constructions generalize well-known features of the Burnside ring. It will be obvious from the constructions that

$$\underline{k}_+ \cong k \otimes \Omega(G) \quad \text{and} \quad \underline{k}^+(G) \cong \Big(\prod_{H \leq G} k\Big)^G,$$

where G acts by permuting the components in the last product according to the conjugation action on the subgroups $H \leq G$, and that under these identifications the map $\rho_G^{\mathbb{Z}}$ is just the classical mark homomorphism

$$\rho_G \colon \Omega(G) \to \big(\prod_{H \leq G} \mathbb{Z}\big)^G, \quad [S] \mapsto (|S^H|)_{H \leq G}.$$

The functor $-^+$ is part of Thévenaz' definition of the twin functor of a Mackey functor (cf. [28, Sect. 4]). The functor $-_+$ was considered in special cases by Deligne in [13] and also by Dress in [14].

2.1 For $X \in k\text{-}\mathbf{Con}(G)$ we define $(X^+, c^+, \mathrm{res}^+, \mathrm{ind}^+) \in k\text{-}\mathbf{Mack}(G)$ by

$$X^+(H) := \big(\prod_{K \leq H} X(K)\big)^H$$

for $H \leq G$, where $h \in H$ acts on $\prod_{K \leq H} X(K)$ by the conjugation maps $(c_{h,K})_{K \leq H}$. Each $g \in G$ induces for $H \leq G$ a map

$$c_{g,H}^+ := \prod_{K \leq H} c_{g,K} \colon \big(\prod_{K \leq H} X(K)\big)^H \to \big(\prod_{K \leq H} X({}^gK)\big)^{{}^gH}.$$

For $K \leq H \leq G$ we define

$$\mathrm{res}^{+H}_{K} \colon \big(\prod_{L \leq H} X(L)\big)^H \to \big(\prod_{L \leq K} X(L)\big)^K$$

as the obvious projection map, and

$$\mathrm{ind}^{+H}_{K} := \sum_{h \in H/K} c_{h,K}^+ \colon \big(\prod_{L \leq K} X(L)\big)^K \to \big(\prod_{L \leq H} X(L)\big)^H$$

as the relative norm map, where we view the image of $c_{h,K}^+$ as contained in $\prod_{L \leq H} X(L)$ filling up the additional components with zeros.

For a morphism $f \colon X \to Y$ in $k\text{-}\mathbf{Con}(G)$ we define

$$f_H^+ := \prod_{K \leq H} f_K \colon \big(\prod_{K \leq H} X(K)\big)^H \to \big(\prod_{K \leq H} Y(K)\big)^H$$

for $H \leq G$. It is a straight forward verification to show that $-^+$ is a functor from $k\text{-}\mathbf{Con}(G)$ to $k\text{-}\mathbf{Mack}(G)$ and that the same definitions yield a functor $-^+ \colon k\text{-}\mathbf{Con}_{\mathrm{alg}}(G) \to k\text{-}\mathbf{Mack}_{\mathrm{alg}}(G)$.

2.2 For $(A, c, \mathrm{res}) \in k\text{-}\mathbf{Res}(G)$ we define $(A_+, c_+, \mathrm{res}_+, \mathrm{ind}_+) \in k\text{-}\mathbf{Mack}(G)$ as follows. For $H \leq G$ let

$$A_+(H) := \big(\bigoplus_{K \leq H} A(K)\big)_H,$$

where we view $\bigoplus_{K\leq H} A(K)$ as a kH-module via the sum of the conjugation maps $c_{h,K}\colon A(K) \to A({}^hK)$, $h \in H$, and where for a kH-module M the k-module M_H of coinvariants is defined as M/N, N being the smallest KH-submodule such that H acts trivially on M/N, i.e. $N = I(kH)M$, where $I(kH)$ is the augmentation ideal of kH. We will abbreviate the image of $a \in A(K)$, $K \leq H$, in $A_+(H)$ by $[K,a]_H$. Thus, if $\mathcal{R}_H$ is a set of representatives for the conjugacy classes of subgroups of H, we can write each element $x \in A_+(H)$ as

$$x = \sum_{K\in\mathcal{R}_H} [K, a_K]_H$$

for certain elements $a_K \in A(K)$, $K \in \mathcal{R}_H$, with

$$\sum_{K\in\mathcal{R}_H} [K, a_K]_H = \sum_{K\in\mathcal{R}_H} [K, a'_K]_H$$

for elements $a'_K \in A(K)$, $K \in \mathcal{R}_H$, if and only if there exist $n_K \in N_H(K)$ for each $K \in \mathcal{R}_H$ with $a_K = {}^{n_K}(a'_K)$.

For $K \leq H \leq G$ and $g \in G$ we define maps

$$c_{+g,H}\colon A_+(H) \to A_+({}^gH), \quad [U,a]_H \mapsto [{}^gU, {}^ga]_{{}^gH},$$
$$\mathrm{res}_{+K}^{\ H}\colon A_+(H) \to A_+(K), \quad [U,a]_H \mapsto \sum_{h\in K\backslash H/U} [K\cap {}^hU, \mathrm{res}^{{}^hU}_{K\cap {}^hU}({}^ha)]_K,$$
$$\mathrm{ind}_{+K}^{\ H}\colon A_+(K) \to A_+(H), \quad [V,b]_K \mapsto [V,b]_H,$$

where $U \leq H$, $V \leq K$, $a \in A(U)$, and $b \in A(V)$.

For a morphism $f\colon A \to B$ in $k\mathrm{-}\mathbf{Res}(G)$ and $H \leq G$ we set

$$f_{+H}\colon A_+(H) \to B_+(H), \quad [K,a]_H \mapsto [K, f_K(a)]_H,$$

where $K \leq H$ and $a \in A(K)$. Then again, straight forward calculations show that $-_+$ is a functor from $k\mathrm{-}\mathbf{Res}(G)$ to $k\mathrm{-}\mathbf{Mack}(G)$, and that the same definitions yield a functor $-_+\colon k\mathrm{-}\mathbf{Res}_{\mathrm{alg}}(G) \to k\mathrm{-}\mathbf{Mack}_{\mathrm{alg}}(G)$, where the k-algebra structure on $A_+(H)$ for $A \in k\mathrm{-}\mathbf{Res}_{\mathrm{alg}}(G)$ and $H \leq G$ is defined by

$$[U,a]_H \cdot [V,b]_H := \sum_{h\in U\backslash H/V} \left[U\cap {}^hV, \mathrm{res}^U_{U\cap {}^hV}(a)\cdot \mathrm{res}^{{}^hV}_{U\cap {}^hV}({}^hb)\right]_H,$$

where $U, V \leq H$, $a \in A(U)$, and $b \in A(V)$. If each $A(H)$, $H \leq G$, is commutative, so is each $A_+(H)$.

For $A \in k\mathrm{-}\mathbf{Res}(G)$ (resp. $A \in k\mathrm{-}\mathbf{Res}_{\mathrm{alg}}(G)$) and $H \leq G$ we define the map

$$\iota^A_H\colon A(H) \to A_+(H), \quad a \mapsto [H,a]_H$$

which is injective by Axiom (C1). This defines a morphism $\iota^A\colon A \to A_+$ of k-restriction functors (resp. k-algebra restriction functors) on G, which is functorial in A.

Note that if $A \in k\mathrm{-}\mathbf{Res}_{\mathrm{alg}}(G)$ is commutative then $A_+(H)$ is an $A(H)$-algebra via ι^A. If A is not commutative, this map is still a unitary ring homomorphism providing $A_+(H)$ with an $A(H)$-module structure: $a \cdot [K,b]_H = [K, \mathrm{res}^H_K(a)b]_H$, for $a \in A(H)$, $b \in A(K)$, $K \le H \le G$.

2.3 For $A \in k\mathrm{-}\mathbf{Res}(G)$ and $H \le G$ we define

$$\pi^A_H \colon A_+(H) \to A(H), \quad [K,a]_H \mapsto \begin{cases} a, & \text{if } K = H, \\ 0, & \text{if } K < H. \end{cases}$$

By Axiom (C1), π^A_H is well-defined, and it is clear that $\pi^A \colon A_+ \to A$ is a morphism of k-conjugation functors on G. Since $\pi^A_H \colon A_+(H) \to A(H)$ is surjective with kernel $\sum_{K<H} \mathrm{ind}_+K^H(A_+(K))$ for $H \le G$, we call π^A the ***Brauer morphism*** in analogy of the Brauer map in modular representation theory, cf. [12, §58A], and also [28, p. 29]. Moreover, for $H \le G$ and $x \in A_+(H)$ we call $\pi^A_H(x) \in A(H)$ the ***residue*** of x, a notion introduced and studied by Puig in [20] and by Thévenaz in [28].

Next, for $H \le G$, we define

$$\rho^A_H := (\pi^A_K \circ \mathrm{res}_+{}^H_K)_{K \le H} \colon A_+(H) \to A^+(H) = \Big(\prod_{K \le H} A(K)\Big)^H.$$

Again by routine calculations one can show that $\rho^A \colon A_+ \to A^+$ is a morphism in $k\mathrm{-}\mathbf{Mack}(G)$ which is natural in $A \in k\mathrm{-}\mathbf{Res}(G)$. If $A \in k\mathrm{-}\mathbf{Res}_{\mathrm{alg}}(G)$, then $\pi^A \colon A_+ \to A$ is in $k\mathrm{-}\mathbf{Con}_{\mathrm{alg}}(G)$, and therefore, $\rho^A \colon A_+ \to A^+$ is a morphism in $k\mathrm{-}\mathbf{Mack}_{\mathrm{alg}}(G)$.

As the following proposition shows, for each $H \le G$, the map

$$\begin{aligned} \sigma^A_H \colon A^+(H) &\to A_+(H), \\ (a_K)_{K \le H} &\mapsto \sum_{L \le K \le H} |L|\mu(L,K)[L, \mathrm{res}^K_L(a_K)]_H, \end{aligned} \tag{2.3.a}$$

is almost inverse to ρ^A_H, where $\mu(L,K)$ denotes the Möbius function of the poset of subgroups of H evaluated at (L,K), cf. [21] for generalities on Möbius functions.

2.4 Proposition *For $A \in k\mathrm{-}\mathbf{Res}(G)$ and $H \le G$ one has*

$$\sigma^A_H \circ \rho^A_H = |H| \cdot \mathrm{id}_{A_+(H)} \quad \textit{and} \quad \rho^A_H \circ \sigma^A_H = |H| \cdot \mathrm{id}_{A^+(H)}.$$

In particular, the kernels and cokernels of ρ^A_H and σ^A_H are annihilated by $|H|$. If $A_+(H)$ has trivial $|H|$-torsion, then ρ^A_H is injective, and if $|H|$ is invertible in k, then ρ^A_H is an isomorphism.

Proof For $U \le H$ and $a \in A(U)$ we have

$$\begin{aligned}\rho_H^A([U,a]_H) &= \big((\pi_K^A \circ \mathrm{res}_{+}{}_K^H)([U,a]_H)\big)_{K\le H}\\ &= \Big(\sum_{h\in K\backslash H/U} \pi_K^A([K\cap {}^hU, \mathrm{res}_{K\cap {}^hU}^{{}^hU}({}^ha)]_K)\Big)_{K\le H}\\ &= \Big(\sum_{\substack{h\in H/U\\ K\le {}^hU}} \mathrm{res}_K^{{}^hU}({}^ha)\Big)_{K\le H},\end{aligned}$$

since for $K \le H$ the relation $K \le {}^hU$ implies $KhU = hU$. Applying σ_H^A to this family we obtain

$$\begin{aligned}(\sigma_H^A \circ \rho_H^A)([U,a]_H) &= \sum_{L\le K\le H} |L|\mu(L,K) \sum_{\substack{h\in H/U\\ K\le {}^hU}} [L, \mathrm{res}_L^{{}^hU}({}^ha)]_H\\ &= \frac{1}{|U|}\sum_{h\in H}\sum_{L\le K\le {}^hU} \mu(L,K)|L|[L,\mathrm{res}_L^{{}^hU}({}^ha)]_H.\end{aligned}$$

Considering for $h \in H$ the map

$$f\colon \{L \le {}^hU\} \to A_+(H), \quad L \mapsto |L|[L, \mathrm{res}_L^{{}^hU}({}^ha)]_H,$$

the inner sum collapses by Möbius inversion (cf. [21, 3., Prop. 2]) to the element $|{}^hU|[{}^hU, {}^ha]_H = |U|[U,a]_H$ which yields

$$(\sigma_H^A \circ \rho_H^A)([U,a]_H) = \frac{1}{|U|}\sum_{h\in H} |U|[U,a]_H = |H|[U,a]_H.$$

Conversely, let $(a_K)_{K\le H} \in A^+(H)$. Then the U-component, $U \le H$, of the element $(\rho_H^A \circ \sigma_H^A)((a_K)_{K\le H}) \in A^+(H)$ is given by

$$\begin{aligned}&(\pi_U^A \circ \mathrm{res}_{+}{}_U^H \circ \sigma_H^A)((a_K)_{K\le H})\\ &\quad= \sum_{L\le K\le H} |L|\mu(L,K)\pi_U^A\big(\mathrm{res}_{+}{}_U^H([L,\mathrm{res}_L^K(a_K)]_H)\big)\\ &\quad= \sum_{L\le K\le H} |L|\mu(L,K) \sum_{\substack{h\in H/L\\ U\le {}^hL}} \mathrm{res}_U^{{}^hL}\big({}^h(\mathrm{res}_L^K(a_K))\big)\\ &\quad= \sum_{L\le K\le H} \mu(L,K) \sum_{\substack{h\in H\\ U\le {}^hL}} \mathrm{res}_U^{{}^hK}({}^h(a_K))\\ &\quad= \sum_{h\in H}\sum_{U^h\le L\le K\le H} \mu(L,K)\mathrm{res}_U^{{}^hK}({}^h(a_K)).\end{aligned}$$

Considering for each $h \in H$ the map

$$f\colon \{U^h \le K \le H\} \to A(U), \quad K \mapsto \mathrm{res}_U^{{}^hK}({}^ha_K),$$

the inner sum collapses by Möbius inversion (cf. [21, 3., Prop. 2]) to ${}^h a_{(U^h)} = a_U$, and we have

$$(\pi_U^A \circ \mathrm{res}_{+U}^H \circ \sigma_H^A)((a_K)_{K \le H}) = \sum_{h \in H} a_U = |H| a_U,$$

which completes the proof of the proposition. □

2.5 Remark Note that the above proposition reproves the explicit formula of Gluck, cf. [16] and [32], for the idempotents $e_H^{(G)} \in \mathbb{Q} \otimes \Omega(G)$, $H \le G$, (and also in $k \otimes \Omega(G)$, if $|G|$ is invertible in G), where $e_H^{(G)} = \rho_G^{-1}((a_K)_{K \le G})$ with $a_K = 1$ if $K =_G H$, and $a_K = 0$ otherwise:

$$e_H^{(G)} = \frac{1}{|N_G(H)|} \sum_{L \le H} |L| \mu(L, H)[G/L]. \tag{2.5.a}$$

3 The definition of a canonical induction formula

Throughout this section let k be a commutative ring, G a finite group, M a k-Mackey functor on G, and $A \subseteq M$ a k-restriction subfunctor of M, i.e. $A(H) \subseteq M(H)$, $H \le G$, are k-submodules and stable under the conjugation and restriction maps of M.

3.1 For $H \le G$ we define

$$b_H^{M,A} \colon A_+(H) \to M(H), \quad [K, a]_H \mapsto \mathrm{ind}_K^H(a),$$

where $K \le H$ and $a \in A(K)$. It is easy to see that $b^{M,A} \colon A_+ \to M$ is a morphism of k-Mackey functors on G, which we will call the ***induction morphism*** of M from A. If M is a k-Green functor and $A \subseteq M$ a k-algebra restriction subfunctor, then $b^{M,A}$ is a morphism of k-Green functors on G.

3.2 Remark and Example Let $k = \mathbb{Z}$, $M = R$ the character ring Green functor, and $A = R^{\mathrm{ab}}$ as in Example 1.8 (a). By Brauer's induction theorem (cf. [22, Théorème 20]), the induction morphism $b^{R,R^{\mathrm{ab}}} \colon R_+^{\mathrm{ab}} \to R$ is surjective. For fixed $H \le G$ and $\chi \in R(H)$, the 'different ways' of writing χ as a $\mathbb{Z}$-linear combination of induced linear characters correspond bijectively to the set of elements in $A_+(H)$ that are mapped to χ under $b_H^{R,R^{\mathrm{ab}}}$, provided that one agrees to identify two such linear combinations, if one arises from the other by replacing sumands $\mathrm{ind}_K^H(\psi)$ by conjugate summands $\mathrm{ind}_{{}^hK}^H({}^h\psi)$ for $K \le H$, $\psi \in \widehat{K}$, and $h \in H$.

Now we fix $H \le G$. If we want to specify for each $\chi \in R(H)$ a preferred way of writing χ as a $\mathbb{Z}$-linear combination of induced linear characters, this amounts to choosing a map $a_H \colon R(H) \to R_+^{\mathrm{ab}}(H)$ such that $b_H^{R,R^{\mathrm{ab}}} \circ a_H = \mathrm{id}_{R(H)}$. Of course there are many different possible choices for a_H in general.

Next we consider all subgroups $H \leq G$ simultaneously. Knowing that R_+^{ab} and R are Mackey functors on G and that $b^{R,R^{\mathrm{ab}}}$ is a morphism of Mackey functors, it is just natural to require that $a\colon R \to R_+^{\mathrm{ab}}$ be also a morphism of Mackey functors. However, there is no family of maps $a_H\colon R(H) \to R_+^{\mathrm{ab}}(H)$, $H \leq G$, commuting with induction maps and satisfying $b_H^{R,R^{\mathrm{ab}}} \circ a_H = \mathrm{id}_{R(H)}$ for all $H \leq G$. In fact, if we assume that $(a_H)_{H \leq G}$ is such a family, then Artin's induction theorem (cf. [12, 15.4]), namely

$$|H| \cdot R(H) \subseteq \sum_{\substack{K \leq H \\ K \text{ cyclic}}} \mathrm{ind}_K^H(R(K))$$

for $H \leq G$ implies that

$$|H| \cdot a_H(R(H)) \subseteq \sum_{\substack{K \leq H \\ K \text{ cyclic}}} \mathrm{ind}_{+}{}_K^H(R_+^{\mathrm{ab}}(K))$$

for $H \leq G$. Recalling the definition of $\mathrm{ind}_{+}{}_K^H$ for $K \leq H \leq G$ from 2.2 we observe from this inclusion that $|H| \cdot a_H(R(H))$ is contained in the $\mathbb{Z}$-span of the elements $[K, \psi]_H$, where $K \leq H$ is a cyclic group and $\psi \in \hat{K}$. Since these elements are part of a $\mathbb{Z}$-basis of $R_+^{\mathrm{ab}}(G)$ (as we will see in Lemma 7.2), also $a_H(R(H))$ must be contained in this span. But then $b_H^{R,R^{\mathrm{ab}}} \circ a_H = \mathrm{id}_{R(H)}$ implies

$$R(H) = \sum_{\substack{K \leq H \\ K \text{ cyclic}}} \mathrm{ind}_K^H(R(K))$$

which is certainly not true for arbitrary finite groups H. The best we can hope for in this example is a morphism $a\colon R \to R_+^{\mathrm{ab}}$ of $\mathbb{Z}$-*restriction* functors on G with $b^{R,R^{\mathrm{ab}}} \circ a = \mathrm{id}_R$. Such a morphism was constructed in [2].

The above considerations motivate the following definition.

3.3 Definition A ***canonical induction formula*** for M from A is a morphism $a \in k\text{–}\mathbf{Res}(G)(M, A_+)$ with $b^{M,A} \circ a = \mathrm{id}_M$.

3.4 Remark As examples we will introduce in this paper canonical induction formulae for R from R^{ab} (as already done in [2]), for R_F from R_F^{ab}, for $L_{\mathcal{O}}$ from $L_{\mathcal{O}}^{\mathrm{ab}}$, for $T_{\mathcal{O}}$ from $T_{\mathcal{O}}^{\mathrm{ab}}$, and for $P_{\mathcal{O}}$ from $P_{\mathcal{O}}^{\mathrm{ab}}$, the notation being the one introduced in Example 1.8.

Note that our notation is different from the one in [2]. What we denote here by $R_+^{\mathrm{ab}}(G)$ was denoted there by $R_+(G)$, which in view of 2.2 has a different meaning here.

4 An application: Extending morphisms

In this section we will indicate how to use canonical induction formulae in order to extend certain morphisms. In subsequent papers we will apply this to construct Adams operations and Chern classes on various representation rings.

4.1 Throughout this section we assume that K is a commutative ring and G is a finite group. We fix a k-restriction functor M on G (not necessarily a Mackey functor) and a k-restriction subfunctor $A \subseteq M$. Furthermore we assume that there is a morphism $a \in k\mathbf{-Res}(G)(M, A_+)$ such that $a_H(\varphi) = [H, \varphi]_H$ for all $H \leq G$ and $\varphi \in A(H)$, a condition which will be satisfied in all the examples of canonical induction formulae considered in this paper, namely for M and A as listed in Remark 3.4, cf. Proposition 6.12.

4.2 Definition For M, A, and $a\colon M \to A_+$ as in 4.1 and for any $N \in k\mathbf{-Mack}(G)$ we define a k-linear map

$$\Phi_N^{M,A}\colon k\mathbf{-Res}(G)(M,N) \to k\mathbf{-Res}(G)(A,N), \quad F \mapsto F|_A,$$

by restricting a morphism on M to one on A. More interestingly, in the other direction, we define a k-linear map

$$\Sigma_N^{M,A,a}\colon k\mathbf{-Res}(G)(A,N) \to k\mathbf{-Res}(G)(M,N), \quad f \mapsto b^{N,N} \circ f_+ \circ a,$$

and call $\Sigma_N^{M,A,a}(f)$ the ***canonical extension*** of $f \in k\mathbf{-Res}(G)(A,N)$ with respect to a.

This terminology is justified by the following theorem.

4.3 Theorem *Let M, A, and $a\colon M \to A_+$ be as in 4.1, and let $N \in k\mathbf{-Mack}(G)$. Then $\Phi_N^{M,A} \circ \Sigma_N^{M,A,a} = \mathrm{id}$. In particular, the map $\Phi_N^{M,A}$ is surjective, i.e. each morphism $f\colon A \to N$ of k-restriction functors on G can be extended to a morphism $F\colon M \to N$ of restriction functors on G.*

Proof We have to show that $(b_H^{N,N} \circ f_{+H} \circ a_H)(\varphi) = f_H(\varphi)$ for all $f \in k\mathbf{-Res}(G)(A,N)$, $H \leq G$, and $\varphi \in A(H)$. Since $a_H(\varphi) = [H,\varphi]_H$, we obtain

$$\begin{aligned}(b_H^{N,N} \circ f_{+H} \circ a_H)(\varphi) &= b_H^{N,N}(f_{+H}([H,\varphi]_H)) = b_H^{N,N}([H, f_H(\varphi)]_H)\\ &= \mathrm{ind}_H^H(f_H(\varphi)) = f_H(\varphi),\end{aligned}$$

which completes the proof. □

The following theorem describes a situation where $\Phi_N^{M,A}$ and $\Sigma_N^{M,A,a}$ are inverse isomorphisms, in particular where the canonical extension map $\Sigma_N^{M,A,a}$ does not depend on a.

4.4 Theorem *Let M, A, and $a\colon M \to A_+$ be as in 4.1, and let $N \in k\mathbf{-Mack}(G)$ be such that $A(H) = M(H)$ for all $H \in \mathcal{C}(N)$. Then $\Phi_N^{M,A}$ and $\Sigma_N^{M,A,a}$ are inverse isomorphisms.*

Proof It suffices to show that $\Sigma_N^{M,A,a} \circ \Phi_N^{M,A} = \mathrm{id}$, i.e. $b_H^{N,N} \circ (F|_A)_{+H} \circ a_H = F_H$ for all $F \in k\mathbf{-Res}(G)(M,N)$ and $H \leq G$. By the last property of $\mathcal{C}(N)$ mentioned in 1.7, it is enough to show that

$$\mathrm{res}_K^H \circ b_H^{N,N} \circ f_{+H} \circ a_H = \mathrm{res}_K^H \circ F_H$$

for all $K \leq H$ with $K \in \mathcal{C}(N)$, where $f := F|_A$. Since b, f_+, a, and F commute with restrictions, it suffices to show $b_K^{N,N} \circ f_{+K} \circ a_K = F_K$ for all $K \in \mathcal{C}(N)$. But for $K \in \mathcal{C}(N)$ we have $A(K) = M(K)$, hence $(b_K^{N,N} \circ f_{+K} \circ a_K)(\varphi) = f_K(\varphi) = F_K(\varphi)$ for all $\varphi \in M(K)$ as in the proof of Theorem 4.3. □

5 Adjunctions

Throughout this section let k be a commutative ring and G a finite group. We will try to obtain an overview of the set of morphisms $a\colon M \to A_+$ of k-restriction functors on G in the situation of Definition 3.3.

5.1 For $A \in k\text{-}\mathbf{Res}(G)$ and $H \leq G$ we recall the map

$$\iota_H^A\colon A(H) \to A_+(H), \quad a \mapsto [H, a]_H,$$

from 2.2, which is induced by the inclusion $A(H) \to \bigoplus_{K \leq H} A(K)$. Note that $\pi^A \circ \iota^A = \mathrm{id}_A$.

For $X \in k\text{-}\mathbf{Con}(G)$ and $H \leq G$ we set

$$\omega_H^X\colon X^+(H) \to X(H), \quad (x_K)_{K \leq H} \mapsto x_H,$$

i.e. ω_H^X is induced by the projection $\prod_{K \leq H} X(K) \to X(H)$. Then $\omega^X\colon X^+ \to X$ is a morphism of k-conjugation functors on G, even of k-algebra conjugation functors, if $X \in k\text{-}\mathbf{Con}_{\mathrm{alg}}(G)$. Moreover, ω^X is functorial in X.

A statement similar to part (i) of the following proposition can be found in [7, Lemme 2.10].

5.2 Proposition (i) *The forgetful functor $k\text{-}\mathbf{Mack}(G) \to k\text{-}\mathbf{Res}(G)$ is right adjoint to $-_+\colon k\text{-}\mathbf{Res}(G) \to k\text{-}\mathbf{Mack}(G)$. More precisely, for $A \in k\text{-}\mathbf{Res}(G)$ and $M \in k\text{-}\mathbf{Mack}(G)$ one has k-linear inverse isomorphisms*

$$\begin{aligned} \kappa_{A,M}\colon k\text{-}\mathbf{Mack}(G)(A_+, M) &\to k\text{-}\mathbf{Res}(G)(A, M) \\ (f_H)_{H \leq G} &\mapsto (f_H \circ \iota_H^A)_{H \leq G} \\ \big([K, a]_H \mapsto \mathrm{ind}_K^H(g_K(a))\big)_{H \leq G} &\leftarrow\!\shortmid (g_H)_{H \leq G} \end{aligned}$$

which are natural in A and M. The same maps yield inverse natural bijection between $k\text{-}\mathbf{Mack}_{\mathrm{alg}}(G)(A_+, M)$ and $k\text{-}\mathbf{Res}_{\mathrm{alg}}(G)(A, M)$, if $A \in k\text{-}\mathbf{Res}_{\mathrm{alg}}(G)$ and $M \in k\text{-}\mathbf{Mack}_{\mathrm{alg}}(G)$.

(ii) *The forgetful functor $k\text{-}\mathbf{Res}(G) \to k\text{-}\mathbf{Con}(G)$ is left adjoint to the composition $k\text{-}\mathbf{Con}(G) \to k\text{-}\mathbf{Mack}(G) \to k\text{-}\mathbf{Res}(G)$ of $-^+$ with the forgetful functor of part (i). More precisely, for $B \in k\text{-}\mathbf{Res}(G)$ and $X \in k\text{-}\mathbf{Con}(G)$ one has k-linear inverse isomorphism*

$$\begin{aligned} \lambda_{B,X}\colon k\text{-}\mathbf{Res}(G)(B, X^+) &\to k\text{-}\mathbf{Con}(G)(B, X) \\ (r_H)_{H \leq G} &\mapsto (\omega_H^X \circ r_H)_{H \leq G} \\ \big(b \mapsto (p_K(\mathrm{res}_K^H(b)))_{K \leq H}\big)_{H \leq G} &\leftarrow\!\shortmid (p_H)_{H \leq G} \end{aligned}$$

which are natural in B an X. The same maps yield inverse natural bijections between $k-\mathbf{Res}_{\mathrm{alg}}(G)(B, X^+)$ and $k-\mathbf{Con}_{\mathrm{alg}}(G)(B, X)$, if $B \in k-\mathbf{Res}_{\mathrm{alg}}(G)$ and $X \in k-\mathbf{Con}_{\mathrm{alg}}(G)$.

Proof All assertions are immediate consequences of the very definitions of the categories and functors involved. □

5.3 Our incentive from Section 3 is to obtain an overview of the morphisms in $k-\mathbf{Res}(G)(M, A_+)$ for $M \in k-\mathbf{Mack}(G)$ and $A \subseteq M$ a k-restriction subfunctor. More generally, for arbitrary $M, A \in k-\mathbf{Res}(G)$, we can now define a map

$$\Theta_{M,A} \colon k-\mathbf{Res}(G)(M, A_+) \to k-\mathbf{Res}(G)(M, A^+) \overset{\lambda_{M,A}}{\to} k-\mathbf{Con}(G)(M, A),$$
$$a \mapsto \rho^A \circ a,$$

where the first map in this definition is composition which ρ^A. Since $\omega^A \circ \rho^A = \pi^A$, we have $\Theta_{M,A}(a) = \pi^A \circ a$ for $a \in k-\mathbf{Res}(G)(M, A_+)$, and we call $p := \Theta(a)$ the ***residue*** of a. For $H \leq G$ we can recover a_H from p up to $|H|$-torsion by

$$|H| a_H(m) = \sum_{L \leq K \leq H} |L| \mu(L, K)[L, \mathrm{res}^K_L(p_K(\mathrm{res}^H_K(m)))]_H, \tag{5.3.a}$$

for $m \in M(H)$. In fact, we just apply the inverse of $\lambda_{M,A}$ to p and compose $\lambda^{-1}_{M,A}(p)_H = \rho^A_H \circ a_H$ with σ^A_H, using $\sigma^A_H \circ \rho^A_H = |H| \mathrm{id}_{A_+(H)}$ from Proposition 2.4.

5.4 Corollary *Let $M, A \in k-\mathbf{Res}(G)$, $a \in k-\mathbf{Res}(G)(M, A_+)$, and let $p := \Theta_{M,A}(a) = \pi^A \circ a \in k-\mathbf{Con}(G)(M, A)$ be the residue of a.*

(i) *For each $H \leq G$ the diagram*

$$\begin{array}{ccc} M(H) & \overset{a_H}{\longrightarrow} & A_+(H) \\ & {\scriptstyle (p_K \circ \mathrm{res}^H_K)_{K \leq H}} \searrow & \downarrow {\scriptstyle \rho^A_H = (\pi^A_K \circ \mathrm{res}_+{}^H_K)_{K \leq H}} \\ & & A^+(H) \end{array} \tag{5.4.a}$$

is commutative.

(ii) *If $A_+(H)$ has trivial $|H|$-torsion for all $H \leq G$, then $\Theta_{M,A}$ is injective.*

(iii) *If $|G|$ is invertible in k, then $\Theta_{M,A}$ is an isomorphism.*

(iv) *Assume that $M, A \in k-\mathbf{Res}_{\mathrm{alg}}(G)$ and that ρ^A is injective. Then $a \in k-\mathbf{Res}_{\mathrm{alg}}(G)(M, A_+)$, if and only if $p \in k-\mathbf{Con}_{\mathrm{alg}}(G)(M, A)$.*

Proof (i) For $K \leq H \leq G$ we have $\pi^A_K \circ \mathrm{res}_+{}^H_K \circ a_H = \pi^A_K \circ a_K \circ \mathrm{res}^H_K = p_K \circ \mathrm{res}^H_K$.

(ii) This is immediate from Equation (5.3.a).

(iii) If $|G|$ is invertible in k, then ρ^A is an isomorphism by Proposition 2.4, and hence $\Theta_{M,A}$ is an isomorphism by definition.

(iv) If $a \in k-\mathbf{Res}_{\mathrm{alg}}(G)(M, A_+)$, then $p = \pi^A \circ a \in k-\mathbf{Con}_{\mathrm{alg}}(G)(M, A)$, since $\pi^A \in k-\mathbf{Con}_{\mathrm{alg}}(G)(A_+, A)$, cf. 2.3. Now let $p \in k-\mathbf{Con}_{\mathrm{alg}}(G)(M, A)$. Since ρ^A is an injective morphism of k-algebra restriction functors, it suffices to show that $\rho^A \circ a \in k-\mathbf{Res}_{\mathrm{alg}}(G)(M, A^+)$. But this follows immediately from the commutativity of Diagram (5.4.a), since res_K^H, $K \le H \le G$, are k-algebra homomorphisms and since the k-algebra structure of $A^+(H)$ is defined componentwise. □

5.5 Definition Let $M, A \in k\mathrm{-}\mathbf{Res}(G)$ and assume that $A_+(H)$ has trivial $|H|$-torsion for each $H \le G$. Then

$$\Theta_{M,A} \colon k\mathrm{-}\mathbf{Res}(G)(M, A_+) \to k\mathrm{-}\mathbf{Con}(G)(M, A), \quad a \mapsto \pi^A \circ a,$$

is injective by Corollary 5.4 (ii) and for each $p \in \mathrm{im}(\Theta_{M,A})$ we define

$$a^{M,A,p} \in k\mathrm{-}\mathbf{Res}(G)(M, A_+)$$

as the unique preimage of p under $\Theta_{M,A}$.

6 Invertible group order

Throughout this section let G be a finite group and k a commutative ring such that $|G|$ is *invertible* in k.

6.1 Our aim is to find canonical induction formulae $a \in k\mathrm{-}\mathbf{Res}(G)(M, A_+)$ for given $M \in k\mathrm{-}\mathbf{Mack}(G)$ and given k-restriction subfunctor $A \subseteq M$. By Corollary 5.4 (iii) we have an isomorphism

$$\Theta_{M,A} \colon k\mathrm{-}\mathbf{Res}(G)(M, A_+) \to k\mathrm{-}\mathbf{Con}(G)(M, A), \quad a \mapsto \pi^A \circ a.$$

For $p \in k\mathrm{-}\mathbf{Con}(G)(M, A)$ and $a^{M,A,p} := \Theta_{M,A}^{-1}(p)$, as defined in 5.5, we observe from Equation (5.3.a) the explicit formula

$$a_H^{M,A,p}(m) = \frac{1}{|H|} \sum_{L \le K \le H} |L| \mu(L, K) [L, \mathrm{res}_L^K(p_K(\mathrm{res}_K^H(m)))]_H, \tag{6.1.a}$$

for all $H \le G$ and $m \in M(H)$, and from Corollary 5.4 (i) the commutativity of the diagram

$$\begin{array}{ccc} M(H) & \xrightarrow{a_H^{M,A,p}} & A_+(H) \\ (p_K \circ \mathrm{res}_K^H)_{K \le H} & \searrow & \big\downarrow \rho_H^A = (\pi_K^A \circ \mathrm{res}_{+K}^{\;H})_{K \le H} \\ & & A^+(H) \end{array} \tag{6.1.b}$$

for all $H \le G$, where ρ_H^A is an isomorphism by Proposition 2.4.

In a first step we will determine those $p \in k\mathrm{-}\mathbf{Con}(G)(M, A)$ which correspond under $\Theta_{M,A}$ to canonical induction formulae, i.e. which satisfy $b^{M,A} \circ a^{M,A,p} = \mathrm{id}_M$. Before we will do this we have to state a result about a natural decomposition of a Mackey functor in the invertible group order case, parts of which can be found in [15, Thm. 2] and [32, Thm. 4.1'].

For the notation of the following proposition we refer to Remark 2.5, Proposition 1.5 (i), and to 1.6 and 1.7.

6.2 Proposition *Let $M \in k-\mathbf{Mack}(G)$ (resp. $M \in k-\mathbf{Mack}_{\mathrm{alg}}(G)$). For each $H \leq G$ we have a decomposition into k-submodules (resp. ideals)*

$$M(H) = e_H^{(H)} \cdot M(H) \oplus (1 - e_H^{(H)}) \cdot M(H) \tag{6.2.a}$$

and the summands are given by

$$e_H^{(H)} \cdot M(H) = \mathcal{K}(M)(H) \quad \text{and} \quad (1 - e_H^{(H)}) \cdot M(H) = \mathcal{I}(M)(H). \tag{6.2.b}$$

In particular, M decomposes as k-conjugation functor (resp. k-algebra conjugation functor) as $M = \mathcal{K}(M) \oplus \mathcal{I}(M)$ (resp. $M = \mathcal{K}(M) \times \mathcal{I}(M)$), and $\mathcal{C}(M) = \mathcal{P}(M)$.

Moreover, for an inclusion $M \subseteq N$ of k-Green functors we have $\mathcal{C}(M) = \mathcal{C}(N)$.

Proof Since $|G|$ is invertible in k we have $e_H^{(H)} \in k \otimes \Omega(H)$ by Remark 2.5. Since the pairing $(k \otimes \Omega) \otimes_k M \to M$ of Proposition 1.5 (i) provides $M(H)$ with the structure of a $k \otimes \Omega(H)$-module, we clearly have a decomposition as in (6.2.a).

For the proof of the remaining equations we first claim that for $K < H \leq G$ we have $\mathrm{res}_K^H(e_H^{(H)}) = 0$. In fact, Since $\rho^{\underline{k}}$ is an isomorphism by Proposition 2.4, it suffices to show that $\rho_K^{\underline{k}}(\mathrm{res}_K^H(e_H^{(H)})) = 0$ in $\underline{k}^+(K)$. But $\rho^{\underline{k}}$ commutes with restrictions, and the definition of $e_H^{(H)}$ in Remark 2.5 together with the definition of restriction on $\underline{k}^+(H)$ proves the claim.

Using this we have $\mathrm{res}_K^H(e_H^{(H)} \cdot M(H)) = \mathrm{res}_K^H(e_H^{(H)}) \cdot \mathrm{res}_K^H(M(H)) = 0$ for all $K < H \leq G$. Conversely, the explicit formula (2.5.a) yields

$$1 - e_H^{(H)} = -\frac{1}{|H|} \sum_{K<H} |K| \mu(K,H)[H/K], \tag{6.2.c}$$

and the definition of the pairing $(k \otimes \Omega) \otimes_k M \to M$ shows that $\mathcal{K}(M)(H) = \bigcap_{K<H} \ker(\mathrm{res}_K^H \colon M(H) \to M(K))$ is annihilated by $1 - e_H^{(H)}$, and hence contained in $e_H^{(H)} \cdot M(H)$. This proves the first part of (6.2.b).

Again the definition of the pairing $(k \otimes \Omega) \otimes_k M \to M$ and Equation (6.2.c) imply $(1 - e_H^{(H)}) \cdot M(H) \subseteq \mathcal{I}(M)(H)$. For the opposite inclusion it suffices to show that $e_H^{(H)} \cdot \mathrm{ind}_K^H(M(K)) = 0$ for $K < H$. But this is immediate from the Frobenius axioms (P3) in Definition 1.4 and the claim above.

If $M \subseteq N$ is an inclusion of k-Green functors, then we have

$$H \in \mathcal{C}(M) \iff \mathcal{K}(M)(H) \neq 0 \iff e_H^{(H)} \cdot M(H) \neq 0 \iff e_H^{(H)} \cdot 1_{M(H)} \neq 0,$$

and similar for N. Since $1_{M(H)} = 1_{N(H)}$, this shows $\mathcal{C}(M) = \mathcal{C}(N)$. □

6.3 Corollary *Let $M \in k-\mathbf{Mack}(G)$, $H \leq G$, and $m \in M(H)$. Then $m = 0$, if and only if $e_H^{(H)} \cdot m = 0$ and $\mathrm{res}_K^H(m) = 0$ for all $K < H$.*

Proof The condition $\mathrm{res}^H_K(m) = 0$ for all $K < H$ is equivalent to $m \in \mathcal{K}(M)(H)$, hence, by Proposition 6.2, also to $m = e_H^{(H)} \cdot m$, and the result follows. □

6.4 Proposition *Let $M \in k\mathrm{-}\mathbf{Mack}(G)$, $A \subseteq M$ a k-restriction subfunctor, $p \in k\mathrm{-}\mathbf{Con}(G)(M,A)$, and $a^{M,A,p}$ the corresponding morphism in $k\mathrm{-}\mathbf{Res}(G)(M,A_+)$, cf. Definition 5.5. Then the following are equivalent:*

(i) *The morphism $a^{M,A,p}$ is a canonical indution formula, i.e. $b^{M,A} \circ a^{M,A,p} = \mathrm{id}_M$.*

(ii) *For all $H \leq G$ and $m \in M(H)$ one has $e_H^{(H)} \cdot (p_H(m) - m) = 0$, i.e. $p_H(m) - m \in \sum_{K<H} \mathrm{ind}^H_K(M(K))$ by Proposition 6.2.*

Proof We set $a := a^{M,A,p}$. From Equation (6.1.a) we obtain

$$
\begin{aligned}
e_H^{(H)} \cdot b_H^{M,A}(a_H(m)) &= \qquad\qquad\qquad (6.4.a)\\
&= \frac{1}{|H|} e_H^{(H)} \sum_{L \leq K \leq H} |L| \mu(L,K) (\mathrm{ind}^H_L \circ \mathrm{res}^K_L \circ p_K \circ \mathrm{res}^H_K)(m)\\
&= e_H^{(H)} \cdot p_H(m),
\end{aligned}
$$

for $H \leq G$ and $m \in M(H)$, since $e_H^{(H)}$ annihilates all summands with $L < H$ by Proposition 6.2.

If a is a canonical induction formula for M from A, then Equation (6.4.a) shows that $e_H^{(H)} \cdot m = e_H^{(H)} \cdot p_H(m)$.

Conversely, if $e_H^{(H)} \cdot m = e_H^{(H)} \cdot p_H(m)$ for all $H \leq G$ and $m \in M(H)$, then by Corollary 6.3 it suffices to show that

$$
e_H^{(H)} \cdot b_H^{M,A}(a_H(m)) = e_H^{(H)} \cdot m \quad \text{and} \quad \mathrm{res}^H_K(b_H^{M,A}(a_H(m))) = \mathrm{res}^H_K(m)
$$

for all $K < H \leq G$ and $m \in M(H)$. Under our assumption, the first equation is just a restatement of Equation (6.4.a), and therefore holds. The second equation follows by induction on $|H|$. In fact, since $b^{M,A}$ and a commute with restrictions, it suffices to show that $(b_K^{M,A} \circ a_K)(m') = m'$ for all $m' \in M(K)$, $K < H$. but this follows from the first equation and the induction hypothesis on the second one. □

6.5 Corollary *Let M and A be as in the above Proposition. If there exists a canonical induction formula for M from A, then $e_H^{(H)} \cdot M(H) \subseteq A(H)$ for all $H \leq G$.* □

6.6 Corollary *Let M, A, and p be as in Proposition 6.4 and assume that $A(H) = M(H)$ and $p_H = \mathrm{id}_{M(H)}$ for all $H \in \mathcal{C}(M)$. Then $a^{M,A,p}$ is a canonical induction formula for M from A.*

Proof Condition (ii) in Proposition 6.4 is obviously satisfied for $H \in \mathcal{C}(M)$, since under our assumption $p_H(m) - m = 0$, and also generally for $H \notin \mathcal{C}(M)$, since $e_H^{(H)} \cdot M(H) = 0$ for $H \notin \mathcal{C}(M)$ by Proposition 6.2. □

Assuming the notation of 6.1, we next determine those $p \in k\mathbf{-Con}(G)(M,A)$ whose corresponding morphism $a^{M,A,p} \in k\mathbf{-Res}(G)(M,A_+)$ is even a morphism of Mackey functors.

6.7 Proposition *Let $M \in k\mathbf{-Mack}(G)$, $A \subseteq M$ a k-restriction subfunctor, $a \in k\mathbf{-Res}(G)(M,A_+)$, and $p := \pi^A \circ a$, i.e. $a = a^{M,A,p}$. Then the following are equivalent:*
(i) $a \in k\mathbf{-Mack}(G)(M,A_+)$.
(ii) $p_H(\mathrm{ind}_K^H(M(K))) = 0$ *for all* $K < H \le G$.
(iii) $p_H((1 - e_H^{(H)}) \cdot M(H)) = 0$ *for all* $H \le G$.

Proof Statements (ii) and (iii) are equivalent by Proposition 6.2. We show that (i) and (ii) are equivalent, just assuming that ρ^A is injective which is of course satisfied if $|G|$ is invertible in k by Proposition 2.4. By definition, a is a morphism of Mackey functors, if and only if

$$\mathrm{ind}_{+}{}_K^H \circ a_K = a_H \circ \mathrm{ind}_K^H \colon M(K) \to A_+(H)$$

for all $K \le H \le G$. Since ρ_H^A is injective, this is equivalent to

$$\pi_U^A \circ \mathrm{res}_{+}{}_U^H \circ \mathrm{ind}_{+}{}_K^H \circ a_K = \pi_U^A \circ \mathrm{res}_{+}{}_U^H \circ a_H \circ \mathrm{ind}_K^H \colon M(K) \to A(U)$$

for all $H \le G$ and $U, K \le H$. We transform the left hand side by using the Mackey axiom and commutativity of a with restrictions and conjugations, and we transform the right hand side by using the commutativity of a with restrictions, the Mackey axiom, and the relation $p_U = \pi_U^A \circ a_U$ to obtain the equivalent condition

$$\sum_{h \in U\backslash H/K} \pi_U^A \circ \mathrm{ind}_{+}{}_{U\cap {}^hK}^U \circ a_{U\cap {}^hK} \circ \mathrm{res}_{U\cap {}^hK}^{{}^hK} \circ c_{h,K}$$
$$= \sum_{h \in U\backslash H/K} p_U \circ \mathrm{ind}_{U\cap {}^hK}^U \circ \mathrm{res}_{U\cap {}^hK}^{{}^hK} \circ c_{h,K} \colon M(K) \to A(U)$$

for all $H \le G$ and $K, U \le H$. Since $\pi_U^A \circ \mathrm{ind}_{+}{}_{U\cap {}^hK}^U = 0$ unless $U \le {}^hK$, this is equivalent to

$$\sum_{\substack{h \in U\backslash H/K \\ U \le {}^hK}} p_U \circ \mathrm{res}_U^{{}^hK} \circ c_{h,K} = \sum_{h \in U\backslash H/K} p_U \circ \mathrm{ind}_{U\cap {}^hK}^U \circ \mathrm{res}_{U\cap {}^hK}^{{}^hK} \circ c_{h,K} \qquad (6.7.\mathrm{a})$$

for all $H \le G$ and $U, K \le H$.

Now Equation (6.7.a) implies $0 = p_H \circ \mathrm{ind}_K^H$ for $K < H \le G$ by choosing $U = H$. Conversely, if $p_H \circ \mathrm{ind}_K^H = 0$ for all $K < H \le G$, then Equation (6.7.a) holds, since the right hand side reduces to the left hand side. □

6.8 Corollary *Let $M \in k\mathbf{-Mack}(G)$, $A \subseteq M$ a k-restriction subfunctor, and $p \in k\mathbf{-Con}(G)(M,A)$. Then $a^{M,A,p}$ is a canonical induction formula and a morphism of k-Mackey functors on G if and only if*

$$(\mathrm{id}_{M(H)} - p_H)(M(H)) \subseteq (1 - e_H^{(H)}) \cdot M(H) \subseteq \ker(p_H)$$

for all $H \leq G$.

Proof This is immediate from Proposition 6.4 and Proposition 6.7. □

6.9 Example For each k-Mackey functor M on G there is a canonical induction formula for M of a minimal type (cf. Corollary 6.5), namely from A with $A(H) := e_H^{(H)} \cdot M(H)$, $H \leq G$, and associated to $p_H \colon M(H) \to A(H)$, $m \mapsto e_H^{(H)} \cdot m$, $H \leq G$. In fact, this choice of p satisfies the condition in Corollary 6.8, hence $a^{M,A,p} \in k\text{--}\mathbf{Mack}(G)$. From Equation (6.1.a) we have the explicit formula

$$a_H^{M,A,p}(m) = \frac{1}{|H|} \sum_{\substack{K \leq H \\ K \in \mathcal{C}(M)}} |K| [K, e_K^{(K)} \cdot \mathrm{res}_K^H(m)]_H,$$

for $H \leq G$ and $m \in M(H)$, since res_L^K is trivial on $e_K^{(K)}$ for $L < K$ (as shown in the proof of Proposition 6.2) and $e_K^{(K)} \cdot M(K) = 0$ for $K \notin \mathcal{C}(M)$. Moreover, applying $b^{M,A}$ and expanding $e_K^{(K)}$ as in Equation (2.5.a) we obtain

$$m = \frac{1}{|H|} \sum_{\substack{L \leq K \leq H \\ K \in \mathcal{C}(M)}} |L| \mu(L, K) \mathrm{ind}_L^H(\mathrm{res}_L^H(m)) \qquad (6.9.\text{a})$$

for $m \in M(H)$, $H \leq G$. This generalizes Brauer's explicit version (cf. [9] or [11, 15.4]) of Artin's induction theorem for the character ring tensored with $\mathbb{Q}$, where $\mathcal{C}(M)$ consists of the set of cyclic subgroups of G, and also an explicit version of Conlon's induction theorem for the Green ring and various subrings tensored with $\mathbb{Q}$, where $\mathcal{C}(M)$ consists of the set of l-hypo-elementary subgroups of G (cf. [29, Theorem D'] and see also [12, 81.31]).

Recall from 2.2 that for $A \in k\text{--}\mathbf{Res}_{\mathrm{alg}}(G)$ the ring $A_+(H)$ is a natural $A(H)$-module for all $H \leq G$.

6.10 Proposition *Let $M \in k\text{--}\mathbf{Mack}_{\mathrm{alg}}(G)$, $A \subseteq M$ a k-algebra restriction subfunctor, and $p \in k\text{--}\mathbf{Con}(G)(M, A)$. Then the maps $a_H^{M,A,p} \colon M(H) \to A_+(H)$ are k-algebra (resp. $A(H)$-module) homomorphisms for all $H \leq G$, if and only if the maps $p_H \colon M(H) \to A(H)$ are k-algebra (resp. $A(H)$-module) homomorphisms for all $H \leq G$.*

Proof The statement about the k-algebra structure follows immediately from Corollary 5.4 (iv).

We set $a := a^{M,A,p}$. The maps a_H, $H \leq G$, are $A(H)$-linear, if and only if $a_H(b \cdot m) = [H, b]_H \cdot a_H(m)$ for all $b \in A(H)$ and $m \in M(H)$. By the injectivity and multiplicativity of ρ^A this is equivalent to $(\rho_H^A \circ a_H)(b \cdot m) = \rho_H^A([H, b]_H) \cdot \rho_H^A(a_H(m))$. Since $\rho_H^A = (\pi_K^A \circ \mathrm{res}_{+K}^{\ H})_{K \leq H}$, since a commutes with restrictions, since $\pi_K^A \circ a_K = p_K$, and since $\mathrm{res}_K^H \colon M(H) \to M(K)$ is multiplicative for $K \leq H \leq G$, this is equivalent to

$$p_K(\mathrm{res}_K^H(b) \cdot \mathrm{res}_K^H(m)) = \mathrm{res}_K^H(b) \cdot p_K(\mathrm{res}_K^H(m))$$

for all $K \le H \le G$, $m \in M(H)$, and $b \in A(H)$. This in turn is equivalent to p_H being $A(H)$-linear for all $H \le G$. □

6.11 Proposition *Let $M, M' \in k-\mathbf{Mack}(G)$, $A \subseteq M$ and $A' \subseteq M'$ be k-restriction subfunctors, $p\colon M \to A$ and $p'\colon M' \to A'$ be morphisms of k-conjugation functors on G, and let $f\colon M \to M'$ be a morphism of k-restriction functors with $f(A) \subseteq A'$. Then the diagram*

$$\begin{array}{ccc} M & \xrightarrow{a^{M,A,p}} & A_+ \\ f\downarrow & & \downarrow f_+ \\ M' & \xrightarrow{a^{M',A',p'}} & A'_+ \end{array}$$

is commutative, if and only if the diagram

$$\begin{array}{ccc} M & \xrightarrow{p} & A \\ f\downarrow & & \downarrow f \\ M' & \xrightarrow{p'} & A' \end{array}$$

is commutative.

Proof We write a and a' for $a^{M,A,p}$ and $a^{M',A',p'}$ respectively. If the first diagram commutes we compose it with $\pi^{A'}\colon A'_+ \to A'$ and obtain $\pi^{A'} \circ f_+ \circ a = \pi^{A'} \circ a' \circ f$. Since $\pi^{A'}$ is natural in A', we have $\pi^{A'} \circ f_+ = f \circ \pi^A$, and we also have $\pi^A \circ a = p$ and $\pi^{A'} \circ a' = p'$, showing that $f \circ p = p' \circ f$.

Now we assume that $f \circ p = p' \circ f\colon M \to A'$. By Proposition 2.4, $\rho^{A'}$ is injective, and for the commutativity of the first diagram it suffices to show that

$$\pi_K^{A'} \circ \mathrm{res}_{+K}^{\ H} \circ f_{+H} \circ a_H = \pi_K^{A'} \circ \mathrm{res}_{+K}^{\ H} \circ a'_H \circ f_H$$

for all $K \le H \le G$. Now the left hand side is equal to

$$\pi_K^{A'} \circ f_{+K} \circ a_K \circ \mathrm{res}_K^H = f_K \circ \pi_K^A \circ a_K \circ \mathrm{res}_K^H = f_K \circ p_K \circ \mathrm{res}_K^H$$

and the right hand side equals

$$\pi_K^{A'} \circ a'_K \circ f_K \circ \mathrm{res}_K^H = p'_K \circ f_K \circ \mathrm{res}_K^H,$$

and the result follows. □

6.12 Proposition *Let $M \in k-\mathbf{Mack}(G)$, $A \subseteq M$ be a k-restriction subfunctor, $p \in k-\mathbf{Con}(G)(M, A)$, $H \le G$, and $m \in M(H)$ such that*

$$\mathrm{res}_K^H(p_H(m)) = p_K(\mathrm{res}_K^H(m)) \qquad (6.12.a)$$

for all $K \le H$, then $a_H^{M,A,p}(m) = [H, m]_H$. In particular this holds for all $m \in A(H)$ and all $H \le G$, if $p_H|_{A(H)} = \mathrm{id}_{A(H)}$ for all $H \le G$.

Proof In view of the explicit formula (6.1.a) we have to show that

$$\sum_{L\le K\le H} \mu(L,K)|L|[L,\mathrm{res}^H_L(p_H(m))]_H = |H|[H,p_H(m)]_H.$$

But this follows by Möbius inversion of the function $K\mapsto |K|[K,\mathrm{res}^H_K(p_H(m))]_H$ on the poset of subgroups of H, cf. [21, 3., Prop. 2]. □

6.13 Example We assume the notation of Example 1.8.
(a) We define $p\in\mathbb{Z}\text{-}\mathbf{Con}(G)(R,R^{\mathrm{ab}})$ for $H\le G$ and $\chi\in\mathrm{Irr}(G)$ by

$$p_H(\chi)=\begin{cases}\chi, & \text{if } \chi(1)=1,\\ 0, & \text{if } \chi(1)>1.\end{cases}$$

Since $R(H)=R^{\mathrm{ab}}(H)$ and $p_H=\mathrm{id}_{R(H)}$ for all cyclic subgroups of G, Corollary 6.5 implies that $a=a^{\mathbb{Q}\otimes R,\mathbb{Q}\otimes R^{\mathrm{ab}},\mathbb{Q}\otimes p}$ is a canonical induction formula; but a is not a morphism of Mackey functors, since in general $p_H(\mathrm{ind}^H_K(\chi))\ne 0$ for $K<H\le G$ and $\chi\in R(K)$, cf. Proposition 6.7 and Remark 3.2. Moreover, a_H is $R^{\mathrm{ab}}(H)$-linear, but not a ring homomorphism, cf. Proposition 6.10. By Proposition 6.12, $a_H(\varphi)=[H,\varphi]_H$ for all $H\le G$ and $\varphi\in R^{\mathrm{ab}}(H)$.

(b) We define $p\in\mathbb{Z}\text{-}\mathbf{Con}(G)(R_F,R_F^{\mathrm{ab}})$ for $H\le G$ and a simple FH-module V by

$$p_H([V])=\begin{cases}[V], & \text{if } \dim_F(V)=1,\\ 0, & \text{if } \dim_F(V)>1.\end{cases}$$

Since $R_F(H)=R_F^{\mathrm{ab}}(H)$ for all cyclic l'-subgroups H of G, Corollary 6.5 implies that $a=a^{\mathbb{Q}\otimes R_F,\mathbb{Q}\otimes R_F^{\mathrm{ab}},\mathbb{Q}\otimes p}$ is a canonical induction formula. As in part (a), a is not a morphism of Mackey functors, and a_H is $R_F^{\mathrm{ab}}(H)$-linear but not a ring homomorphism for $H\le G$. Note that $a_H([F_\varphi])=[H,[F_\varphi]]_H$ for all $H\le G$ and all $\varphi\in\widehat{H}(F)$ by Proposition 6.12.

(c) We define $p\in\mathbb{Z}\text{-}\mathbf{Con}(G)(L_{\mathcal{O}},L_{\mathcal{O}}^{\mathrm{ab}})$ for $H\le G$ and an indecomposable linear source FH-module V by

$$p_H([V])=\begin{cases}[V], & \text{if } \mathrm{rk}_{\mathcal{O}}V=1,\\ 0, & \text{if } \mathrm{rk}_{\mathcal{O}}V>1.\end{cases}$$

Note that $p_H(T_{\mathcal{O}}(H))\subseteq T_{\mathcal{O}}^{\mathrm{ab}}(H)$ and $p_H(P_{\mathcal{O}}(H))\subseteq P_{\mathcal{O}}^{\mathrm{ab}}(H)$. Since $P_{\mathcal{O}}(H)=P_{\mathcal{O}}^{\mathrm{ab}}(H)$ and $p_H=\mathrm{id}_{P_{\mathcal{O}}(H)}$ for all cyclic l'-subgroups of G, Corollary 6.5 implies that $a=a^{\mathbb{Q}\otimes P_{\mathcal{O}},\mathbb{Q}\otimes P_{\mathcal{O}}^{\mathrm{ab}},\mathbb{Q}\otimes p}$ is a canonical induction formula. By the results of this section we see that a is not a morphism of Mackey functors, but that a_H is $P_{\mathcal{O}}^{\mathrm{ab}}(H)$-linear for all l'-subgroups $H\le G$ (note that $P_{\mathcal{O}}^{\mathrm{ab}}(H)=0$, if l divides $|H|$), and that $a_H([\mathcal{O}_\varphi])=[H,[\mathcal{O}_\varphi]]_H$ for all l'-subgroups $H\le G$ and all $\varphi\in\widehat{H}(\mathcal{O})$.

In order to show that $a^{\mathbb{Q}\otimes L_{\mathcal{O}},\mathbb{Q}\otimes L^{\text{ab}}_{\mathcal{O}},\mathbb{Q}\otimes p}$ and $a^{\mathbb{Q}\otimes T_{\mathcal{O}},\mathbb{Q}\otimes T^{\text{ab}}_{\mathcal{O}},\mathbb{Q}\otimes p}$ are canonical induction formulae we have to go deeper into modular representation theory. This is done in [4] where we also show that $a^{\mathbb{Q}\otimes L_{\mathcal{O}},\mathbb{Q}\otimes L^{\text{ab}}_{\mathcal{O}},\mathbb{Q}\otimes p}$ is integral, i.e. maps $L_{\mathcal{O}}$ to $L^{\text{ab}}_{\mathcal{O}}$. Note that by Proposition 6.11 these three induction formulae are restrictions of one-another with respect to the inclusions $P_{\mathcal{O}} \subseteq T_{\mathcal{O}} \subseteq L_{\mathcal{O}}$ and $P^{\text{ab}}_{\mathcal{O}+} \subseteq T^{\text{ab}}_{\mathcal{O}+} \subseteq L^{\text{ab}}_{\mathcal{O}+}$, cf. also Lemma 7.2 and Lemma 7.5.

7 Change of base ring and stable basis

Throughout this section let k be a commutative ring and G a finite group.

In the previous section we could give complete answers to some questions about canonical induction formulae in the situation where $|G|$ is invertible in the base ring k (cf. Propositions 6.4, 6.7, 6.10, and 6.11). However, we are mainly interested in the base ring $\mathbb{Z}$ and therefore have to study how these results over $\mathbb{Q}$ can be used for the base ring $\mathbb{Z}$. More generally we fix a homomorphism $k \to k'$ between commutative rings for this section and study how the constructions A_+ and A^+ for $A \in k\text{--}\mathbf{Res}(G)$ behave under this extension of base rings.

Moreover, in Poposition 2.4 and Corollary 5.4 (ii) we needed the hypothesis that $A_+(H)$ has trivial $|H|$-torsion for all $H \leq G$ in order to obtain injectivity of ρ^A and $\Theta_{M,A}$. First we will establish this situation.

7.1 Definition Let $A \in k\text{--}\mathbf{Res}(G)$. A ***stable*** k-basis of A is a family $\mathcal{B} = (\mathcal{B}(H))_{H\leq G}$ of subsets $\mathcal{B}(H) \subset A(H)$, $H \leq G$, such that $\mathcal{B}(H)$ is a k-basis of $A(H)$ for each $H \leq G$ and $c_{g,H}(\mathcal{B}(H)) = \mathcal{B}({}^gH)$, for all $g \in G$ and $H \leq G$.

7.2 Lemma *Let $A \in k\text{--}\mathbf{Res}(G)$ and let $\mathcal{B} = (\mathcal{B}(H))_{H\leq G}$ be a stable k-basis of A. For each $H \leq G$ the set $\mathcal{M}(H) := \{(K,b) \mid K \leq H,\, b \in \mathcal{B}(K)\}$ is a left H-set via the conjugation maps and isomorphic to the disjoint union $\bigcup_{K\leq H} \mathcal{B}(K)$ as H-set. Moreover, the elements $[K,b]_H \in A_+(H)$, where (K,b) runs through a set $\mathcal{R}(H) \subseteq \mathcal{M}(H)$ of representatives for the H-orbits of $\mathcal{M}(H)$, form a k-basis of $A_+(H)$.*

Proof The first statement is obvious. For each $H \leq G$ we have a kH-isomorphism $\bigoplus_{K\leq H} A(H) \cong k\mathcal{M}(H)$ and a decomposition

$$k\mathcal{M}(H) = k\mathcal{R}(H) \oplus k\{b - {}^hb \mid b \in \mathcal{R}(H),\, h \in H\}.$$

It is clear that the second summand on the right hand side is just $I(kH)\cdot k\mathcal{M}(H)$, so that the composition

$$k\mathcal{R}(H) \subseteq k\mathcal{M}(H) \cong \bigoplus_{K\leq H} A(K) \to A_+(H)$$

is an isomorphism of k-modules (cf. 2.2) sending $(K,b) \in \mathcal{R}(H)$ to $[K,b]_H$. □

7.3 Remark Note that in the Examples 1.8 (a)–(e) the k-restriction functors R^{ab}, R^{ab}_F, $L^{\text{ab}}_{\mathcal{O}}$, $T^{\text{ab}}_{\mathcal{O}}$, and $P^{\text{ab}}_{\mathcal{O}}$ have a stable basis, namely, respectively

$\mathcal{B}(H) = \widehat{H}$, $\widehat{H}(F)$, $\widehat{H}(\mathcal{O})$, $\widehat{H}(\mathcal{O})_{p'}$, and $\widehat{H}(\mathcal{O})$ if H is an l'-subgroup, $\emptyset$ otherwise, for $H \leq G$. By Lemma 7.2, the groups $R_+^{\mathrm{ab}}(H)$, $R_{F+}^{\mathrm{ab}}(H)$, $L_{\mathcal{O}+}^{\mathrm{ab}}(H)$, $T_{\mathcal{O}+}^{\mathrm{ab}}(H)$, and $P_{\mathcal{O}+}^{\mathrm{ab}}(H)$ are free abelian with basis $[K,\varphi]_H$, $(K,\varphi) \in H\backslash\mathcal{M}(H)$, for $H \leq G$.

7.4 For any $X \in k\mathbf{-Con}(G)$ (resp. $X \in k\mathbf{-Con}_{\mathrm{alg}}(G)$) there is a cononical morphism (i.e. natural in X) of k'-Mackey functors (resp. k'-Green functors),

$$k' \otimes_k X^+ \to (k' \otimes_k X)^+,$$

which maps $\alpha \otimes_k (x_K)_{K \leq H}$ to $(\alpha \otimes_k x_K)_{K \leq H}$ for $H \leq G$, $\alpha \in k'$, $x_K \in X(K)$, $K \leq H$. Moreover, there is a canonical morphism of k-Mackey functors (resp. k-Green functors) on G, $X^+ \to k' \otimes_k X^+$, mapping $(x_K)_{K \leq H}$ to $1 \otimes_k (x_K)_{K \leq H}$ for $H \leq G$, $x_K \in X(K)$, $K \leq H$.

For any $A \in k\mathbf{-Res}(G)$ (resp. $A \in k\mathbf{-Res}_{\mathrm{alg}}(G)$) there is a canonical morphism (i.e. natural in A) of k'-Mackey functors (resp. k'-Green functors),

$$k' \otimes_k A_+ \to (k' \otimes_k A)_+,$$

which maps $\alpha \otimes_k [K,a]_H$ to $[K, \alpha \otimes_k a]_H$ for $H \leq G$, $\alpha \in k'$, $a \in A(K)$, $K \leq H$. Moreover, there is a canonical morphism of k-Mackey functors (resp. k-Green functors) on G, $A_+ \to k' \otimes_k A_+$, which maps $[K,a]_H$ to $[K, 1 \otimes_k a]_H$ for $H \leq G$, $a \in A(K)$, $K \leq H$.

It is easy to see that for $A \in k\mathbf{-Res}(G)$ the diagram

$$\begin{array}{ccccc} A_+ & \longrightarrow & k' \otimes_k A_+ & \longrightarrow & (k' \otimes_k A)_+ \\ {\scriptstyle \rho^A}\downarrow & & \downarrow{\scriptstyle k' \otimes_k \rho^A} & & \downarrow{\scriptstyle \rho^{k' \otimes_k A}} \\ A^+ & \longrightarrow & k' \otimes_k A^+ & \longrightarrow & (k' \otimes_k A)^+ \end{array}$$

is commutative.

7.5 Lemma *Let $X \in k-\mathbf{Con}(G)$ and $A \in k-\mathbf{Res}(G)$.*

(i) *If k' is flat over k, then the canonical morphism $k' \otimes_k X^+ \to (k' \otimes_k X)^+$ is an isomorphism.*

(ii) *The canonical morphism $k' \otimes_k A_+ \to (k' \otimes_k A)_+$ is always an isomorphism.*

(iii) *If k' and $X(H)$, $H \leq G$, are flat over k and if $k \to k'$ is injective, then the canonical morphism $X^+ \to k' \otimes_k X^+$ is injective.*

(iv) *If $A_+(H)$ is flat over k for all $H \leq G$ and if $k \to k'$ is injective, then the canonical morphism $A_+ \to k' \otimes_k A_+$ is injective.*

Proof (i) We show that if M is any kG-module, then the map $k' \otimes_k M^G \to (k' \otimes_k M)^G$, $\alpha \otimes_k m \mapsto \alpha \otimes_k m$, where $\alpha \in k'$ and $m \in M^G$, is an isomorphism. In fact, this map is the composition of the following sequence of natural isomorphisms

$$\begin{aligned} k' \otimes_k M^G &\cong \mathrm{Hom}_{k'}(k', k') \otimes_k \mathrm{Hom}_{kG}(k, M) \cong \mathrm{Hom}_{k' \otimes_k kG}(k' \otimes_k k, k' \otimes_k M) \\ &\cong \mathrm{Hom}_{k'G}(k', k' \otimes_k M) \cong (k' \otimes_k M)^G, \end{aligned}$$

where we identify k' with $\mathrm{Hom}_{k'}(k', k')$ and M^G with $\mathrm{Hom}_{kG}(k, M^G)$, k being the trivial kG-module, and where the second canonical map is an isomorphism, since k' is flat over k and k is finitely presented as kG-module (cf. [19, Lemma I.4.1 (b)]).

(ii) We show that, for any kG-module M, the map $k' \otimes_k M_G \to (k' \otimes_k M)_G$, $\alpha \otimes_k (m + I(kG) \cdot M) \mapsto (\alpha \otimes_k m) + I(kG) \cdot (k' \otimes_k M)$, where $\alpha \in k'$ and $m \in M$, is an isomorphism. In fact, identifying M_G with $k \otimes_{kG} M$, k being the trivial kG-module, via $\alpha \otimes_{kG} m \mapsto \alpha m + I(kG) \cdot M$ for $\alpha \in k'$ and $m \in M$, the above map is the composition of the following sequence of isomorphisms

$$k' \otimes_k M_G \cong k' \otimes_k (k \otimes_{kG} M) \cong k' \otimes_{k'G} (k' \otimes_k M) \cong (k' \otimes_k M)_G,$$

where the middle isomorphism is given by $\alpha' \otimes_k (\alpha \otimes_{kG} m) \mapsto \alpha' \otimes_{k'G} (\alpha \otimes_k m)$ with inverse $\alpha' \otimes_{k'G} (\beta' \otimes_k m) \mapsto \alpha'\beta' \otimes_k (1 \otimes_{kG} m)$ for $\alpha \in k$, $\alpha', \beta' \in k'$, and $m \in M$.

(iii) This follows from the commutativity of the diagram

$$\begin{array}{ccc} k \otimes_k \big(\prod_{K \leq H} X(K)\big)^H & \longrightarrow & k' \otimes_k \big(\prod_{K \leq H} X(K)\big)^H \\ \downarrow & & \downarrow \\ k \otimes_k \prod_{K \leq H} X(K) & \longrightarrow & k' \otimes_k \prod_{K \leq H} X(K), \end{array}$$

from the injectivity of the vertical maps (since k' is flat over k), and the injectivity of the lower horizontal map (since $\prod_{K \leq H} X(K)$ is flat over k).

(iv) This is obvious. □

7.6 Corollary *Let $A \in k-\mathbf{Res}(G)$. If A has a stable basis, k' is flat over k, and $k \to k'$ is injective, then the left horizontal maps in Diagram (7.4.a)* *are injective, the right horizontal maps are isomorphisms, and the vertical morphisms are injective.*

Proof The statement about the vertical morphisms follows from Proposition 2.4 and Lemma 7.2, and the one about the horizontal morphisms follows from Lemma 7.5 and Lemma 7.2. □

8 The standard situation

In the sequel we will often assume the following standard situation.

8.1 Hypothesis M is a $\mathbb{Z}$-Mackey functor on a finite group G, A is a $\mathbb{Z}$-restriction subfunctor of M on G, and $p \in \mathbb{Z}\!-\!\mathbf{Con}(G)(M, A)$ such that the following conditions are satisfied:

(i) $M(H)$ is a free abelian group for all $H \leq G$.

(ii) A has a stable basis $\mathcal{B}$ such that for all $K \leq H \leq G$ and $\varphi \in \mathcal{B}(H)$, the element $\mathrm{res}^H_K(\varphi) \in A(K)$ is a linear combination

$$\mathrm{res}^H_K(\varphi) = \sum_{\psi \in \mathcal{B}(K)} m^{(H,\varphi)}_{(K,\psi)} \cdot \psi$$

of the basis elements $\psi \in \mathcal{B}(K)$ with *non-negative* coefficients $m^{(H,\varphi)}_{(K,\psi)} \in \mathbb{N}_0$.

8.2 Remark We assume $A \subseteq M$, $\mathcal{B}$, and p as in Hypothesis 8.1.

(a) Hypothesis 8.1 is clearly designed for the various representation rings of G. Note that the Examples 6.13, namely $R^{\mathrm{ab}} \subseteq R$, $R^{\mathrm{ab}}_F \subseteq R_F$, $L^{\mathrm{ab}}_{\mathcal{O}} \subseteq L_{\mathcal{O}}$, $T^{\mathrm{ab}}_{\mathcal{O}} \subseteq T_{\mathcal{O}}$, and $P^{\mathrm{ab}}_{\mathcal{O}} \subseteq P_{\mathcal{O}}$, together with the respective morphisms p and the stable bases from Remark 7.3, satisfy this hypothesis.

(b) As a convention, the letters χ, θ, ξ, ζ will always denote elements of $M(H)$, $H \leq G$, and $\varphi, \psi, \lambda, \mu$ will always denote elements of $A(H)$, $H \leq G$. This should help the reader to switch from the abstract setting to the standard example $R^{\mathrm{ab}} \subseteq R$.

(c) We call a pair (H, φ) with $H \leq G$ and $\varphi \in \mathcal{B}(H)$ a ***monomial pair***. For $H \leq G$ the set of monomial pairs (K, ψ) with $K \leq H$ will be denoted by $\mathcal{M}(H)$, cf. the notation in Lemma 7.2. Note that $\mathcal{M}(H)$ is an H-poset (i.e. H acts via poset automorphisms) by the following definition

$$(L, \lambda) \leq (K, \psi) :\Longleftrightarrow L \leq K \text{ and } m^{(K,\psi)}_{(L,\lambda)} > 0,$$
$${}^h(K, \psi) := ({}^hK, c_{h,K}(\psi)),$$

for $(L, \lambda), (K, \psi) \in \mathcal{M}(H)$ and $h \in H$. Note that we need the non-negativity property in Hypothesis 8.1 (ii) to ensure that the relation $\leq$ is transitive. We write $(L, \lambda) =_H (K, \psi)$ if (L, λ) and (K, ψ) lie in the same H-orbit, and we denote by $N_H(K,\psi)$ the stabilizer of (K,ψ) in H. The posets $\mathcal{M}(H)$, $H \le G$, are subposets of $\mathcal{M}(G)$ and inherit the H-action from $\mathcal{M}(G)$.

(d) We recollect some results that hold in the situation of Hypothesis 8.1:

For $H \le G$, $A_+(H)$ is a free $\mathbb{Z}$-module with basis $\{[K,\psi]_H\}$, where (K,ψ) runs through a set of representatives of the H-orbits $H\backslash\mathcal{M}(H)$, cf. Lemma 7.2. We identify this $\mathbb{Z}$-basis of $A_+(H)$ with the $\mathbb{Q}$-basis $\{1\otimes[K,\psi]_H\}$ of $\mathbb{Q}\otimes A_+(H)$ and write just $[K,\psi]_H \in \mathbb{Q}\otimes A_+(H)$. We will always identify $\mathbb{Q}\otimes A_+$ with $(\mathbb{Q}\otimes A)_+$ and $\mathbb{Q}\otimes A^+$ with $(\mathbb{Q}\otimes A)^+$ via the natural isomorphisms definied in 7.4, cf. Corollary 7.6. Under these isomorphisms of Mackey functors, the structure morphisms $\pi^{\mathbb{Q}\otimes A}$ and $\mathbb{Q}\otimes\pi^A$ are also identified, as well as the maps $\mathbb{Q}\otimes r^p_H\colon \mathbb{Q}\otimes M(H)\to\mathbb{Q}\otimes A^+(H)$ and $r^{\mathbb{Q}\otimes p}_H\colon \mathbb{Q}\otimes M(H)\to(\mathbb{Q}\otimes A)^+(H)$, where we set $r^p_H := \lambda^{-1}_{M,A}(p)_H = (p_K\circ\mathrm{res}^H_K)_{K\le H}$, cf. Proposition 5.2 (ii). We will work with the Mackey functors $\mathbb{Q}\otimes A_+$ and $\mathbb{Q}\otimes A^+$ and denote the corresponding structural maps again by $c_+, \mathrm{res}_+, \mathrm{ind}_+, c^+, \mathrm{res}^+, \mathrm{ind}^+, \pi^A, \rho^A, r^p, b^{M,A}$. Note that $\rho^A\colon A_+\to A^+$ is injective and $\rho^A\colon \mathbb{Q}\otimes A_+\to\mathbb{Q}\otimes A^+$ is an isomorphism by Proposition 2.4. Under the above identifications we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}-\mathbf{Res}(G)(\mathbb{Q}\otimes M,\mathbb{Q}\otimes A_+) & \stackrel{\Theta_{\mathbb{Q}\otimes M,\mathbb{Q}\otimes A}}{\longrightarrow} & \mathbb{Q}-\mathbf{Con}(G)(\mathbb{Q}\otimes M,\mathbb{Q}\otimes A) \\ \cup & & \cup \\ \mathbb{Z}-\mathbf{Res}(G)(M,A_+) & \stackrel{\Theta_{M,A}}{\longrightarrow} & \mathbb{Z}-\mathbf{Con}(G)(M,A), \end{array}$$

where the vertical inclusions are described by extensions of scalars, and $\Theta_{M,A}$ and $\Theta_{\mathbb{Q}\otimes M,\mathbb{Q}\otimes A}$ are the maps of taking residues (cf. 5.3). By Proposition 5.4 (ii) and (iii), $\Theta_{M,A}$ is injective and $\Theta_{\mathbb{Q}\otimes M,\mathbb{Q}\otimes A}$ is an isomorphism. The set of canonical induction formulae for M from A is a subset of $\mathbb{Z}-\mathbf{Res}(G)(M,A_+)$ and can be identified via $\Theta_{M,A}$ with a subset of $\mathbb{Z}-\mathbf{Con}(G)(M,A)$. In the next section we will derive sufficient conditions for $p\in\mathbb{Z}-\mathbf{Con}(G)(M,A)$ to be the residue of a canonical induction formula for M from A, i.e. that $a^{\mathbb{Q}\otimes M,\mathbb{Q}\otimes A,\mathbb{Q}\otimes p}(M)\subseteq A_+$, in which case we call $a^{\mathbb{Q}\otimes M,\mathbb{Q}\otimes A,\mathbb{Q}\otimes p}$ ***integral***. Note that this is equivalent to the condition $p\in\mathrm{im}(\Theta_{M,A})$, i.e. $p=\Theta_{M,A}(a)$ for a unique $a\in\mathbb{Z}-\mathbf{Res}(G)(M,A_+)$, namely $a=a^{M,A,p}$ with the notation of Definition 5.5.

(e) Now we assume the integrality condition that $p=\Theta_{M,A}(a)$ for some $a=a^{M,A,p}\in\mathbb{Z}-\mathbf{Res}(G)(M,A_+)$. Then $\mathbb{Q}\otimes a = a^{\mathbb{Q}\otimes M,\mathbb{Q}\otimes A,\mathbb{Q}\otimes p}$ under the identifications of part (d). Since we have the commutative diagram

$$\begin{array}{ccccccc} M & & \stackrel{a^{M,A,p}}{\longrightarrow} & & A_+ & \stackrel{b^{M,A}}{\longrightarrow} & M \\ & r^p\searrow & & \swarrow\rho^A & & & \\ & & A_+ & & & & \\ \cap & & \cap & & \cap & & \cap \\ & & \mathbb{Q}\otimes A_+ & & & & \\ & \mathbb{Q}\otimes r^p\nearrow & & \nwarrow\mathbb{Q}\otimes\rho^A & & & \\ \mathbb{Q}\otimes M & & \underset{\mathbb{Q}\otimes a^{M,A,p}}{\longrightarrow} & & \mathbb{Q}\otimes A_+ & \underset{\mathbb{Q}\otimes b^{M,A}}{\longrightarrow} & \mathbb{Q}\otimes M \end{array}$$

with vertical inclusions, we know that a is an induction formula (resp. a morphism of Mackey functors, resp. a morphism of k-algebra restriction functors, resp. $A(H)$-linear for all $H \leq G$, resp. commuting with a morphism $f \in \mathbb{Z}-\mathbf{Res}(G)(M, M'$ with $f(A) \subseteq A'$ for some other M', A' satisfying the standard hypothesis, cf. Proposition 6.11) if and only if the $\mathbb{Q}$-tensored statement holds for $\mathbb{Q} \otimes a$. Hence we can apply all the results developed in Section 6 to $\mathbb{Q} \otimes a$ and obtain the desired information about a.

(f) From now on we will write $a^{M,A,p}$ also for $a^{\mathbb{Q}\otimes M, \mathbb{Q}\otimes A, \mathbb{Q}\otimes p}$. With Proposition 6.4 we already have a nice criterion to decide, whether $a^{M,A,p}\colon \mathbb{Q} \otimes M \to \mathbb{Q} \otimes A_+$ is a canonical induction formula. What we don't have so far, are conditions on M, A, $\mathcal{B}$, and p implying that $a^{M,A,p}$ is integral. Note that there is the factor $1/|H|$ in (6.1.a) which makes this task difficult. We will show in the next section that under suitable conditions one can get rid of the denominator $|H|$. As a first step towards this result we have to refine the summation in Equation (6.1.a) to an alternating sum by expanding the Möbius coefficient and using the simplicial complex of chains of the poset $\mathcal{M}(G)$ as index set for the summation.

8.3 We assume M, A, $\mathcal{B}$, and p as in Hypothesis 8.1. For $H \leq G$ and $\chi \in M(H)$ we write

$$p_H(\chi) = \sum_{\varphi \in \mathcal{B}(H)} m_\varphi(\chi) \cdot \varphi \in A(H)$$

and call $m_\varphi(\chi) \in \mathbb{Z}$ the ***multiplicity*** of φ in χ. Note that these multiplicities depend on p without being apparent from the notation. Note also that $m_\psi(\mathrm{res}^H_K(\varphi)) = m^{(H,\varphi)}_{(K,\psi)}$ for $K \leq H$, $\varphi \in \mathcal{B}(H)$, and $\psi \in \mathcal{B}(K)$, if $p_K|_{A(K)} = \mathrm{id}_{A(K)}$, but that in general this equality does not hold.

We denote the set of strictly ascending chains $\sigma = ((H_0, \varphi_0) < \cdots < (H_n, \varphi_n))$ of elements in $\mathcal{M}(H)$ by $\Delta(\mathcal{M}(H))$. We write $|\sigma| = n$ for the ***length*** of the above chain. Note that $\Delta(\mathcal{M}(H))$ is an H-set and $\Delta(\mathcal{M}(H)) \subseteq \Delta(\mathcal{M}(G))$ for all $H \leq G$. For σ as above we define the ***multiplicity*** m_σ of σ by

$$m_\sigma := m^{(H_1,\varphi_1)}_{(H_0,\varphi_0)} \cdots m^{(H_n,\varphi_n)}_{(H_{n-1},\varphi_{n-1})} \in \mathbb{N}_0,$$

cf. 8.1 (ii). This multiplicity does not depend on p.

8.4 Lemma *Assuming Hypothesis 8.1 and the notation from 8.3, we have*

$$a^{M,A,p}_H(\chi) = \frac{1}{|H|} \sum_{\substack{\sigma=((H_0,\varphi_0)<\cdots<(H_n,\varphi_n)) \\ \in \Delta(\mathcal{M}(H))}} (-1)^n |H_0| m_\sigma m_{\varphi_n}(\mathrm{res}^H_{H_n}(\chi))[H_0, \varphi_0]_H \tag{8.4.a}$$

for all $H \leq G$ and $\chi \in M(H)$.

Proof We expand the explicit formula (6.1.a) using

$$\mu(L, K) = \sum_{L=H_0<\cdots<H_n=K} (-1)^n,$$

the sum running over all chains connecting L and K, cf. [21, 3, Prop. 6]. This yields

$$\begin{aligned} a_H^{M,A,p}(\chi) &= \frac{1}{|H|} \sum_{H_0<\cdots<H_n\le H} (-1)^n |H_0| [H_0, (\mathrm{res}^{H_n}_{H_0} \circ p_{H_n} \circ \mathrm{res}^H_{H_n})(\chi)]_H \\ &= \frac{1}{|H|} \sum_{\substack{H_0<\cdots<H_n\le H \\ \varphi_n\in\mathcal{B}(H_n)}} (-1)^n |H_0| m_{\varphi_n}(\mathrm{res}^H_{H_n}(\chi)) [H_0, \mathrm{res}^{H_n}_{H_0}(\varphi_n)]_H \end{aligned}$$

for $H \le G$ and $\chi \in M(H)$. For a fixed chain $H_0 < \cdots < H_n$ of subgroups of H and $\varphi_n \in \mathcal{B}(H_n)$ we have

$$\begin{aligned} \mathrm{res}^{H_n}_{H_0}(\varphi_n) &= (\mathrm{res}^{H_1}_{H_0} \circ \cdots \circ \mathrm{res}^{H_n}_{H_{n-1}})(\varphi_n) \\ &= \sum_{\varphi_{n-1}\in\mathcal{B}(H_{n-1})} m^{(H_n,\varphi_n)}_{(H_{n-1},\varphi_{n-1})} \cdot (\mathrm{res}^{H_1}_{H_0} \circ \cdots \circ \mathrm{res}^{H_{n-1}}_{H_{n-2}})(\varphi_{n-1}) \\ &\;\vdots \\ &= \sum_{\varphi_0\in\mathcal{B}(H_0)} \cdots \sum_{\varphi_{n-1}\in\mathcal{B}(H_{n-1})} m^{(H_n,\varphi_n)}_{(H_{n-1},\varphi_{n-1})} \cdots m^{(H_1,\varphi_1)}_{(H_0,\varphi_0)} \cdot \varphi_0 \\ &= \sum_{\sigma=((H_0,\varphi_0)<\cdots<(H_n,\varphi_n))\in\Delta(\mathcal{M}(H))} m_\sigma \cdot \varphi_0, \end{aligned}$$

since $m^{(H_i,\varphi_i)}_{(H_{i-1},\varphi_{i-1})} = 0$ unless $(H_{i-1}, \varphi_{i-1}) < (H_i, \varphi_i)$ by definition. Substituting this formula for $\mathrm{res}^{H_n}_{H_0}(\varphi_n)$ into the last alternating sum we obtain the result. $\square$

9 An integrality theorem

In this section we are going to transform the alternating sum formula (8.4.a) for $a_H^{M,A,p}$ in order to derive an integrality criterion in Theorem 9.3. Throughout this section we assume $A \subseteq M$, $\mathcal{B}$, and $p\colon M \to A$ as in Hypothesis 8.1 and use the associated multiplicities $m^{(H,\varphi)}_{(K,\psi)}$, m_σ, and $m_\varphi(\chi)$ for $(K,\psi), (H,\varphi) \in \mathcal{M}(G)$, $\chi \in M(U)$, $\varphi \in \mathcal{B}(U)$, $U \le G$, $\sigma \in \Delta(\mathcal{M}(G))$.

9.1 Let $H \le G$. For

$$\sigma = ((H_0,\varphi_0) < \cdots < (H_n,\varphi_n)) \in \Delta(\mathcal{M}(H)) \tag{9.1.a}$$

and a set π of primes we define

$$N_H^\pi(\sigma) := \{s \in N_H(\sigma) \mid s_{\pi'} \in H_0\},$$

where $N_H(\sigma)$ denotes the stabilizer of σ in H and $s_{\pi'}$ the π'-part of s, π' being the complement of π in the set of all primes. Note that $H_0 \trianglelefteq N_H(\sigma)$ by axiom (C1) in Definition 1.1 and that for $s \in N_H(\sigma)$ the condition $s_{\pi'} \in H_0$ is equivalent to sH_0 being a π-element in $N_H(\sigma)/H_0$. Hence, we have

$$|N_H^\pi(\sigma)| = |H_0| \cdot |(N_H(\sigma)/H_0)_\pi|, \tag{9.1.b}$$

where G_π denotes the set of π-elements in an arbitrary group G.

Our aim is to replace under suitable conditions, which will arise along the way, the factor $|H_0|$ in Formula (8.4.a) with the bigger factor $|N_H^\pi(\sigma)|$ without changing the result of the alternating sum; thereby obtaining in the case, where π is the set of all primes, an integral expression after passing to a sum over H-orbits in $\Delta(\mathcal{M}(H))$.

For $H \leq G$ and π as above we enlarge $\Delta(\mathcal{M}(H))$ by defining

$$\widetilde{\Delta}^\pi(\mathcal{M}(H)) := \{(s,\sigma) \in H \times \Delta(\mathcal{M}(H)) \mid s \in N_H^\pi(\sigma)\}$$

and we give a partition

$$\widetilde{\Delta}^\pi(\mathcal{M}(H)) = \widetilde{\Delta}_0^\pi(\mathcal{M}(H)) \cup \widetilde{\Delta}_1^\pi(\mathcal{M}(H)) \cup \widetilde{\Delta}_2^\pi(\mathcal{M}(H)) \cup \widetilde{\Delta}_3^\pi(\mathcal{M}(H)) \tag{9.1.c}$$

of $\widetilde{\Delta}^\pi(\mathcal{M}(H))$ into four disjoint subsets by defining for $(s,\sigma) \in \widetilde{\Delta}^\pi(\mathcal{M}(H))$ with σ as in (9.1.a):

$$\begin{aligned}
(s,\sigma) \in \widetilde{\Delta}_0^\pi(\mathcal{M}(H)) &:\Longleftrightarrow s \in H_0,\\
(s,\sigma) \in \widetilde{\Delta}_1^\pi(\mathcal{M}(H)) &:\Longleftrightarrow s \notin H_n,\\
(s,\sigma) \in \widetilde{\Delta}_2^\pi(\mathcal{M}(H)) &:\Longleftrightarrow s \in H_{i+1} \smallsetminus H_i \text{ and } H_i(s) < H_{i+1}\\
&\qquad\quad \text{for some } i \in \{0,\dots,n-1\},\\
(s,\sigma) \in \widetilde{\Delta}_3^\pi(\mathcal{M}(H)) &:\Longleftrightarrow s \in H_{i+1} \smallsetminus H_i \text{ and } H_i(s) = H_{i+1}\\
&\qquad\quad \text{for some } i \in \{0,\dots,n-1\},
\end{aligned}$$

where $H_i(s)$ denotes the subgroup of G generated by H_i and s (s normalizes H_i, since $s \in N_H(\sigma)$). Note that the index i in the definition of $\widetilde{\Delta}_2^\pi(\mathcal{M}(H))$ and $\widetilde{\Delta}_3^\pi(\mathcal{M}(H))$ is uniquely determined by the condition $s \in H_{i+1} \smallsetminus H_i$.

Next we consider the map

$$\begin{aligned}
f\colon \widetilde{\Delta}_3^\pi(\mathcal{M}(H)) &\to \widetilde{\Delta}_1^\pi(\mathcal{M}(H)) \cup \widetilde{\Delta}_2^\pi(\mathcal{M}(H)),\\
(s,\sigma) &\mapsto (s, (H_0,\varphi_0) < \cdots < \widehat{(H_{i+1},\varphi_{i+1})} < \cdots < (H_n,\varphi_n)),
\end{aligned}$$

for σ as in (9.1.a), where $i \in \{0,\dots,n-1\}$ is given by $s \in H_{i+1} \smallsetminus H_i$, and the pair $(H_{i+1},\varphi_{i+1}) = (H_i(s),\varphi_{i+1})$ is ommitted from σ.

For $(s,\sigma) \in \widetilde{\Delta}_1^\pi(\mathcal{M}(H))$ with σ as in (9.1.a) we have

$$\begin{aligned}
f^{-1}((s,\sigma)) = \bigl\{(s, (H_0,\varphi_0) < \cdots < (H_n,\varphi_n) < (H_n(s),\varphi))\bigm|\\
\varphi \in \mathcal{B}(H_n(s)), m_{(H_n,\varphi_n)}^{(H_n(s),\varphi)} > 0\bigr\},
\end{aligned}$$

and for $(s,\sigma) \in \widetilde{\Delta}_2^\pi(\mathcal{M}(H))$ with σ as in (9.1.a) and $i \in \{0,\ldots,n-1\}$ with $s \in H_{i+1} \smallsetminus H_i$, and hence $H_i(s) < H_{i+1}$, we have

$$f^{-1}((s,\sigma)) = \big\{(s,(H_0,\varphi_0) < \cdots < (H_i,\varphi_i) < (H_i(s),\varphi) < \cdots < (H_n,\varphi_n))\big| \\ \varphi \in \mathcal{B}(H_i(s)), m_{(H_i,\varphi_i)}^{(H_i(s),\varphi)} m_{(H_i(s),\varphi)}^{(H_{i+1},\varphi_{i+1})} \neq 0\big\}.$$

9.2 Introducing for $H \leq G$, σ as in (9.1.a), and $\chi \in M(H)$ the abbreviation

$$g(\chi,\sigma) := (-1)^{|\sigma|} m_\sigma m_{\varphi_n}(\mathrm{res}_{H_n}^H(\chi))[H_0,\varphi_0]_H \in A_+(H), \tag{9.2.a}$$

we can rewrite Equation (8.4.a) as

$$|H| \cdot a_H^{M,A,p}(\chi) = \sum_{(s,\sigma)\in\widetilde{\Delta}_0^\pi(\mathcal{M}(H))} g(\chi,\sigma),$$

where the factor $|H_0|$ in Equation (8.4.a) is replaced by summing over pairs (s,σ) with $s \in H_0$, where σ is as in (9.1.a). Using the partition in (9.1.c) and abbreviating $\widetilde{\Delta}^\pi(\mathcal{M}(H))$ (resp. $\widetilde{\Delta}_i^\pi(\mathcal{M}(H))$ for $i = 0,1,2,3$) by $\widetilde{\Delta}^\pi$ (resp. $\widetilde{\Delta}_i^\pi$), we may continue with

$$|H| \cdot a_H^{M,A,p}(\chi) = \sum_{(s,\sigma)\in\widetilde{\Delta}^\pi} g(\chi,\sigma) - \sum_{(s,\sigma)\in\widetilde{\Delta}_1^\pi\cup\widetilde{\Delta}_2^\pi\cup\widetilde{\Delta}_3^\pi} g(\chi,\sigma). \tag{9.2.b}$$

We examine the last sum in (9.2.b) further using the function f from 9.1:

$$\sum_{(s,\sigma)\in\widetilde{\Delta}_1^\pi\cup\widetilde{\Delta}_2^\pi\cup\widetilde{\Delta}_3^\pi} g(\chi,\sigma) = \sum_{(s,\sigma)\in\widetilde{\Delta}_1^\pi} \Big(g(\chi,\sigma) + \sum_{(s',\sigma')\in f^{-1}((s,\sigma))} g(\chi,\sigma')\Big) \tag{9.2.c}$$
$$+ \sum_{(s,\sigma)\in\widetilde{\Delta}_2^\pi} \Big(g(\chi,\sigma) + \sum_{(s',\sigma')\in f^{-1}((s,\sigma))} g(\chi,\sigma')\Big).$$

For $(s,\sigma) \in \widetilde{\Delta}_1^\pi$ with σ as in (9.1.a), i.e. $s \notin H_n$, we have

$$g(\chi,\sigma) + \sum_{(s',\sigma')\in f^{-1}((s,\sigma))} g(\chi,\sigma') = \tag{9.2.d}$$
$$(-1)^{|\sigma|} m_\sigma \Big(m_{\varphi_n}(\mathrm{res}_{H_n}^H(\chi)) - \sum_{\varphi\in\mathcal{B}(H_n(s))} m_{(H_n,\varphi_n)}^{(H_n(s),\varphi)} m_\varphi(\mathrm{res}_{H_n(s)}^H(\chi))\Big)[H_0,\varphi_0]_H,$$

and for $(s,\sigma) \in \widetilde{\Delta}_2^\pi$ with σ as in (9.1.a) and $i \in \{0,\ldots,n-1\}$ with $s \in H_{i+1} \smallsetminus H_i$, hence $H_i(s) < H_{i+1}$, we have

$$g(\chi,\sigma) + \sum_{(s',\sigma')\in f^{-1}((s,\sigma))} g(\chi,\sigma') \tag{9.2.e}$$
$$= (-1)^{|\sigma|} \frac{m_\sigma}{m_{(H_i,\varphi_i)}^{(H_{i+1},\varphi_{i+1})}} m_{\varphi_n}(\mathrm{res}_{H_n}^H(\chi)) \times$$
$$\times \Big(m_{(H_i,\varphi_i)}^{(H_{i+1},\varphi_{i+1})} - \sum_{\varphi\in\mathcal{B}(H_i(s))} m_{(H_i,\varphi_i)}^{(H_i(s),\varphi)} m_{(H_i(s),\varphi)}^{(H_{i+1},\varphi_{i+1})}\Big)[H_0,\varphi_0]_H$$
$$= 0,$$

since

$$m_{(H_i,\varphi_i)}^{(H_{i+1},\varphi_{i+1})} = \sum_{\varphi\in\mathcal{B}(H_i(s))} m_{(H_i,\varphi_i)}^{(H_i(s),\varphi)} m_{(H_i(s),\varphi)}^{(H_{i+1},\varphi_{i+1})}$$

by transitivity of restriction.

Finally substituting (9.2.d) and (9.2.e) in (9.2.c) we can rewrite (9.2.b) as

$$\begin{aligned} &|H|\cdot a_H^{M,A,p}(\chi) \qquad (9.2.f)\\ &= \sum_{(s,\sigma)\in\widetilde{\Delta}^\pi} g(\chi,\sigma) - \sum_{\left(s,\sigma=((H_0,\varphi_0)<\cdots<(H_n,\varphi_n))\right)\in\widetilde{\Delta}_1^\pi} (-1)^{|\sigma|} m_\sigma h(\chi,s,\sigma)[H_0,\varphi_0]_H \end{aligned}$$

for $H\leq G$ and $\chi\in M(H)$, with

$$h(\chi,s,\sigma) = m_{\varphi_n}(\mathrm{res}_{H_n}^H(\chi)) - \sum_{\varphi\in\mathcal{B}(H_n(s))} m_{(H_n,\varphi_n)}^{(H_n(s),\varphi)} m_\varphi(\mathrm{res}_{H_n(s)}^H(\chi)). \qquad (9.2.g)$$

9.3 Theorem *Let $H\leq G$ and $\chi\in M(H)$. Assume that for a set π of primes the following condition holds:*

($*_\pi$) *For all $T\trianglelefteq U\leq H$ such that U/T is a cyclic π-group and all $\psi\in\mathcal{B}(T)$ which are fixed under U (i.e. ${}^u\psi=\psi$ for all $u\in U$), the coefficients of ψ in the two elements $p_T(\mathrm{res}_T^U(\vartheta))$ and $\mathrm{res}_T^U(p_U(\vartheta))$ in $A(T)$ with respect to the basis $\mathcal{B}(T)$, where $\vartheta:=\mathrm{res}_U^H(\chi)$, coincide, i.e.*

$$m_\psi(\mathrm{res}_T^U(\vartheta)) = \sum_{\varphi\in\mathcal{B}(U)} m_{(T,\psi)}^{(U,\varphi)}\cdot m_\varphi(\vartheta). \qquad (9.3.a)$$

Then one has

$$\begin{aligned} a_H^{M,A,p}(\chi) &= \frac{1}{|H|}\sum_{\sigma=((H_0,\varphi_0)<\cdots<(H_n,\varphi_n))\in\Delta(\mathcal{M}(H))} (-1)^{|\sigma|}|N_H^\pi(\sigma)|m_\sigma\times \qquad (9.3.b)\\ &\qquad\times m_{\varphi_n}(\mathrm{res}_{H_n}^H(\chi))[H_0,\varphi_0]_H\\ &= \sum_{\sigma=((H_0,\varphi_0)<\cdots<(H_n,\varphi_n))\in H\backslash\Delta(\mathcal{M}(H))} (-1)^{|\sigma|}\frac{|(N_H(\sigma)/H_0)_\pi|}{|N_H(\sigma)/H_0|}m_\sigma\times\\ &\qquad\times m_{\varphi_n}(\mathrm{res}_{H_n}^H(\chi))[H_0,\varphi_0]_H, \end{aligned}$$

where the second sum runs over a set of representatives for the H-orbits of $\Delta(\mathcal{M}(H))$.

Proof We consider the term $h(\chi,s,\sigma)$ in (9.2.g). With $U=H_n(s)$, $T=H_n$, and $\psi=\varphi_n$, condition ($*_\pi$) implies that $h(\chi,s,\sigma)=0$. Therefore we obtain from (9.2.f):

$$a_H^{M,A,p}(\chi) = \frac{1}{|H|} \sum_{(s,\sigma)\in\tilde{\Delta}^\pi(\mathcal{M}(H))} g(\chi,\sigma)$$

$$= \frac{1}{|H|} \sum_{\sigma=((H_0,\varphi_0)<\cdots<(H_n,\varphi_n))\in\Delta(\mathcal{M}(H))} (-1)^{|\sigma|}|N_H^\pi(\sigma)|m_\sigma \times$$

$$\times\, m_{\varphi_n}(\mathrm{res}_{H_n}^H(\chi))[H_0,\varphi_0]_H.$$

This is the first equation of the theorem. Since the summands of the above sum are constant on H-orbits of $\Delta(\mathcal{M}(H))$, we may collect them and obtain further:

$$a_H^{M,A,p}(\chi) = \frac{1}{|H|} \sum_{\substack{\sigma=((H_0,\varphi_0)<\cdots<(H_n,\varphi_n))\\ \in H\backslash\Delta(\mathcal{M}(H))}} (-1)^{|\sigma|}\frac{|H|}{|N_H(\sigma)|}|N_H^\pi(\sigma)|m_\sigma \times$$

$$\times\, m_{\varphi_n}(\mathrm{res}_{H_n}^H(\chi))[H_0,\varphi_0]_H.$$

Now the second equation of the theorem follows from Equation (9.1.b):

$$\frac{|N_H^\pi(\sigma)|}{|N_H(\sigma)|} = \frac{|H_0|\cdot|(N_H(\sigma)/H_0)_\pi|}{|H_0|\cdot|N_H(\sigma)/H_0|} = \frac{|(N_H(\sigma)/H_0)_\pi|}{|N_H(\sigma)/H_0|}.$$

□

9.4 Corollary *Let $H \leq G$, $\chi \in M(H)$, and assume that condition $(*_\pi)$ holds for some set π of primes. Then*

$$|H|_{\pi'} \cdot a_H^{M,A,p}(\chi) \in A_+(H),$$

i.e. in $a_H^{M,A,p}(\chi)$ occur only π'-numbers as denominators.

Proof It is well-known (cf. [17, V.19.14]) that $|G_\pi|$ is a multiple of $|G|_\pi$ for any finite group G. Therefore, the factor $|(N_H(\sigma)/H_0)_\pi|/|N_H(\sigma)/H_0|$ in the second sum in (9.3.b) splits into an integer and a rational number whose denominator divides $|H|_{\pi'}$:

$$\frac{|(N_H(\sigma)/H_0)_\pi|}{|N_H(\sigma)/H_0|} = \frac{|(N_H(\sigma)/H_0)_\pi|}{|N_H(\sigma)/H_0|_\pi} \cdot \frac{1}{|N_H(\sigma)/H_0|_{\pi'}}.$$

□

9.5 Corollary *If $(*_\pi)$ holds for the set π of all primes and for all $H \leq G$ and $\chi \in M(H)$, then we have*

$$a_H^{M,A,p}(\chi) \tag{9.5.a}$$

$$= \sum_{\sigma=((H_0,\varphi_0)<\cdots<(H_n,\varphi_n))\in H\backslash\Delta(\mathcal{M}(H))} (-1)^n m_\sigma m_{\varphi_n}(\mathrm{res}_{H_n}^H(\chi))[H_0,\varphi_0]_H$$

for all $H \leq G$ *and all* $\chi \in M(H)$*; in particular,* $a^{M,A,p}$ *is integral.*

If additionally $A(H) = M(H)$ *and* $p_H = \mathrm{id}_{M(H)}$ *for all* $H \in \mathcal{C}(M)$*, then* $b^{M,A} \circ a^{M,A,p} = \mathrm{id}_M$*, i.e.* $a^{M,A,p}$ *is an integral canonical induction formula.*

Proof Formula (9.5.a) is an immediate consequence of Formula (9.3.b). Since $M(H)$ is free for all $H \leq G$, we have $\mathcal{C}(M) = \mathcal{C}(\mathbb{Q} \otimes M)$. Therefore, Corollary 6.6 yields the last statement. □

9.6 Remark Comparing the general formula for $a_H^{M,A,p}(\chi)$ in Lemma 8.4 to the second one in (9.3.b), one realizes that under the hypothesis $(*_\pi)$ one can replace the factor $|H_0|$ with the bigger factor $|N_H^\pi(\sigma)|$ without changing the whole alternating sum. In the standard example, i.e. $M = R$, $A = R^{\mathrm{ab}}$, and $p_H(\chi) = \chi$ for linear $\chi \in \mathrm{Irr}(H)$ and $p_H(\chi) = 0$ for non-linear $\chi \in \mathrm{Irr}(H)$, this was observed in [3] for the set π of all primes by interpreting both alternating sums as Euler characteristics of certain chain complexes which could be proven to be homotopy equivalent. Theorem 9.3 will be the main tool for integrality proofs of all the canonical induction formulae in Example 6.13.

9.7 Example We show that for $M = R$, $A = R^{\mathrm{ab}}$, $\mathcal{B}$, and p as in Example 6.13 (a) and Remark 7.3, condition $(*_\pi)$ is satisfied for all $H \leq G$ and all $\chi \in R(H)$, where π is the set of all primes. Then Corollary 9.5 implies that $a^{M,A,p}$ is an integral canonical induction formula, and that the explicit formula (9.5.a) holds. Note that in this example $m_\sigma = 1$ for all chains $\sigma \in \mathcal{M}(G)$. This formula is the same as the one obtained in [2] with a different proof for the integrality.

So let U be a finite group and $T \trianglelefteq U$ such that U/T is cyclic. Let furthermore $\vartheta \in \mathrm{Irr}(U)$ and $\psi \in \widehat{T}$ such that $\psi\colon T \to \mathbb{C}^\times$ is U-stable. Then we have to show that the multiplicities of ψ in $p_T(\mathrm{res}_T^U(\vartheta))$ and $\mathrm{res}_T^U(p_U(\vartheta))$ coincide. If ϑ is linear, then $p_T(\mathrm{res}_T^U(\vartheta)) = \mathrm{res}_T^U(\vartheta) = \mathrm{res}_T^U(p_U(\vartheta))$ and we are done. If $\vartheta(1) > 1$, then $\mathrm{res}_T^U(p_U(\vartheta)) = 0$ and we have to show that ψ does not occur in $p_T(\mathrm{res}_T^U(\vartheta))$ which is equivalent to

$$0 = (\psi, \mathrm{res}_T^U(\vartheta))_T = (\mathrm{ind}_T^U(\psi), \vartheta)_U$$

by Frobenius reciprocity. Since ψ is U-stable, the cyclic subgroup $T/\ker(\psi)$ is central in $U/\ker(\psi)$, and since U/T is cyclic, $U/\ker(\psi)$ is abelian. Therefore, $\mathrm{ind}_T^U(\psi)$, being a character which comes via inflation from a character of the abelian group $U/\ker(\psi)$, splits into linear characters, and hence $(\mathrm{ind}_T^U(\psi), \vartheta)_U = 0$.

Note that by exactly the same proofs we obtain for any field K of characteristic zero containing a primitive $\exp(G)$-th root of unity, an equivalent canonical induction formula for $R_K \cong R$ from $R_K^{\mathrm{ab}} \cong R^{\mathrm{ab}}$, $R_K(H)$ being the Grothendieck group of KH**−mod** or equivalently the ring of virtual K-characters, and $R_K^{\mathrm{ab}}(H) \subseteq R_K(H)$ the span of $\widehat{H}(K)$ for $H \leq G$.

9.8 Example We show that for $M = R_F$, $A = R_F^{\mathrm{ab}}$, $\mathcal{B}$, and p as in Example 6.13 (b) and Remark 7.3, condition $(*_\pi)$ is satisfied for all $H \leq G$

and all $[V] \in R_F(H)$, $V \in FH\mathbf{-mod}$, where π is the set of all primes. Again Corollary 9.5 applies, showing that $a^{R_F, R_F^{\mathrm{ab}}, p}$ is an integral canonical induction formula. Note that in Equation (9.5.a) we have $m_\sigma = 1$ for all $\sigma \in \Delta(\mathcal{M}(G))$, and that $m_\varphi([V])$ counts the multiplicity of F_φ as a composition factor in $V \in FH\mathbf{-mod}$ for $H \leq G$ and $\varphi \in \widehat{H}(F)$.

So let $T \trianglelefteq U$ be as in the previous example, let $S \in FU\mathbf{-mod}$ be irreducible, and let $\psi \in \widehat{T}(F)$ be U-stable. We have to show that the multiplicities of $[F_\psi]$ in $p_T(\mathrm{res}_T^U([S]))$ and in $\mathrm{res}_T^U(p_U([S]))$ coincide. If $\dim_F S = 1$, this holds trivially as in Example 9.7. If $\dim_F S > 1$, then $\mathrm{res}_T^U(p_U([S])) = 0$, and we have to show that F_ψ is not a composition factor of $\mathrm{res}_T^U([S])$. But since T is normal in U, $\mathrm{res}_T^U(S)$ is semisimple and we have to show that

$$0 = \mathrm{Hom}_{FT}(F_\psi, \mathrm{res}_T^U(S)) \cong \mathrm{Hom}_{FU}(\mathrm{ind}_T^U(F_\psi), S).$$

As in the previous example, $U/\ker(\psi)$ is abelian, and $\mathrm{ind}_T^U(F_\psi)$ is the inflation of an $FU/\ker(\psi)$-module. Since $U/\ker(\psi)$ is abelian, $\mathrm{ind}_T^U(F_\psi)$ has only one-dimensional composition factors, and therefore, $\mathrm{Hom}_{FU}(\mathrm{ind}_T^U(F_\psi), S) = 0$ as required.

Let $d\colon R \to R_F$ be the decomposition map (cf. [22, 15.2]) which is a morphism of Mackey functors on G with $d(R^{\mathrm{ab}}) \subseteq R_F^{\mathrm{ab}}$. We remark that the diagram

$$\begin{array}{ccc} R & \xrightarrow{a} & R_+^{\mathrm{ab}} \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d_+} \\ R_F & \xrightarrow{\tilde{a}} & R_{F+}^{\mathrm{ab}} \end{array}$$

with the canonical induction formulae a from Example 9.7 and $\tilde{a}$ from the present example, is *not* commutative. In fact, there can't be any such canonical induction formula $\tilde{a}\colon R_F \to R_{F+}^{\mathrm{ab}}$, because by Proposition 6.11 we needed

$$\begin{array}{ccc} R & \xrightarrow{p} & R^{\mathrm{ab}} \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ R_F & \xrightarrow{\tilde{p}} & R_F^{\mathrm{ab}} \end{array}$$

to commute for some $\tilde{p} \in \mathbb{Z}\mathbf{-Con}(G)(R_F, R_F^{\mathrm{ab}})$. But for $G = S_4$, ε the sign character, χ_2 the unique irreducible character of degree 2, and $l = \mathrm{char}(F) = 3$, the virtual character $\chi = 1 + \varepsilon - \chi_2$ is in the kernel of d_G, and $d_G(p_G(\chi)) = d_G(1 + \varepsilon) \neq 0$.

9.9 Example Let $M = P_{\mathcal{O}}$, $A = P_{\mathcal{O}}^{\mathrm{ab}}$, $\mathcal{B}$, and p be as in Example 6.13 (c) and Remark 7.3, i.e. $\mathcal{B}(H) = \widehat{H}(\mathcal{O})$for $H \leq G$ an l'-group and $\mathcal{B}(H) = \emptyset$ otherwise. We will show that $a^{P_{\mathcal{O}}, P_{\mathcal{O}}^{\mathrm{ab}}, p}$ is integral. First we show that, with $\pi = l'$ being the set of all primes distinct from $l = \mathrm{char}(\mathcal{O}/\mathrm{rad}(\mathcal{O}))$, condition $(*_{l'})$ is satisfied for all $H \leq G$ and $[V] \in P_{\mathcal{O}}(H)$, $V \in \mathcal{O}H\mathbf{-proj}$.

Let $T \trianglelefteq U \leq G$ with U/T a cyclic l'-group, $\psi\colon T \to \mathcal{O}^\times$ a U-invariant homomorphism such that $\mathcal{O}_\psi$ is projective. Then in particular, T is an l'-group, and with T also U. Let $V \in \mathcal{O}U-\mathbf{proj}$ be indecomposable. We have to show that the multiplicites of $[\mathcal{O}_\psi]$ in $p_T(\mathrm{res}_T^U([V]))$ and $\mathrm{res}_T^U(p_U([V]))$ coincide. But since U is an l'-group, we have $P_\mathcal{O}(U) \cong R(U)$, $P_\mathcal{O}^{\mathrm{ab}}(U) = R^{\mathrm{ab}}(U)$, and the proof of Example 9.7 can be repeated.

Hence, from Theorem 9.3 we have

$$a_H^{P_\mathcal{O},P_\mathcal{O}^{\mathrm{ab}},p}([V]) = \sum_{\sigma=((H_0,\varphi_0)<\cdots<(H_n,\varphi_n))\in H\backslash\Delta(\mathcal{M}(H))} (-1)^n \frac{|(N_H(\sigma)/H_0)_{l'}|}{|N_H(\sigma)/H_0|} m_\sigma \times$$
$$\times\, m_{\varphi_n}(\mathrm{res}_{H_n}^H([V]))[H_0,[\mathcal{O}_{\varphi_0}]]_H.$$

Since all occurring groups H_i are l'-groups, we have $|(N_H(\sigma)/H_0)_{l'}| = |N_H(\sigma)_{l'}| \cdot |H_0|$ and it suffices to show that

$$\frac{|N_H(\sigma)_{l'}|}{|N_H(\sigma)|_{l'}} \quad \text{and} \quad \frac{m_{\varphi_n}(\mathrm{res}_{H_n}^H([V]))}{|N_H(\sigma)|_l}$$

are natural numbers. However, $|N_H(\sigma)|_{l'}$ divides $|N_H(\sigma)_{l'}|$ by [17, V.19.14], and $|N_H(\sigma)|_l$ divides $m_{\varphi_n}(\mathrm{res}_{H_n}^H([V]))$ by the following argument. Let Q be a Sylow l-subgroup of $N_H(\sigma)$, $H' := QH_n$, and let W be an indecomposable summand of $\mathrm{res}_{H'}^H(V)$. It suffices to show that $|Q|$ divides the multiplicity $m_{\varphi_n}(\mathrm{res}_{H_n}^{H'}([W]))$, which we may assume to be non-zero. Since W is projective, W is a direct summand of $\mathrm{ind}_1^{H'}(\mathcal{O}) \cong \mathrm{ind}_{H_n}^{H'}(\mathrm{ind}_1^{H_n}(\mathcal{O}))$. Since $H'/H_n \cong Q$ is an l-group, we have $W \cong \mathrm{ind}_{H_n}^{H'}(X)$ for some indecomposable $\mathcal{O}H_n$-module X by Green's indecomposability theorem. Mackey's decomposition formula then yields

$$\mathrm{res}_{H_n}^{H'}(W) \cong \bigoplus_{s\in H_n\backslash H'/H_n} \mathrm{ind}_{H_n\cap {}^sH_n}^{H_n}(\mathrm{res}_{H_n\cap {}^sH_n}^{{}^sH_n}({}^sX)) \cong \bigoplus_{s\in H'/H_n} {}^sX.$$

Since $\mathcal{O}_{\varphi_n}$ is a direct summand of $\mathrm{res}_{H_n}^{H'}(W)$, and since φ_n is stable under $H' \leq N_H(\sigma)$, we have $\mathrm{res}_{H_n}^{H'}(W) \cong \bigoplus_{i=1}^{|H'/H_n|} \mathcal{O}_{\varphi_n}$, and therefore $m_{\varphi_n}(\mathrm{res}_{H_n}^{H'}([W])) = |H'/H_n| = |Q|$.

10 A global point of view

10.1 Often it is possible to define a Mackey functor on any finite group not only on the subgroups of a given group. So let us assume that we have a free abelian group $M(G)$ for each $G \in \mathbf{gr}$, the category of finite groups, such that we have induction and restriction maps whenever there is a subgroup inclusion $H \leq G$, and that there are conjugation maps $c_{g,H}\colon M(H) \to M({}^gH)$, whenever $H \leq G$ and $g \in G$. Moreover let us assume that the axioms in the definition of a Mackey functor are satisfied. We also assume that with respect to restrictions, we can extend M to a contravariant functor $M\colon \mathbf{gr} \to \mathbf{Ab}$ to the category of abelian groups, i.e. that we now have a map $\mathrm{res}_f\colon M(G') \to M(G)$ for every group homomorphism $f\colon G \to G'$. We assume that in the situation $H \leq G$, $g \in G$, for $f\colon {}^gH \to H$, $h' \mapsto g^{-1}hg$, the map $\mathrm{res}_f\colon M(H) \to M({}^gH)$ coincides with the conjugation map $c_{g,H}$. Let us furthermore assume that we are given a subfunctor $A \subseteq M\colon \mathbf{gr} \to \mathbf{Ab}$, such that $A(G)$ has a stable $\mathbb{Z}$-basis $\mathcal{B}(G)$ for every $G \in \mathbf{gr}$ with respect to group isomorphisms and such that the restriction of a basis element in $\mathcal{B}(G)$ to a subgroup $H \leq G$ is a $\mathbb{Z}$-linear combination of $\mathcal{B}(H)$ with non-negative coefficients. Finally suppose we have for each $G \in \mathbf{gr}$ a morphism $p_G\colon M(G) \to A(G)$ commuting with res_f for each group *isomorphism* f. To this situation we will refer in the sequel as the ***global standard situation***. When restricting our attention to subgroups of a given finite group G we are brought back to the situation of Hypothesis 8.1. Note that we can consider each of the Examples in 1.8 (a), (b), (d), (e) as a global standard situation; in part (d) and (e) we can choose $\mathcal{O}$ as the ring of integers of the maximal unramified extension of $\mathbb{Q}_l$ in an algebraic closure.

10.2 Assume that $A \subseteq M\colon \mathbf{gr} \to \mathbf{Ab}$ is given as in 10.1. Then each group homomorphism $f\colon G \to G'$ induces a map

$$\mathrm{res}_{+f}\colon A_+(G') \to A_+(G), \quad [H',a]_{G'} \mapsto \sum_{g' \in f(G)\backslash G'/H'} [f^{-1}({}^{g'}H'), \mathrm{res}_{f_{g',H'}}(a)]_G, \tag{10.2.a}$$

where $f_{g',H'}\colon f^{-1}({}^{g'}H') \to {}^{g'}H' \to H'$ is the restriction of f to $f^{-1}({}^{g'}H')$ followed by the conjugation map $\tilde{h} \mapsto g'^{-1}\tilde{h}g'$ for $\tilde{h} \in {}^{g'}H'$. It is not difficult to see that res_f is a ring homomorphism, if A is a ring valued functor, and that $\mathrm{res}_{+f} \circ \mathrm{res}_{+f'} = \mathrm{res}_{+(f' \circ f)}$, if $f'\colon G' \to G''$ is another group homomorphism, so that A_+ is again a contravariant functor $\mathbf{gr} \to \mathbf{Ab}$ and $-_+$ is a functor from the contravariant functor category $\mathcal{F}^\circ(\mathbf{gr}, \mathbf{Ab})$, resp. $\mathcal{F}^\circ(\mathbf{gr}, \mathbf{Ri})$ for the category $\mathbf{Ri}$ of rings, to itself. Moreover it is a straight forward calculation that $b^{M,A}\colon A_+ \to M$ is a functorial morphism. Assuming M, A, $\mathcal{B}$, and p as in the global standard situation we can define $a_G^{M,A,p}\colon \mathbb{Q} \otimes M(G) \to \mathbb{Q} \otimes A_+(G)$ exactly as in Section 8 for each $G \in \mathbf{gr}$, and all our previous results which hold under Hypothesis 8.1 are still valid.

10.3 Proposition *Let M, A, $\mathcal{B}$, and p be given as in the global standard situation 10.1, and assume that $p_G|_{A(G)} = \mathrm{id}_{A(G)}$ for all $G \in \mathbf{gr}$ and that $\mathrm{res}_\nu(\ker(p_{G/N})) \subseteq \ker(p_G)$ for all canonical epimorphisms $\nu\colon G \to G/N$, $N \trianglelefteq G$. Then the diagram*

$$\begin{array}{ccc} \mathbb{Q} \otimes M(G') & \xrightarrow{a_{G'}^{M,A,p}} & \mathbb{Q} \otimes A_+(G') \\ {\scriptstyle \mathrm{res}_f}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{+f}} \\ \mathbb{Q} \otimes M(G) & \xrightarrow{a_G^{M,A,p}} & \mathbb{Q} \otimes A_+(G) \end{array}$$

commutes for all group homomorphisms $f\colon G \to G'$, i.e. $a^{M,A,p}$ is a functorial morphism from $\mathbb{Q} \otimes M$ to $\mathbb{Q} \otimes A_+$, considered as functors $\mathbf{gr} \to \mathbb{Q}\text{–}\mathbf{Mod}$.

Proof Since f can be written as a composition $G \xrightarrow{f} f(G) \leq G'$ of an epimorphism and an inclusion we may assume that $f\colon G \to G'$ is surjective. Note that, since p commutes with isomorphisms, res_f maps $\ker(p_{G'})$ to $\ker(p_G)$. We will write a_U instead of $a_U^{M,A,p}$ for $U \in \mathbf{gr}$, and proceed by induction on $|G|$. If $|G| = 1$, then the result holds trivially. So let $|G| > 1$. By hypothesis we have $M(G') = A(G') \oplus \ker(p_{G'})$. If $\chi' \in A(G')$, then by Proposition 6.12 we have $a_{G'}(\chi') = [G', \chi']_{G'}$ and also $a_G(\mathrm{res}_f(\chi')) = [G, \mathrm{res}_f(\chi')]_G$ which is equal to $\mathrm{res}_{+f}([G', \chi']_{G'}) = \mathrm{res}_{+f}(a_{G'}(\chi'))$ by (10.2.a).

Now let $\chi' \in \ker(p_{G'})$. Since ρ_G^A is injective, it suffices to show that

$$(\pi_H^A \circ \mathrm{res}_{+H}^{\ G} \circ \mathrm{res}_{+f} \circ a_{G'})(\chi') = (\pi_H^A \circ \mathrm{res}_{+H}^{\ G} \circ a_G \circ \mathrm{res}_f)(\chi') \qquad (10.3.a)$$

for all $H \leq G$. We first assume that $H < G$. The right hand side of (10.3.a) is equal to $(\pi_H^A \circ a_H \circ \mathrm{res}_H^G \circ \mathrm{res}_f)(\chi') = (p_H \circ \mathrm{res}_H^G \circ \mathrm{res}_f)(\chi')$. The left hand side is equal to $(\pi_H^A \circ \mathrm{res}_{+f\colon H \to f(H)} \circ \mathrm{res}_{+f(H)}^{\ G'} \circ a_G)(\chi') = (\pi_H^A \circ \mathrm{res}_{+f\colon H \to f(H)} \circ a_{f(H)} \circ \mathrm{res}_{f(H)}^{G'})(\chi')$ which is equal to $(\pi_H^A \circ a_H \circ \mathrm{res}_{f\colon H \to f(H)} \circ \mathrm{res}_{f(H)}^{G'})(\chi) = (p_H \circ \mathrm{res}_H^G \circ \mathrm{res}_f)(\chi')$ by induction and (10.3.a) holds in this case.

Now let $H = G$. The right hand side of (10.3.a) is equal to $p_G(\mathrm{res}_f(\chi'))$ which is 0, since res_f maps $\ker(p_{G'})$ to $\ker(p_G)$. On the other hand, if we write

$$a_{G'}(\chi') = \sum_{(H',\varphi') \in G' \backslash \mathcal{M}(G')} \alpha_{(H',\varphi')}^{G'}(\chi')[H', \varphi']_{G'}$$

with $\alpha_{(H',\varphi')}^{G'}(\chi') \in \mathbb{Q}$ as a linear combination of the basis elements $[H', \varphi']_{G'}$, we obtain for the left hand side of (10.3.a),

$$(\pi_G^A \circ \mathrm{res}_{+f} \circ a_{G'})(\chi') = \sum_{\varphi' \in \mathcal{B}(G')} \alpha_{(G',\varphi')}^{G'}(\chi') \mathrm{res}_f(\varphi'),$$

since $(\pi_G^A \circ \mathrm{res}_{+f})([H', \varphi']_{G'}) = 0$ for $(H', \varphi') \in \mathcal{M}(G')$ with $H' < G'$. But looking at the explicit formula (6.1.a) we see that $\alpha_{(G',\varphi')}^{G'}(\chi') = 0$ for all $\varphi' \in \mathcal{B}(G')$, since $p_{G'}(\chi') = 0$, and the proof is complete. □

10.4 Remark In the present treatise we avoided to place ourselves into the framework of global Mackey functors as they are defined for example by Webb in [30] or by Bouc in [6], because it would require even more elaborate technicalities, and because we would like to apply the theory also to the absolute Galois group G of a field K and work with all subgroups $H \leq G$ of finite index. It is possible to establish the notion of a Mackey functor as in Definition 1.1 also in this context.

However, the reader who prefers to work with global Mackey functors should have no difficulties to translate our results into the language of global Mackey functors, where one has also induction maps along arbitrary group homomorphisms.

11 Computation of canonical induction formulae

Throughout this section let M, A, $\mathcal{B}$, and $p\colon M \to A$ be given as in Hypothesis 8.1. Let furthermore $a := a^{M,A,p}\colon \mathbb{Q}\otimes M \to \mathbb{Q}\otimes A_+$ be the associated morphism of $\mathbb{Q}$-restriction functors on G.

11.1 For each $H \leq G$, $\chi \in M(H)$, and $(K,\psi) \in \mathcal{M}(H)$ we denote by $\alpha^H_{(K,\psi)}(\chi)$ the coefficient of $a_H(\chi)$ at the basis element $[K,\psi]_H$, i.e.

$$a_H^{M,A,p}(\chi) = \sum_{(K,\psi)\in H\backslash\mathcal{M}(H)} \alpha^H_{(K,\psi)}(\chi)[K,\psi]_H. \tag{11.1.a}$$

The aim of this section is to describe how to compute the coefficients $\alpha^H_{(K,\psi)}(\chi)$. Usually, even for small groups it takes too long to use the explicit formula (6.1.a). More efficient is the use of the commutative diagram (6.1.b). Note that $\mathbb{Q}\otimes A^+(H)$ has a $\mathbb{Q}$-basis $c^{(H)}_{(L,\lambda)} \in (\prod_{K\leq H} A(K))^H$, $(L,\lambda)\in H\backslash\mathcal{M}(H)$, with $c^{(H)}_{(L,\lambda)}$ having entry $\sum_{n\in N_H(L)/N_H(L,\lambda)} {}^n\lambda$ at the component L, the appropriate conjugate entries at all components which are H-conjugate to L, and zero entries everywhere else. This means that each H-conjugate of λ occurs exactly once in $c^{(H)}_{(L,\lambda)}$. Obviously, $c^{(H)}_{(K,\psi)} = c^{(H)}_{(L,\lambda)}$, if and only if $(K,\psi) =_H (L,\lambda)$. Let $\gamma^H_{(L,\lambda),(K,\psi)} \in \mathbb{Z}$ denote the coefficient of $\rho^A_H([K,\psi]_H)$ at the basis element $c^{(H)}_{(L,\lambda)}$ for $(K,\psi),(L,\lambda)\in\mathcal{M}(H)$. Then $\gamma^H_{(L,\lambda),(K,\psi)}$ depends only on the H-orbits of (L,λ) and (K,ψ) in $\mathcal{M}(H)$, and we denote by $\Gamma_{\mathcal{M}(H)}$ the square matrix $(\gamma^H_{(L,\lambda),(K,\psi)})$ indexed by $(L,\lambda),(K,\psi)\in H\backslash\mathcal{M}(H)$, i.e. representatives of the H-orbits of $\mathcal{M}(H)$. Then Diagram (6.1.b) translates into the matrix equation

$$\Gamma_{\mathcal{M}(H)}\cdot\big(\alpha^H_{(K,\psi)}(\chi)\big)_{(K,\psi)} = \big(m_\psi(\mathrm{res}^H_K(\chi))\big)_{(K,\psi)}, \tag{11.1.b}$$

where (K,ψ) runs through $H\backslash\mathcal{M}(H)$. For $(L,\lambda),(K,\psi)\in\mathcal{M}(H)$ we write $(L,\lambda)\leq_H (K,\psi)$, if $(L,\lambda)\leq {}^h(K,\psi)$ for some $h\in H$.

11.2 Proposition *With the notation of 11.1, the coefficients* $\gamma^H_{(L,\lambda),(K,\psi)}$ *for* $H\leq G$ *and* $(L,\lambda),(K,\psi)\in\mathcal{M}(H)$ *are given by*

$$\gamma^H_{(L,\lambda),(K,\psi)} = \sum_{h\in L\backslash H/K} m^{{}^h(K,\psi)}_{(L,\lambda)} = \sum_{h\in H/K} m^{{}^h(K,\psi)}_{(L,\lambda)}. \tag{11.2.a}$$

In particular, $\gamma^H_{(L,\lambda),(K,\psi)} = 0$ *unless* $(L,\lambda)\leq_H (K,\psi)$, *and after a suitable ordering of* $H\backslash\mathcal{M}(H)$, *the matrix* $\Gamma_{\mathcal{M}(H)}$ *is upper triangular with diagonal entry* $|N_H(K,\psi)/K|$ *at position* (K,ψ).

Proof By the definitions of ρ_H^A and $c_{(L,\lambda)}^{(H)}$, the number $\gamma_{(L,\lambda),(K,\psi)}^H$ is the coefficient of λ with respect to $\mathcal{B}(L)$ in

$$\pi_L^A(\mathrm{res}_{+L}^H([K,\psi]_H)) = \sum_{h\in L\backslash H/K} \pi_L([L\cap {}^hK, \mathrm{res}_{L\cap {}^hK}^{{}^hK}({}^h\psi)]_L)$$
$$= \sum_{\substack{h\in L\backslash H/K \\ L\cap {}^hK=L}} \mathrm{res}_L^{{}^hK}({}^h\psi) = \sum_{\substack{h\in L\backslash H/K \\ L\leq {}^hK}} \sum_{\mu\in\mathcal{B}(L)} m_{(L,\mu)}^{{}^h(K,\psi)} \cdot \mu.$$

Now the first equation in 11.2.a follows, since $L \leq {}^hK$ whenever $m_{(L,\lambda)}^{{}^h(K,\psi)} \neq 0$, and the second equation follows, since if $L \leq {}^hK$ then $LhK = hK$. □

11.3 Lemma *With the notation from 11.1 we have*

$$\alpha_{(K,\psi)}^H(\chi) = \frac{|K|}{|N_H(K,\psi)|} \sum_{\substack{\sigma=((H_0,\varphi_0)<\cdots<(H_n,\varphi_n))\in\Delta(\mathcal{M}(H)) \\ (H_0,\varphi_0)=(K,\psi)}} (-1)^n m_\sigma m_{\varphi_n}(\mathrm{res}_{H_n}^H(\chi)) \tag{11.3.a}$$

for all $H \leq G$ and all $\chi \in M(H)$.

Proof By (8.4.a) we have

$$|H|\alpha_{(K,\psi)}^H(\chi) = \sum_{\substack{\sigma=((H_0,\varphi_0)<\cdots<(H_n,\varphi_n))\in\Delta(\mathcal{M}(H)) \\ (H_0,\varphi_0)=_H(K,\psi)}} (-1)^n |H_0| m_\sigma m_{\varphi_n}(\mathrm{res}_{H_n}^H(\chi)).$$

Since in the H-orbit of $\sigma = ((H_0,\varphi_0) < \cdots < (H_n,\varphi_n)) \in \Delta(\mathcal{M}(H))$ with $(H_0,\varphi_0) =_H (K,\psi)$ there are precisely $|N_H(K,\psi)/N_H(\sigma)|$ chains among its $|H/N_H(\sigma)|$ elements which start with (K,ψ), the result follows. □

There is one general result about the vanishing of some of the coefficients $\alpha_{(K,\psi)}^H(\chi)$.

11.4 Proposition *Assume the notation of 11.1 and assume further that there is a stable $\mathbb{Z}$-basis $\widetilde{\mathcal{B}}(H)$ of $M(H)$ for all $H \leq G$, containing $\mathcal{B}(H)$, such that $\mathrm{res}_K^H(\chi)$ is a $\mathbb{Z}$-linear combination of $\widetilde{\mathcal{B}}(K)$ with non-negative coefficients for each $K \leq H$ and $\chi \in \widetilde{\mathcal{B}}(H)$, and that p_H is the identity on $\mathcal{B}(H)$ and zero on $\widetilde{\mathcal{B}}(H) \smallsetminus \mathcal{B}(H)$ for all $H \leq G$.*

Then for each $H \leq G$ and $\chi \in \widetilde{\mathcal{B}}(H)$ the following holds: If $(K,\psi) \in \mathcal{M}(H)$ such that $m_\psi(\mathrm{res}_K^H(\chi)) = 0$, (i.e. ψ does not occur in $\mathrm{res}_K^H(\chi)$), then $\alpha_{(K,\psi)}^H(\chi) = 0$.

Proof We can extend the definitions from 8.1 and 8.3 to multiplicities $m_{(K,\vartheta)}^{(H,\chi)} \in \mathbb{N}_0$ for $K \leq H \leq G$, $\chi \in \widetilde{\mathcal{B}}(H)$, $\vartheta \in \widetilde{\mathcal{B}}(K)$, and $m_\vartheta(\zeta) \in \mathbb{N}_0$ for $\zeta \in M(K)$ with respect to $\widetilde{\mathcal{B}}(K)$. For $\vartheta \in \mathcal{B}(K)$ this definition of $m_\vartheta(\zeta)$ coincides with the old one by the hypothesis on p_K.

We show that each summand in (11.3.a) is zero, if $m_\psi(\mathrm{res}^H_K(\chi)) = 0$. Let $\sigma = ((H_0, \varphi_0) < \cdots < (H_n, \varphi_n)) \in \Delta(\mathcal{M}(H))$ with $(H_0, \varphi_0) = (K, \psi)$. Then

$$0 = m_\psi(\mathrm{res}^H_K(\chi)) = \sum_{\vartheta \in \widetilde{\mathcal{B}}(H_n)} m^{(H_n,\vartheta)}_{(K,\psi)} m_\vartheta(\mathrm{res}^H_{H_n}(\chi))$$

by transitivity of restriction. Since all the factors in the sum are non-negative, and since $m^{(H_n,\varphi_n)}_{(K,\psi)} \neq 0$, we have $m_{\varphi_n}(\mathrm{res}^H_{H_n}(\chi)) = 0$, and the summand in (11.3.a) associated to σ vanishes. □

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